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Analysis of elasticity theory problem for a radially inhomogeneous sphere with fixed lateral surface

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Abstract. *We study an axially symmetric problem of elasticity theory for a small thickness radially-inhomogeneous sphere no of the poles* 0 *and* π. *The case when the modules of elasticity changes in radius by the linear law. It is supposed that the lateral surfaces of the sphere were fixed and on the end of the sphere (on conical sections) the stresses keeping it in equilibrium are given. A characteristic equation was obtained and its roots were classified. The homogeneous solutions dependent on the roots of the characteristic equation were constructed. Asymptotic formulas for displacements and stress that allow to calculate the stress-strain state of a small thickness radially inhomogeneous sphere we found. An istul on satisfaction of boundary conditions on the end (on conical sections) of the sphere using the Lagrange variational principle, was considered.*

Keywords. equilibriums equation · boundary layer · edge effect · Legendre function · characterictic equation

Mathematics Subject Classification (2010): 74D05

1 Introduction.

Study of inhomogeneous shell occupies a special plane in shell theory. Complexity of phenomena arising in deformation of inhomogeneous shells generated a number of applied theories. For studying the field of applicability of the existing applied theories of inhomogeneous shells and for creating new, more specified applied theories, it is required to analyze the stress-strain state of inhomogeneous shells from the position of there dimensional equations of elasticity theory. Furthermore, number of problems related to study of stress-strain state for inhomogeneous shells, may be solved correctly only with elasticity theory (problems of stress concentration at the holes in the vicinity of local loads).

Analysis of inhomogeneous shell on the basics of there-dimensional equations of elasticity theory is connected with significant mathematical difficulties. A spherical shell is one of the main elements of the series of technical theory for a sphere is a subject of a number of researches. An elasticity theory problem was considered shill by Saint- Venant [10] . the equilibrium of an elastic, symmetrically loaded spherical shell was studied in the papers [7, 8]. [17] studies a spatial problem of elasticity theory for a small thickness spherical shell and

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asymptotic solutions are compared with the solutions obtained in applied theories. In [13, 16], deformation of a transversally-isotropic thin sphere is studied. In [6] a spatial problem of elasticity theory for a sandwich spherical shell is studied by means of matrix theory. In [5], the analysis of a three-dimensional stress-strain state for a three layered sphere is conducted . [2] studies a torsion problem of a radially-laminate spherical shell with arbitrary number of alternating right and soft layers. Interpretation of penetrating solutions with a weak boundary layer one given. Applied theory of torsion of a radially-laminate spherical shell essentially specifying Saint-Venant theory is constructed.

2 Statement of boundary value problems for a radially -inhomogeneous sphere

Let us consider an axially symmetric problem of elasticity theory for a small-thickness radially-inhomogeneous isotropic sphere. Assume, that the sphere contains no of the poles 0 and π. In the spherical system of coordinates, we denote the domain occupied by the sphere through $\Gamma = \{r \in [r_1; r_2], \theta \in [\theta_1; \theta_2], \varphi \in [0; 2\pi]\}$.

Assume that the change of the elasticity modules in radius is is described by the linear function

$$G(r) = G_* r, \lambda(r) = \lambda_* r$$

where G_*, λ_* are some constants.

The equilibrium equations in displacements are of the from:

$$\begin{cases} (2G_0+\lambda_0)\frac{\partial^2 u_\rho}{\partial \rho^2} + 2\varepsilon(2G_0+\lambda_0)\frac{\partial u_\rho}{\partial \rho} - 4G_0\varepsilon^2 u_\rho - 3G_0\varepsilon^2\left(\frac{\partial u_\theta}{\partial \theta} + u_\theta ctg\theta\right) + \\ +\varepsilon(G_0+\lambda_0)\left(\frac{\partial u_\theta}{\partial \rho} ctg\theta + \frac{\partial^2 u_\theta}{\partial\theta\partial\rho}\right) + \varepsilon^2 G_0\left(\frac{\partial^2 u_\rho}{\partial \theta^2} + \frac{\partial u_\rho}{\partial \theta} ctg\theta\right) = 0, \\ G_0\frac{\partial^2 u_\theta}{\partial \rho^2} + 2\varepsilon G_0\frac{\partial u_\theta}{\partial \rho} + (5G_0+2\lambda_0)\varepsilon^2\frac{\partial u_\rho}{\partial \theta} + \varepsilon(G_0+\lambda_0)\frac{\partial^2 u_\rho}{\partial\rho\partial\theta} + \\ +\varepsilon^2(2G_0+\lambda_0)\left(\frac{\partial u_\theta}{\partial \theta} ctg\theta + \frac{\partial^2 u_\theta}{\partial^2 \theta} - u_\theta ctg^2\theta\right) - \varepsilon^2(\lambda_0+3G_0)u_\theta = 0. \end{cases} \tag{2.1}$$

Here $u_\rho, \quad u_\theta-$ are displacementst vector components; $G_0 = \frac{r_0 G_*}{t}, \quad \lambda_0 = \frac{\lambda_* r_0}{t}-$ are dimensionaless quantities; $t-$ is some characteristic parameter with dimension of shear modules; $\rho = \frac{1}{\varepsilon} \cdot \ln\left(\frac{r}{r_0}\right)$ – is a new radial variable; $\varepsilon = \frac{1}{2}\ln\left(\frac{r_2}{r_1}\right)$ – is a small parameter characterizing the thickness the sphere; $r_0 = \sqrt{r_1 r_2}$; $\rho \in [-1; 1]$.

Suppose that the lateral sides of the boundary of the sphere are fixed

$$u_\rho = 0, \ u_\theta = 0, \text{ for } \rho = \pm 1, \tag{2.2}$$

and on the ends of the sphere (on conical sections) the following stresses are given

$$\sigma_{\theta\theta}|_{\theta=\theta_n} = f_{1n}(\rho), \sigma_{\rho\theta}|_{\theta=\theta_n} = f_{2n}(\rho). \tag{2.3}$$

Here $f_{1n}(\rho)$. $f_{2n}(\rho) \quad (n = 1; 2)$ are rather smooth functions with order $O(1)$ with respect to ε and satisfy the equilbrium conditions.

3 Construction of solutions

We will book for the solution of (2.1),(2.2) in the form

$$u_\rho(\rho,\theta) = a(\rho)m(\theta), \ \ u_\theta(\rho,\theta) = d(\rho)m'(\theta), \tag{3.1}$$

where the function $m(\theta)$ satisfies the Lagrange equation [4]:

$$m''(\theta)+ctg\theta\cdot m'(\theta)+\left(z^2-\frac{1}{4}\right)m(\theta)=0. \tag{3.2}$$

Substituting (3.1) in (2.1), allowing for (3.2), ïafter separating variables with respect to the functions $a(\rho), d(\rho)$ we get the following system of ordinary differential equations

$$\begin{aligned}&(2G_0+\lambda_0)(a''(\rho)+2\varepsilon a'(\rho))-\varepsilon^2G_0\left(z^2+\frac{15}{4}\right)a(\rho)-\\&-\varepsilon(G_0+\lambda_0)\left(z^2-\frac{1}{4}\right)d'(\rho)+3\varepsilon^2G_0\left(z^2-\frac{1}{4}\right)d(\rho)=0,\\&G_0\left(d''(\rho)+2\varepsilon d'(\rho)\right)-\varepsilon^2\left(\left(z^2-\frac{1}{4}\right)(2G_0+\lambda_0)+G_0\right)d(\rho)+\\&+\varepsilon^2(5G_0+2\lambda_0)a(\rho)+\varepsilon(G_0+\lambda_0)a'(\rho)=0,\end{aligned} \tag{3.3}$$

Substituting (3.1) in (2.2), we get the following homogeneous boundary conditions

$$a(\rho)=0,\ d(\rho)=0 \text{ for } \rho=\pm1 \tag{3.4}$$

The solution of the system (3.3) is as follows:

$$\begin{aligned}a(\rho)&=e^{-\varepsilon\rho}\left[p_1e^{\varepsilon s_1\rho}A_1+p_1e^{-\varepsilon s_1\rho}A_2+p_2e^{\varepsilon s_2\rho}A_3+p_2e^{-\varepsilon s_2\rho}A_4\right],\\d(\rho)&=e^{-\varepsilon\rho}\left[t_1e^{\varepsilon s_1\rho}A_1+q_1e^{-\varepsilon s_1\rho}A_2+t_2e^{\varepsilon s_2\rho}A_3+q_2e^{-\varepsilon s_2\rho}A_4\right],\end{aligned} \tag{3.5}$$

where $A_n(n=\overline{1,4})$ are arbitrary constants; $q_k=(G_0+\lambda_0)s_k-(4G_0+\lambda_0)$; $p_k=G_0s_k^2-\left(z^2-\frac{1}{4}\right)(2G_0+\lambda_0)-2G_0$; $t_k=-(G_0+\lambda_0)s_k-(4G_0+\lambda_0)$; s_k are the roots of the equation

$$\begin{aligned}&(2G_0+\lambda_0)G_0s^4-\left[2G_0(\lambda_0+2G_0)\left(z^2-\tfrac{1}{4}\right)+10G_0^2+3G_0\lambda_0\right]s^2+\\&+G_0(2G_0+\lambda_0)\left(z^2-\tfrac{1}{4}\right)^2-2G_0^2\left(z^2-\tfrac{1}{4}\right)+2G_0(6G_0+\lambda_0)=0.\end{aligned}$$

Satisfying homogeneous system of linear algebraic equations (3.4), with respect to $A_n(n=\overline{1,4})$ we get a homogeneous system of linear algebraic equations:

$$\begin{cases}p_1e^{-\varepsilon s_1}A_1+p_1e^{\varepsilon s_1}A_2+p_2e^{-\varepsilon s_2}A_3+p_2e^{\varepsilon s_2}A_4=0,\\t_1e^{-\varepsilon s_1}A_1+q_1e^{\varepsilon s_1}A_2+t_2e^{-\varepsilon s_2}A_3+q_2e^{\varepsilon s_2}A_4=0,\\p_1e^{\varepsilon s_1}A_1+p_1e^{-\varepsilon s_1}A_2+p_2e^{\varepsilon s_2}A_3+p_2e^{-\varepsilon s_2}A_4=0,\\t_1e^{\varepsilon s_1}A_1+q_1e^{-\varepsilon s_1}A_2+t_2e^{\varepsilon s_2}A_3+q_2e^{-\varepsilon s_2}A_4=0\end{cases} \tag{3.6}$$

From the condition of non-trivial solutions o f the system (3.6) we get a characteristic equation to determine the spectral parameter z :

$$\begin{aligned}\varDelta(z,\varepsilon)&=(p_1q_2-p_2q_1)\cdot(t_2p_2-p_1t_2)\,sh^2(\varepsilon(s_1+s_2))+\\&+(p_1t_2-p_2q_1)\cdot(p_1q_2-p_2t_1)\,sh^2(\varepsilon(s_2-s_1))=0.\end{aligned} \tag{3.7}$$

The transcendental equation (3.7) is determined by a denumuerable set of roots, and the appropriate constant z_k, $A_{1k}, A_{2k}, A_{3k}, A_{4k}$ are proportional to algebraic sofactor of the elements of any row of the main determinant of the system (3.6). Choosing the algebraic sofactor of the elements of the first row of the main determinary of the system (3.6), we get:

$$A_{1k}=F_k\varDelta_{11},\ A_{2k}=-F_k\varDelta_{12},\ A_{3k}=F_k\varDelta_{13},\ A_{4k}=-F_k\varDelta_{14}. \tag{3.8}$$

where

$$\Delta_{11} = p_2 q_1 (q_2 - t_2) e^{\varepsilon s_1} - t_2 (p_1 q_2 - p_2 q_1) e^{-\varepsilon(2s_2+s_1)} + \\ +q_2 (p_1 t_2 - p_2 q_1) e^{\varepsilon(2s_2-s_1)},$$

$$\Delta_{12} = p_2 t_1 (q_2 - t_2) e^{-\varepsilon s_1} - t_2 (p_1 q_2 - p_2 t_1) e^{\varepsilon(s_1-2s_2)} + \\ +q_2 (p_1 t_2 - p_2 t_1) e^{\varepsilon(s_1+2s_2)};$$

$$\Delta_{13} = p_1 q_2 (q_1 - t_1) e^{\varepsilon s_2} + t_1 (p_1 q_2 - p_2 q_1) e^{-\varepsilon(2s_1+s_2)} - \\ -q_1 (p_1 q_2 - p_2 t_1) e^{\varepsilon(2s_1-s_2)},$$

$$\Delta_{14} = p_1 t_2 (q_1 - t_1) e^{-\varepsilon s_2} + t_1 (p_1 t_2 - p_2 q_1) e^{\varepsilon(s_2-2s_1)} - \\ -q_1 (p_1 t_2 - p_2 t_1) e^{\varepsilon(2s_1+s_2)}.$$

Substituting (3.8) in (3.5), summing over all the roots of the characteristic equation (3.7), allowing for (3.1) we get the solution of the form:

$$u_\rho = \sum_{k=1}^{\infty} F_k a_k(\rho) m_k(\theta), \\ u_\theta = \sum_{k=1}^{\infty} F_k d_k(\rho) m_k^{'}(\theta), \tag{3.9}$$

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$$a_k(\rho) = e^{-\varepsilon\rho} \left[p_1 \left(\Delta_{11} e^{\varepsilon s_1 \rho} - \Delta_{12} e^{-\varepsilon s_1 \rho}\right) + p_2 \left(\Delta_{13} e^{\varepsilon s_2 \rho} - \Delta_{14} e^{-\varepsilon s_2 \rho}\right)\right],$$

$$d_k(\rho) = e^{-\varepsilon\rho} \left[t_1 \Delta_{11} e^{\varepsilon s_1 \rho} - q_1 \Delta_{12} e^{-\varepsilon s_1 \rho} + t_2 \Delta_{13} e^{\varepsilon s_2 \rho} - q_2 \Delta_{14} e^{-\varepsilon s_2 \rho}\right].$$

4 Analysis of the characteristic equation

$\Delta(z,\varepsilon)$ is an even function of its own arguments.

Expand $\Delta(z,\varepsilon)$ in series of ε :

$$\Delta(z,\varepsilon) = 4\varepsilon^2 (s_2^2 - s_1^2)^2 s_1 s_2 (G_0 + \lambda_0)^2 \langle 4G_0 \left[(2G_0+\lambda_0)\left(z^2 - \tfrac{1}{4}\right) + 2G_0 - \right. \\ \left. - G_0 \left(\tfrac{4G_0+\lambda_0}{G_0+\lambda_0}\right)^2\right] + \tfrac{\varepsilon^2}{3} \left\{4G_0^2 \left[\left(z^2 - \tfrac{1}{4}\right)^2 - \left(z^2 - \tfrac{1}{4}\right)\tfrac{2G_0}{2G_0+\lambda_0} + \tfrac{2(6G_0+\lambda_0)}{2G_0+\lambda_0}\right] + \right. \\ + \left[4G_0(2G_0+\lambda_0)\left(z^2 - \tfrac{1}{4}\right) + 8G_0^2 - 8G_0^2 \left(\tfrac{4G_0+\lambda_0}{G_0+\lambda_0}\right)^2\right] \left[2\left(z^2 - \tfrac{1}{4}\right) + \right. \\ \left. \left. + \tfrac{3\lambda_0+10G_0}{2G_0+\lambda_0}\right] + 4(2G_0+\lambda_0)^2 \left(z^2 - \tfrac{1}{4}\right)^2 + 16G_0(2G_0+\lambda_0)\left(z^2 - \tfrac{1}{4}\right) + 16G_0^2\right\} + \ldots \rangle = 0. \tag{4.1}$$

We look for the bounded roots of equation (3.7) as $\varepsilon \to 0$ in the following form:

$$z_k = z_{k_0} + \varepsilon z_{k_1} + \varepsilon^2 z_{k_2} + \ldots. \tag{4.2}$$

After substituting (4.2) in (4.1) we have:

$$z_{k_0}^2 = \frac{G_0(4G_0+\lambda_0)^2}{(2G_0+\lambda_0)\,(G_0+\lambda_0)^2} + \frac{(\lambda_0-6G_0)}{4(2G_0+\lambda_0)}, \quad z_{k1}=0, \tag{4.3}$$

$$z_{k2} = -\frac{1}{24G_0(2G_0+\lambda_0)z_{k0}}\left\{4G_0^2\left[\left(z_{k0}^2-\frac{1}{4}\right)^2-\left(z_{k0}^2-\frac{1}{4}\right)\frac{2G_0}{2G_0+\lambda_0}+\right.\right.$$

$$\left.+\frac{2(6G_0+\lambda_0)}{2G_0+\lambda_0}\right]+\left[4G_0(2G_0+\lambda_0)\left(z_{k0}^2-\frac{1}{4}\right)+8G_0^2-\frac{8G_0^2\,(4G_0+\lambda_0)^2}{(G_0+\lambda_0)^2}\right]\times$$

$$\times\left[2\left(z_{k0}^2-\frac{1}{4}\right)+\frac{10G_0+3\lambda_0}{2G_0+\lambda_0}\right]+4\left(z_{k0}^2-\frac{1}{4}\right)^2(2G_0+\lambda_0)^2+$$

$$\left.+16\left(z_{k0}^2-\frac{1}{4}\right)G_0(2G_0+\lambda_0)+16G_0^2\right\},\cdots$$

By direct verification we can show that the trivial solutions correspond to the root z_k defined by formulas (4.2),(4.3).

Prove that all the zeros of the function $\Delta(z;\varepsilon)$ unboundedly inereace as $\varepsilon\to 0$. Proceed from the contrsry assuming $z_k \to z_k^* \neq \infty$ as $\varepsilon \to 0$. the limit points of the set of zeros z_k as $\varepsilon \to 0$ are determined from the equation

$$\Delta_0(z_k^*) = 16(s_2^2-s_1^2)^2 s_1 s_2 (G_0+\lambda_0)^2 G_0\left[(2G_0+\lambda_0)\left((z_k^*)^2-\frac{1}{4}\right)+\right.$$

$$\left.+2G_0-\frac{G_0(4G_0+\lambda_0)^2}{(G_0+\lambda_0)^2}\right]=0. \tag{4.4}$$

From (4.4) it follows that another bounded roots except the above found by formulas (4.2) do not exist.

So, all the roots of the characteristic equation tend to infinity as $\varepsilon \to 0$. They can be divided into there groups depending on their behavior as $\varepsilon \to 0$:

1^0) $\varepsilon z_k \to 0$ as $\varepsilon \to 0$,

2^0) $\varepsilon z_k \to const$ as $\varepsilon \to 0$,

3^0)$\varepsilon z_k \to \infty$ as $\varepsilon \to 0$.

As in the paper [12] here only the case 2^0)$\varepsilon z_k \to \infty$ as $\varepsilon \to 0$. to possible. We look for z_k in the form.

$$z_k = \frac{\delta_k}{\varepsilon} + O(\varepsilon). \tag{4.5}$$

After substituting (4.5) in characteristic equation (3.7) for δ_k we have:

$$sh^2(2\delta_k) - \frac{4(G_0+\lambda_0)^2}{(3G_0+\lambda_0)^2}\delta_k^2 = 0. \tag{4.6}$$

5 Constructing asymptotic formulas for displacements and stresses

We give asymptotic construction of solutions corresponding to the roots of characteristic equation (3.7).

For displacements and stresses in the first approximation we get two classes of solutions first which corresponds to the zeros of the function

$$sh(2\delta_k) - \frac{2(G_0+\lambda_0)}{3G_0+\lambda_0}\delta_k,$$

the second one to the zeros of the function

$$sh(2\delta_k) + \frac{2(G_0+\lambda_0)}{3G_0+\lambda_0}\delta_k.$$

We respectively have:

1)

$$u_\rho(\rho,\theta) = \sum_{k=1}^{4} D_k u_{\rho k}(\rho,\theta),\ u_\theta(\rho,\theta) = \sum_{k=1}^{4} D_k u_{\theta k}(\rho,\theta),$$

$$\sigma_{\rho\theta} = \sum_{k=1}^{4} D_k T_{\rho k}(\rho,\theta),\quad \sigma_{\rho\rho} = \sum_{k=1}^{4} D_k Q_{\rho k}(\rho,\theta),$$

$$\sigma_{\theta\theta} = \sum_{k=1}^{4} D_k Q_{\theta k}(\rho,\theta),\quad \sigma_{\varphi\varphi} = \sum_{k=1}^{4} D_k Q_{\varphi k}(\rho,\theta). \tag{5.1}$$

where

$$u_{\rho k}(\rho;\theta) = \delta_k\left[(G_0+\lambda_0)\delta_k sh\delta_k sh(\delta_k\rho) - ch\delta_k\left((G_0+\lambda_0)\delta_k\rho ch(\delta_k\rho) - \right.\right.$$
$$\left.\left. -\left(3G_0+\lambda_0\right) sh(\delta_k\rho)\right) + O(\varepsilon)\right] m_k(\theta),$$
$$u_{\theta k}(\rho;\theta) = \varepsilon\delta_k(G_0+\lambda_0)\left[sh\delta_k ch(\delta_k\rho) - \rho ch\delta_k sh(\delta_k\rho) + O(\varepsilon)\right] m_k'(\theta),$$
$$Q_{\rho k} = \frac{2G_0}{\varepsilon}\delta_k^2\left[(G_0+\lambda_0)\delta_k ch(\delta_k\rho) sh\delta_k - ch\delta_k\left(G_0+\lambda_0\right)\delta_k\rho sh(\delta_k\rho) - \right.$$
$$\left. -(2G_0+\lambda_0) ch(\delta_k\rho)) + O(\varepsilon)\right] m_k(\theta), \tag{5.2}$$
$$T_{\rho k} = 2G_0\delta_k\left[(G_0+\lambda_0)\delta_k sh\delta_k sh(\delta_k\rho) - ch\delta_k((G_0+\lambda_0)\delta_k\rho ch(\delta_k\rho) - \right.$$
$$\left. -G_0 sh(\delta_k\rho)) + O(\varepsilon)\right] m_k'(\theta),$$
$$Q_{\theta k} = \frac{2G_0}{\varepsilon}\delta_k^2\left[-\left(G_0+\lambda_0\right)\delta_k sh\delta_k ch(\delta_k\rho) + \left(\left[(G_0+\lambda_0)\delta_k\rho sh(\delta_k\rho) + \right.\right.\right.$$
$$\left.\left.\left. +\lambda_0 ch(\delta_k\rho)\right]\right) ch\delta_k + O(\varepsilon)\right] m_k(\theta),$$
$$Q_{\varphi k} = \frac{\lambda_0}{\varepsilon}\delta_k^2 ch\delta_k[2G_0 ch(\delta_k\rho) + O(\varepsilon)] m_k(\theta)$$

δ_k is the solution of the equation

$$sh2\delta_k - \frac{2(G_0+\lambda_0)}{3G_0+\lambda_0}\delta_k = 0$$

2)

$$u_\rho(\rho,\theta)=\sum_{i=1}^{\infty}F_i u_{\rho i}(\rho,\theta),\ u_\theta(\rho,\theta)=\sum_{i=1}^{\infty}F_i u_{\theta i}(\rho,\theta),$$

$$\sigma_{\rho\theta}=\sum_{i=1}^{\infty}F_i T_{\rho i}(\rho,\theta),\ \sigma_{\rho\rho}=\sum_{i=1}^{\infty}F_i Q_{\rho i}(\rho,\theta), \tag{5.3}$$

$$\sigma_{\theta\theta}=\sum_{i=1}^{\infty}F_i Q_{\theta i}(\rho,\theta),\ \sigma_{\varphi\varphi}=\sum_{i=1}^{\infty}F_k Q_{\varphi i}(\rho,\theta).$$

The expressions for are obtained from $u_{\rho i}, u_{\theta i}, T_{\rho i}, Q_{\rho i}, Q_{\theta i}, Q_{\varphi i}$ changing (5.1),(5.2) chx by shx, shx by chx, δ_k by δ_i and here δ_i are the solutions of the equation

$$sh2\delta_i+\frac{2(G_0+\lambda_0)}{3G_0+\lambda_0}\delta_i=0$$

Note that for the roots of the characteristic equation with asymptotic (4.5) the principal term of the asymptotic solution of equation (3.2) as $\varepsilon\to 0$ takes the form [1]:

$$m_k(\theta)=\begin{cases}\frac{1}{\sqrt{\delta_k\sin\theta}}\exp\left[-\varepsilon^{-1}\sqrt{\delta_k^2}(\theta-\theta_1)\right](1+O(\varepsilon))\text{; in the vicinity of }\theta=\theta_1,\\ \frac{1}{\sqrt{\delta_k\sin\theta}}\exp\left[\varepsilon^{-1}\sqrt{\delta_k^2}(\theta-\theta_2)\right](1+O(\varepsilon))\text{; in the vicinity of }\theta=\theta_2.\end{cases} \tag{5.4}$$

The solutions (5.1), (5.3) are of boundary layer character and in Kirchhoff-Love theory they are absolute. The first terms of its asymptotic expansion are equivalent to the Saint-Venant edge effect of a homogeneous isotropic plate [18]. From (5.4) we get that when removing $\theta=\theta_j$ $(j=1;2)$ from conical sections, the solutions of (5.1), (5.3) exponentially decrease. .

6 Satisfaction of boundary conditions on the ends of the sphere

To determine the constants D_k, F_i we use the Largrange-variational principle [9]. since homogeneous solutions satisfy the equilibrium equation and boundary conditions on lateral surface, the variational principle takes the following from [11, 12]:

$$\sum_{j=1}^{2}\int_{-1}^{1}\left[(\sigma_{\theta\theta}-f_{1j}(\rho))\delta u_\theta+(\sigma_{\rho\theta}-f_{2j}(\rho))\delta u_\rho)\right]\Bigg|_{\theta=\theta_j} e^{2\varepsilon\rho}d\rho=0. \tag{6.1}$$

Substituting (5.1), (5.3) in (6.1), and assembly $\delta D_k, \delta F_i$ independent variations, (6.1) from we get the following systems of linear algebras equations:

$$\sum_{k=1}^{\infty}M_{kj}D_{k0}=\tau_{j1};(j=\overline{1,\infty}) \tag{6.2}$$

$$\sum_{i=1}^{\infty}N_{ij}F_{i0}=\tau_{j2};(j=\overline{1,\infty}) \tag{6.3}$$

where

$$D_k=\varepsilon(D_{k0}+\varepsilon D_{k1}+...),$$

$$F_i = \varepsilon(F_{i0} + \varepsilon F_{i1} + ...),$$

$$M_{kj} = \frac{1}{\sqrt[4]{\delta_k^2\delta_j^2}}\left(\frac{1}{\sin\theta_2} - \frac{1}{\sin\theta_1}\right)2G_0\delta_j\delta_k\left\langle (G_0+\lambda_0)\delta_k\sqrt{-\delta_j^2}\int_{-1}^{1}\{[(G_0+\lambda_0)\delta_k\rho sh(\delta_k\rho) + \right.$$
$$+\lambda_0 ch(\delta_k\rho)]\, ch\delta_k - (G_0+\lambda_0)\delta_k sh\delta_k ch(\delta_k\rho)\}\,[sh\delta_j ch(\delta_j\rho) - \rho ch\delta_j sh(\delta_j\rho)]\,d\rho +$$
$$+\sqrt{-\delta_k^2}\int_{-1}^{1}\{(G_0+\lambda_0)\delta_k sh\delta_k sh(\delta_k\rho) - ch\delta_k\,[(G_0+\lambda_0)\delta_k\rho ch(\delta_k\rho) - G_0 sh(\delta_k\rho)]\}\times$$
$$\left.\times\{(G_0+\lambda_0)\delta_j sh\delta_j sh(\delta_j\rho) - ch\delta_j[(G_0+\lambda_0)\delta_j\rho ch(\delta_j\rho) -\}\ (3G_0+\lambda_0)sh(\delta_j\rho)]\}\,d\rho\right\rangle,$$

$$\tau_{j1} = \sum_{s=1}^{2}\frac{1}{\sqrt{\sin\theta_s}}\frac{\delta_j}{\sqrt[4]{-\delta_j^2}}\left\langle(-1)^s\sqrt{-\delta_j^2}(G_0+\lambda_0)\int_{-1}^{1} f_{1s}(\rho)\,[sh\delta_j ch(\delta_j\rho) - \right.$$
$$-\rho\, ch\delta_j sh(\delta_j\rho)]\,d\rho + \int_{-1}^{1} f_{2s}(\rho)\,\{(G_0+\lambda_0)\delta_j sh\delta_j sh(\delta_j\rho) -$$
$$\left.-ch\delta_j\,[(G_0+\lambda_0)\delta_j\rho ch(\delta_j\rho) - (3G_0+\lambda_0)sh(\delta_j\rho)]\}\,d\rho\right\rangle.$$

The expressions for N_{ij}, τ_{j2} are obtained from the expressions for M_{kj}, τ_{j2} changing by chx by shx, shx by chx, δ_k by δ_i.

The definition of $D_{kn}, F_{in}(n = 1, 2, ...)$ is invariably reduced to the systems whose matrices coincide with the matrices of system (6.2), (6.3).

The solvability and convergence of the reduction method for (6.2) - (6.3) was proved in [3, 14, 15].

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