

## An inverse boundary value problem for the equation of flexural vibrations of a bar

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**Abstract.** *In this work, we study one inverse boundary value problem for the equations of flexural torsional vibrations of a bar with an additional condition. Using the Fourier method, the problem reduces to solving a system of integral equations, and using method of contracting mappings, the existence and uniqueness of a solution to a system of integral equations is proved. The existence and uniqueness of the classical solution of the original problem are proved.*

**Keywords.** equation of vibrations of a bar · inverse problem · uniqueness of a solution

**Mathematics Subject Classification (2010):** 74H45

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### 1 Introduction.

In modern technology, it is necessary to regulate vibration processes in one-dimensional distributed systems, and the relevance of these problems is increasing. For shafts, which are the basic principles of mechanical transmission, dangerous transverse vibrations are not allowed [1]. In aircraft such elements are constructed simultaneously by bending and torsional vibrations. One of the objectives of the project is to prevent the use of shaft vibrations with an adjustable speed [2,3]. For such problems, mathematical models of transverse vibrations of rods are built on the basis of a refined theory [4]. Solutions of unknown parameters in accordance with the known data of its solutions [5, 6]. Such problems are called inverse problems of mathematical physics, which in many works [6] - [10], [12], [14], [15],[17,

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18] were studied for partial differential equations. In problems associated with initial and boundary conditions, additional information is required. The necessary additional information is due to the presence of unknown coefficients or the right-hand sides of the equations [13].

## 2 Problem statement and its reduction to an equivalent task.

Consider the question of the unique solvability of the inverse boundary value problem of determining a pair of functions  $\{u(x, t), a(t)\}$ , that satisfy in the domain  $D_T = \{(x, t) : 0 \leq x \leq 1, 0 \leq t \leq T\}$  the equation [13]

$$u_{tt}(x, t) + u_{xxxx}(x, t) = a(t)u(x, t) + f(x, t) \quad (2.1)$$

with boundary conditions

$$u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x) (0 \leq x \leq 1) \quad (2.2)$$

with periodic conditions

$$u_x(0, t) = 0, u_x(1, t) = 0, u_{xxx}(0, t) = 0, u_{xxx}(1, t) = 0, (0 \leq t \leq T) \quad (2.3)$$

and the additional condition

$$u(0, t) = h(t) (0 \leq t \leq T) \quad (2.4)$$

where  $f(x, t), \varphi(x), \psi(x), h(t)$  “given functions,  $u(x, t)$  and  $a(t)$ ” desired functions.

**Definition 2.1** The pair  $\{u(x, t), a(t)\}$  of functions  $u(x, t) \in C^{2,4}(D_T)$  and  $a(t) \in C[0, T]$  satisfying equation (2.1) in  $D_T$ , condition (2.2) in  $[0, 1]$  and conditions (2.3), (2.4) in  $[0, T]$ , defined as a classical solution of the inverse boundary value problem (2.1)-(2.4).

The following lemma holds:

**Lemma 2.1** Let  $\varphi(x), \psi(x) \in C[0, 1], h(t) \in C^2[0, T], h(t) \neq 0, (0 \leq t \leq T), f(x, t) \in C(D_T)$  and the conditions of approval be fulfilled:

$$\varphi(0) = h(0), \psi(0) = h'(0). \quad (2.5)$$

Then the problem of finding a classical solution to problem (2.1) - (2.4) is equivalent to the problem of determining functions  $u(x, t)$  and  $a(t)$  with properties 1) and 2) the definition of a solution to problem (2.1) - (2.6), from (2.1) - (2.3),

$$h''(t) + u_{xxxx}(0, t) = a(t)h(t) + f(0, t) (0 \leq t \leq T). \quad (2.6)$$

**Proof.** Let  $\{u(x, t), a(t)\}$  is a classical solution of problem (2.1)-(2.4). Since  $h(t) \in C^2[0, T]$ , differentiate (2.4) two times, we obtain:

$$u_{tt}(0, t) = h''(t), (0 \leq t \leq T). \quad (2.7)$$

Substituting  $x = 0$  into equation (2.1), we find:

$$u_{tt}(0, t) + u_{xxxx}(x, t) = a(t)u(0, t) + b(t)u_t(0, t) + f(0, t), (0 \leq t \leq T). \quad (2.8)$$

From (2.8), by virtue of (2.4) and (2.7), it follows that (2.6) holds. Now, suppose that  $u(x, t)$  and  $u(x, t)$  are a solution to problem (2.1)-(2.3), (2.6). Then from (2.6), (2.8), we have:

$$\frac{d^2}{dt^2}(u(0, t) - h(t)) - a(t)(u(0, t) - h(t)) = 0, u_{xxx}(1, t) - u_{xxx}(0, t) = 0 \quad (2.9)$$

By (2.2) and the compatibility condition (2.5), we get: Assuming that

$$\begin{aligned} u(0, 0) - h(0) &= \varphi(0) - h(0) = 0, u_t(0, 0) - h'(0) = \\ &= \psi(0) - h'(0) = 0 (0 \leq t \leq T). \end{aligned} \quad (2.10)$$

From (2.9) and (2.10), due to Lemma 2.1, we conclude that the condition (2.4) is satisfied. The Lemma is proved.

### 3 Solvability of the inverse boundary value problem

The first component  $u(x, t)$  of the solution  $\{u(x, t), a(t)\}$  of the problem (2.1) - (2.3), (2.6) will be sought in the form:

$$u(x, t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x (\lambda_k = \pi k) \quad (3.1)$$

where

$$u_k(t) = m_k \int_0^1 u(x, t) \cos \lambda_k x dx (k = 0, 1, 2, \dots),$$

$$m_k = \begin{cases} 1, & k = 0, \\ 0, & k = 1, 2, \dots \end{cases}$$

Then applying the formal scheme of the Fourier method, from (2.1) and (2.2) we obtained:

$$u_k''(t) + \lambda_k^4 u_k(t) = F_k(t; u, a) (0 \leq t \leq T; k = 1, 2, \dots), \quad (3.2)$$

$$u_k(0) = \varphi_k, u_k'(0) = \psi_k \quad (3.3)$$

where

$$F_k(t; u, a) = f_k(t) + a(t)u_k(t) (k = 0, 1, \dots)$$

$$f_k(t) = m_k \int_0^1 f(x, t) \cos \lambda_k x dx,$$

$$\varphi_k = m_k \int_0^1 \varphi(x) \cos \lambda_k x dx, \psi_k = m_k \int_0^1 \psi(x) \cos \lambda_k x dx, (k = 0, 1, \dots),$$

Further, from (3.2), (3.3) we find:

$$u_0(t) = \varphi_0 + \psi_0 t + \int_0^t (t - \tau) F_0(\tau; u, a) d\tau \quad (3.4)$$

$$u_k(t) = \varphi_k \cos \lambda_k^2 t + \frac{1}{\lambda_k^2} \psi_k \sin \lambda_k^2 t +$$

$$+ \frac{1}{\lambda_k^2} \int_0^t F_k(\tau; u, a) \sin \lambda_k^2 (t - \tau) d\tau, (k = 1, 2, \dots). \quad (3.5)$$

After expression substitution  $u_k(t) (k = 0, 1, \dots)$  to determine the components of the solution to problem (2.1) - (2.3), (2.6) we obtain:

$$u(x, t) = \varphi_0 + t\psi_0 + \int_0^t (t - \tau) F_0(\tau; u) d\tau + \sum_{k=1}^{\infty} \left\{ \varphi_k \cos \lambda_k^2 t + \right. \quad (3.6)$$

$$\left. \frac{1}{\lambda_k^2} \psi_k \sin \lambda_k^2 t + \frac{1}{\lambda_k^2} \int_0^t F_k(\tau; u) \sin \lambda_k^2 (t - \tau) d\tau \right\} \cos \lambda_k x, (k = 1, 2, \dots)$$

Now, from (2.6), considering (3.1), we have:

$$a(t) = [h(t)]^{-1} \left\{ h''(t) - f(0, t) + \sum_{k=1}^{\infty} \lambda_k^4 u_k(t) \right\}. \quad (3.7)$$

Substitute the expression (3.4) in (3.7):

$$a(t) = [h(t)]^{-1} \left\{ h''(t) - f(0, t) + \sum_{k=1}^{\infty} \lambda_k^4 \left[ \varphi_k \cos \lambda_k^2 t + \frac{1}{\lambda_k^2} \psi_k \sin \lambda_k^2 t + \frac{1}{\lambda_k^2} \int_0^t F_k(\tau; u, a) \sin \lambda_k^2 (t - \tau) d\tau \right] \right\}. \quad (3.8)$$

Thus, the solution of the problem (2.1)-(2.3), (2.6) was reduced to the solution of the problem (3.6), (3.8) for the unknown functions  $u(x, t)$  and  $a(t)$ .

**Lemma 3.1** *If  $\{u(x, t), a(t)\}$  any classical solution of the problem (2.1)-(2.3), (2.6) then functions  $u_k(t) = m_k \int_0^1 u(x, t) \cos \lambda_k x dx$ , ( $k = 0, 1, 2, \dots$ ) satisfy the system (3.4), (3.5).*

**Remark 3.1** From Lemma 3.1 it follows that to prove the uniqueness of the solution of the problem (2.1)-(2.3), (2.6) enough to prove the uniqueness of the solution of the problem (3.6), (3.8).

Now, we consider the following spaces:

1 We denote by  $B_{2,T}^\alpha$  [16], a consisting of all functions  $u(x, t)$  of the form

$$u(x, t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x \quad (\lambda_k = \pi k),$$

considered in  $D_T$ , where each of the functions form  $u_k(t)$  ( $k = 0, 1, \dots$ ) is continuous on  $[0, T]$  and

$$J(u) \equiv \|u_0(t)\|_{C[0,T]} + \left( \sum_{k=1}^{\infty} (\lambda_k^\alpha \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} < +\infty.$$

The norm in this set is defined as follows:

$$\|u(x, t)\|_{B_{2,T}^\alpha} = J(u)$$

2 The spaces  $E_T^\alpha$  denote the space consisting of a topological product

$$B_{2,T}^\alpha \times C[0, T].$$

The norm of element  $z(x, t) = \{u, a\}$  is determined by the formula

$$\|z\|_{E_T^\alpha} = \|u(x, t)\|_{B_{2,T}^\alpha} + \|a(t)\|_{C[0,T]}.$$

It is obvious that  $B_{2,T}^\alpha$  and  $E_T^\alpha$  are Banach spaces.

Now in the space  $E_T^\alpha$  consider the operator

$$\Phi(u, a) = \{\Phi_1(u, a), \Phi_2(u, a)\}$$

where

$$\Phi_1(u, a) = \tilde{u}(x, t) \equiv \sum_{k=0}^{\infty} \tilde{u}_k(t) \cos \lambda_k x,$$

$$\Phi_2(u, a) = \tilde{a}(t),$$

where  $\tilde{u}_0(t)$ ,  $\tilde{u}_k(t)$  ( $k = 1, 2, \dots$ ) and  $\tilde{a}(t)$  are equal to the right hand sides of (3.4), (3.5). Now with the help of easy transformations we find:

$$\begin{aligned} \|\tilde{u}_0(t)\|_{C[0,T]} &\leq |\varphi_0| + T|\psi_0| + \\ &+ T\sqrt{T} \left( \int_0^T |f_0(\tau)|^2 d\tau \right)^{\frac{1}{2}} + T^2 \|a(t)\|_{C[0,T]} \|u_0(t)\|_{C[0,T]}, \end{aligned} \quad (3.9)$$

$$\begin{aligned} &\left( \sum_{k=1}^{\infty} (\lambda_k^5 \|\tilde{u}_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq 2 \left( \sum_{k=1}^{\infty} (\lambda_k^5 |\varphi_k|)^2 \right)^{\frac{1}{2}} + \\ &+ 2 \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\psi_k|)^2 \right)^{\frac{1}{2}} + 2\sqrt{T} \left( \int_0^T \sum_{k=1}^{\infty} (\lambda_k^3 |f_k(\tau)|)^2 d\tau \right)^{\frac{1}{2}} + \\ &+ 2T \|a(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^5 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}}, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \|\tilde{a}(t)\|_{C[0,T]} &\leq \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\{ \|h''(t) - f(0, t)\|_{C[0,T]} + \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \times \right. \\ &\left[ \left( \sum_{k=1}^{\infty} (\lambda_k^5 |\varphi_k|)^2 \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\psi_k|)^2 \right)^{\frac{1}{2}} + \sqrt{T} \left( \int_0^T \sum_{k=1}^{\infty} (\lambda_k^3 |f_k(\tau)|)^2 d\tau \right)^{\frac{1}{2}} + \right. \\ &\left. \left. + T \|a(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^5 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right] \right\}. \end{aligned} \quad (3.11)$$

Suppose that the data of the problem (2.1)-(2.3), (2.6), satisfy the following conditions:

- 1  $\varphi(x) \in C^4[0, 1]$ ,  $\varphi^{(5)}(x) \in L_2(0, 1)$ ,  $\varphi'(0) = \varphi'(1) = \varphi'''(0) = \varphi'''(1) = 0$ ;
- 2  $\psi(x) \in C^2[0, 1]$ ,  $\psi^{(3)}(x) \in L_2(0, 1)$ ,  $\psi'(0) = \psi'(1) = 0$ ;
- 3  $f(x, t), f_x(x, t), f_{xx}(x, t) \in C(D_T)$ ,  $f_{xxx}(x, t) \in L_2(D_T)$ ,  $f_x(0, t) = f_x(1, t) = 0$  ( $0 \leq t \leq T$ );
- 4  $h(t) \in C^2[0, T]$ ,  $h(t) \neq 0$  ( $0 \leq t \leq T$ ).

Further, from (3.9)-(3.11) we have:

$$\|\tilde{u}(x, t)\|_{B_{2,T}^5} \leq A_1(T) + B_1(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^5} \quad (3.12)$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^5} \quad (3.13)$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}\|f(x, t)\|_{L_2(D_T)} + \\ + 2\|\varphi^{(5)}(x)\|_{L_2(0,1)} + 2\|\psi^{(3)}(x)\|_{L_2(0,1)} + 2\sqrt{T}\|f_{xxx}(x, t)\|_{L_2(D_T)},$$

$$B_1(T) = T(T + 2),$$

$$A_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\{ \|h''(t) - f(0, t)\|_{C[0,T]} + \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \times \right. \\ \left. \times 2 \left[ \|\varphi^{(5)}(x)\|_{L_2(0,1)} + \|\psi^{(3)}(x)\|_{L_2(0,1)} + \sqrt{T}\|f_{xxx}(x, t)\|_{L_2(D_T)} \right] \right\},$$

$$B_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} T.$$

From inequalities (3.12),(3.13) we conclude:

$$\|\tilde{u}(x, t)\|_{B_{2,T}^5} + \|\tilde{a}(t)\|_{C[0,T]} \leq A(T) + B(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^5}. \quad (3.14)$$

where

$$A(T) = A_1(T) + A_2(T), B(T) = B_1(T) + B_2(T).$$

So, we can prove the following theorem:

**Theorem 3.1** *Let conditions 1-4 be fulfilled and*

$$B(t)(A(T) + 2)^2 < 1 \quad (3.15)$$

*Then the problem (2.1)-(2.3), (2.6) has a unique solution in the sphere  $K = K_R(\|z\|_{E_T^3} \leq R = A(T) + 2)$  of the space  $E_T^5$ .*

**Proof.** In the space  $E_T^5$  consider the equation

$$z = \Phi z, \quad (3.16)$$

where  $z = \{u, a\}$ , the components  $\Phi_i(u, a) (i = 1, 2)$ , of the operator  $\Phi(u, a)$  are determined by the right hand sides of equations (3.6)-(3.8). Consider the operator  $\Phi(u, a)$  in the sphere  $K = K_R$  from  $E_T^5$ . Similar to (3.14) we obtained that for any  $z, z_1, z_2 \in K_R$  the following estimate are valid:

$$\|\Phi z\|_{E_T^3} \leq A(T) + B(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^5} \leq \\ \leq A(T) + B(T) (A(T) + 2)^2, \quad (3.17)$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^5} \leq \\ \leq B(T)R(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x, t) - u_2(x, t)\|_{B_{2,T}^5}) \quad (3.18)$$

Then from (3.17) and (3.18), with considering (3.15), it follows that operator  $\Phi$  acts in the sphere  $K = K_R$  and it is contraction mapping. Therefore, in the sphere  $K = K_R$  the operator  $\Phi$  has a unique fixed point  $\{u, a\}$ , that is a solution of equation (3.6),(3.8). The function  $u(x, t)$ , as the element of the space  $B_{2,T}^5$ , has continuous derivatives  $u_x(x, t), u_{xx}(x, t), u_{xxx}(x, t), u_{xxxx}(x, t)$  in  $D_T$ .

From (3.15) we get:

$$\left( \sum_{k=1}^{\infty} \left( \lambda_k \|u_k''(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} \leq \left[ \left( \sum_{k=1}^{\infty} \left( \lambda_k^5 \|u_k(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} + \sqrt{3} \left\| \|f_x(x, t) + a(t)u_x(x, t)\|_{C[0,T]} \right\|_{L_2(0,1)} \right] \quad (i = 1, 2)..$$

Then it follows that  $u_{tt}(x, t)$  is continuous in  $D_T$ . It is easy to verify that (2.1)-(2.3),(2.6) are satisfied in the ordinary sense. By virtue of the lemma 3.1, it is unique in the sphere  $K = K_R$ . Theorem is proved.

Using the Theorem 1 the following Lemma is proved.

**Theorem 3.2** *Let all the conditions of the theorem 1 be satisfied,*

$$\varphi(0) = h(0), \psi(0) = h'(0).$$

*Then, problem (2.1)-(2.4) has in the sphere  $K = K_R(\|z\|_{E_{T,T}^5} \leq R = A(T) + 2)$  from  $E_T^5$  unique classical solution.*

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