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Studying of elastic equilibrium of a small thickness isotropic cylinder with variable elasticity module

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Abstract. *Based on asymptotic integration of elasticity theory equations, we study axially-symmetric problem of elasticity theory for a radially-inhomogeneous cylinder of small thickness. We consider a case when the elasticity modulus changes in radius by the linear law. It is assumed that the lateral part of the cylinder is fixed, and on the ends of the cylinder the stresses leaving the cylinder in equilibrium, are given.*

Asymptotical formulas for displacements and stresses are written. It is shown that the stress-strain state was made up only from the solution of a boundary layer character and equivalent to the Saint-Venanat edge effect of theory of inhomogeneous plates.

Keywords. radially- inhomogeneous cylinder · asymptotic method · boundary layer · edge effect · variational principle

Mathematics Subject Classification (2010): 74H45

1 Introduction

Study of inhomogeneous shells occupies one of the special places in shell theory. Analysis of inhomogeneous shells on the basis of three-dimensional equations of elasticity theory is a very difficult problem.

Therefore, it is necessary to use different approximate methods allowing to simplify calculation of shells. Complex nature of phenomena arising in deformation of inhomogeneous shells, reduced to formation of a lot of applied theories each of which was constructed on the basis of definite system of assumptions. In modern engineering there arise such new shell constructions whose calculation within the existing applied theories, is impossible.

To establish applicability fields of the existing applied theories of inhomogeneous shells and to create new, more specified applied theories, it is required to analyse the stress-strain state of inhomogeneous shells from the position of three-dimensional equations of elasticity theory.

The asymptotic method [11-14] plays an important role in solving three-dimensional problems of elasticity theory. Frist, in [4] spatial problem of elasticity theory was studied for an isotropic, small thickness cylinder and asymptotic solutions were compared with the solutions obtained by applied theories. In [12] three-dimensional asymptotic theory of a

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small-thickness transversally-isotropic cylinder was developed. An axially-symmetric problem of elasticity theory for a radially-laminated cylinder with alternating rigid and soft layers, was studied in [1]. In [5] an axially-symmetric problem of elasticity theory is analyzed for a radially-inhomogeneous, small thickness hollow cylinder, when the lateral surface of the cylinder is free from stresses . In [10] a semi-analytical method is offered for solving the Almansi-Michell's problem for an inhomogeneous anisotropic cylinder. The influence of inhomogeneity of the material on the stress-strain state of a cylinder was studied in [7, 8].

2 Statement of boundary-value problems for a radially- inhomogeneous cylinder

We consider an axially-symmetric problem of elasticity theory for an inhomogeneous, isotropic, hollow, small thickness cylinder. In the cylindrical system of coordinates, we denote the domain occupied with the cylinder, by

$$\Gamma = \{r \in [r_1; r_2], \varphi \in [0, 2\pi], z \in [-L; L]\}.$$

Assume that alternation of the elasticity modules in radius occurs by the linear law

$$G(r) = G_* r, \lambda(r) = \lambda_* r,$$

where G_*, λ_* are constant variables.

The equilibrium equations in displacements have the form:

$$(L_0 + \partial_1 L_1 + \partial_1^2 L_2)\bar{u} = \bar{0}. \tag{2.1}$$

Here $\bar{u} = \bar{u}(\rho, \xi) = (u_\rho(\rho, \xi), u_\xi(\rho, \xi))^T, L_k$ are matrix differential operators of the form:

$$L_0 = \left\| \begin{matrix} (2G_0 + \lambda_0)(\partial^2 + \varepsilon\partial) - 2G_0\varepsilon^2 & 0 \\ 0 & G_0(\partial^2 + \varepsilon\partial) \end{matrix} \right\|,$$

$$L_0 = \left\| \begin{matrix} 0 & e^{\varepsilon\rho}\left[\varepsilon(G_0 + \lambda_0)\partial + \varepsilon^2\lambda_0)\right] \\ e^{\varepsilon\rho}\left[\varepsilon^2(2G_0 + \lambda_0)\varepsilon(G_0 + \lambda_0)\partial\right] & 0 \end{matrix} \right\|,$$

$$L_2 = \left\| \begin{matrix} \varepsilon^2 G_0 e^{\varepsilon\rho} & 0 \\ 0 & (2G_0 + \lambda_0)\varepsilon^2 e^{\varepsilon\rho} \end{matrix} \right\|,$$

$\partial_1 = \frac{\partial}{\partial\xi}; \partial_1^2 = \frac{\partial^2}{\partial\xi^2}; \partial = \frac{\partial}{\partial\rho}; \rho = \frac{1}{\varepsilon}\ln\left(\frac{r}{r_0}\right), \xi = \frac{z}{r_0}$ are new pure variables; $\varepsilon = \frac{1}{2}\ln\left(\frac{r_2}{r_1}\right)$ is a small parameter characterizing the thickness of the cylinder; $r_0 = \sqrt{r_1 r_2}, \xi \in [-l; l], \rho \in [-1; 1], l = \frac{L}{r_0}; \lambda_0 = \frac{\lambda_* r_0}{G_1}, G_0 = \frac{G_* r_0}{G_1}$ are pure variables and G_1 is a characteristic parameter having dimension of shear modulus. Suppose that the lateral side of the cylinder is rigidly fixed:

$$\bar{u}(\rho, \xi) = \bar{0} \;\; for \;\; \rho = \pm 1. \tag{2.2}$$

Assume that on the ends of the cylinder the following boundary conditions are given

$$\sigma_{\rho\xi}|_{\xi=\pm l} = f_{1s}(\rho), \;\; \sigma_{\xi\xi}|_{\xi=\pm l} = f_{2s}(\rho). \tag{2.3}$$

Here $f_{1s}(\rho), f_{2s}(\rho)(s = 1, 2)$ are rather smooth functions satisfying the equilibrium conditions.

3 Constructing homogeneous solutions for a radially-inhomogeneous, small thickness cylinder

We look for the solution of (2.1), (2.2) in the form:

$$\bar{u}(\rho,\xi)=\bar{a}(\rho)e^{\alpha\xi}, \tag{3.1}$$

where

$$\bar{a}(\rho)=(u(\rho),w(\rho))^T.$$

Substituting (3.1) in (2.1), (2.2), we have:

$$\begin{cases}(L_0+\alpha L_1+\alpha^2 L_2)\bar{a}=\bar{0},\\ \bar{a}|_{\rho=\pm 1}=\bar{0}.\end{cases} \tag{3.2}$$

For solving (3.2) as $\varepsilon\to 0$ we use the asymptotic method [2, 3, 6], based on two iterative processes.

Trivial solutions correspond to the first iterative process. There are no solutions with edge effect character, corresponding to the second iterative process for a radially-inhomogeneous cylinder with a fixed lateral surface.

According to the third iterative process , we have

$$a)\ \ \alpha_k=\varepsilon^{-1}\left(\beta_{0k}+\varepsilon\beta_{1k}+...\right).$$

$$u_\rho^{(1)}=\varepsilon\sum_{k=1}^{\infty}T_k\left[\left(\beta_{0k}\sin\beta_{0k}-\frac{3G_0+\lambda_0}{G_0+\lambda_0}\cos\beta_{0k}\right)\sin(\beta_{0k}\rho)+\right.$$

$$\left.+\beta_{0k}\rho\cos\beta_{0k}\cos(\beta_{0k}\rho)+O(\varepsilon)\right]\exp\left(\frac{1}{\varepsilon}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right), \tag{3.3}$$

$$u_\xi^{(1)}=\varepsilon\sum_{k=1}^{\infty}T_k\beta_{0k}\left[\rho\cos\beta_{0k}\sin(\beta_{0k}\rho)-\sin\,\beta_{0k}\cos(\beta_{0k}\rho)+\right.$$

$$\left.+\,O(\varepsilon)\right]\exp\left(\frac{1}{\varepsilon}\left(\beta_{0k}+\varepsilon\beta_{1k}+...\right)\xi\right).$$

Here β_{0k} is the solution of the equation

$$\sin 2\beta_{0k}-\frac{2(G_0+\lambda_0)}{3G_0+\lambda_0}\beta_{0k}=O. \tag{3.4}$$

The stresses corresponding to the solutions (3.3) are of the form:

$$u_\rho^{(1)}=\varepsilon\sum_{k=1}^{\infty}T_k\beta_{0k}\left[(2G_0+\lambda_0)\left(\beta_{0k}\sin\beta_{0k}-\frac{2G_0}{G_0+\lambda_0}\cos\beta_{0k}\right)\cos(\beta_{0k}\rho)-\right.$$

$$\left.-2G_0\beta_{0k}\rho\cdot\cos\beta_{0k}\sin(\beta_{0k}\rho)-\lambda_0\beta_{0k}\sin\,\beta_{0k}\cos(\beta_{0k}\rho)+O(\varepsilon)\right]\times$$

$$\times\exp\left(\frac{1}{\varepsilon}\left(\beta_{0k}+\varepsilon\beta_{1k}+...\right)\xi\right),$$

$$\sigma_{\rho\xi}^{(1)}=G_0\sum_{k=1}^{\infty}T_k\beta_{0k}\left[\cos\beta_{0k}(\sin\,(\beta_{0k}\rho)+2\beta_{0k}\rho\cdot\cos(\beta_{0k}\rho))+\right.$$

$$+\left(2\beta_{0k}\sin\beta_{0k}-\frac{3G_0+\lambda_0}{G_0+\lambda_0}\cos\beta_{0k}\right)\sin(\beta_{0k}\rho)+$$

$$+ O(\varepsilon)] \exp\left(\frac{1}{\varepsilon}\left(\beta_{0k} + \varepsilon\beta_{1k} + ...\right)\xi\right), \tag{3.5}$$

$$\sigma_{\xi\xi}^{(1)} = \sum_{k=1}^{\infty} T_k\beta_{0k}\left[2G_0\beta_{0k}\rho\cos\beta_{0k}\sin\left(\beta_{0k}\rho\right) - (2G_0\beta_{0k}\sin\beta_{0k} + \right.$$

$$\left. + \frac{2G_0\lambda_0}{G_0+\lambda_0}\cos\beta_{0k}\right)\cos(\beta_{0k}\rho) + O(\varepsilon)] \times$$

$$\times \exp\left(\frac{1}{\varepsilon}\left(\beta_{0k} + \varepsilon\beta_{1k} + ...\right)\xi\right),$$

$$\sigma_{\varphi\varphi}^{(1)} = \sum_{k=1}^{\infty} T_k\beta_{0k}\left[-\frac{2G_0\lambda_0}{G_0+\lambda_0}\cos\beta_{0k}\cos(\beta_{0k}\rho) + O(\varepsilon)\right] \times$$

$$\times \exp\left(\frac{1}{\varepsilon}\left(\beta_{0k} + \varepsilon\beta_{1k} + ...\right)\xi\right).$$

$$b)\ \alpha_k = \varepsilon^{-1}\left(\beta_{0k} + \varepsilon\beta_{1k} + ...\right).$$

$$u_\rho^{(2)} = -\varepsilon\sum_{k=1}^{\infty} F_k\left[\left(\frac{3G_0+\lambda_0}{G_0+\lambda_0}\sin\beta_{0k} + \beta_{0k}\cos\beta_{0k}\right)\cos(\beta_{0k}\rho) + \right.$$

$$+\beta_{0k}\rho\sin\beta_{0k}\sin(\beta_{0k}\rho) + O(\varepsilon)]\exp\left(\frac{1}{\varepsilon}(\beta_{0k} + \varepsilon\beta_{1k} + ...)\xi\right), \tag{3.6}$$

$$u_\xi^{(2)} = \varepsilon\sum_{k=1}^{\infty} F_k\beta_{0k}\left[-\cos\beta_{0k}\sin(\beta_{0k}\rho) + \rho\cos(\beta_{0k}\rho)\sin\beta_{0k} + \right.$$

$$+ O(\varepsilon)]\exp\left(\frac{1}{\varepsilon}\left(\beta_{0k} + \varepsilon\beta_{1k} + ...\right)\xi\right),$$

Here β_{0k} is the solution of the equation

$$\sin 2\beta_{0k} + \frac{2(G_0+\lambda_0)}{3G_0+\lambda_0}\beta_{0k} = 0. \tag{3.7}$$

The stresses corresponding to the solutions (3.6) have the form:

$$u_\rho^{(2)} = \sum_{k=1}^{\infty} F_k\beta_{0k}\left[(2G_0+\lambda_0)\left(\beta_{0k}\cos\beta_{0k} + \frac{2G_0}{G_0+\lambda_0}\sin\beta_{0k}\right)\sin(\beta_{0k}\rho) - \right.$$

$$\left. -2G_0\beta_{0k}\rho\sin\beta_{0k}\cos(\beta_{0k}\rho) - \lambda_0\beta_{0k}\cos\beta_{0k}\sin(\beta_{0k}\rho) + O(\varepsilon)\right] \times$$

$$\times \exp\left(\frac{1}{\varepsilon}(\beta_{0k} + \varepsilon\beta_{1k} + ...)\xi\right),$$

$$\sigma_{\rho\xi}^{(2)} = G_0\sum_{k=1}^{\infty} F_k\beta_{0k}\left[\sin\beta_{0k}(\cos\left(\beta_{0k}\rho\right) - 2\beta_{0k}\rho\cdot\sin(\beta_{0k}\rho)) - \right.$$

$$-\cos(\beta_{0k}\rho)\left(2\beta_{0k}\cos\beta_{0k} + \frac{3G_0+\lambda_0}{G_0+\lambda_0}\sin\beta_{0k}\right) +$$

$$+ O(\varepsilon)] \exp\left(\frac{1}{\varepsilon}\left(\beta_{0k} + \varepsilon\beta_{1k} + ...\right)\xi\right), \tag{3.8}$$

$$\sigma_{\varphi\varphi}^{(2)} = \sum_{k=1}^{\infty} F_k \beta_{0k} \left[\frac{2G_0\lambda_0}{G_0+\lambda_0} \sin\beta_{0k} \sin(\beta_{0k}\rho) + O(\varepsilon)\right] \times$$

$$\times \exp\left(\frac{1}{\varepsilon}\left(\beta_{0k} + \varepsilon\beta_{1k} + ...\right)\xi\right),$$

$$\sigma_{\xi\xi}^{(2)} = \sum_{k=1}^{\infty} F_k \beta_{0k} \left[2G_0\beta_{0k}\rho \sin\beta_{0k} \cos(\beta_{0k}\rho) + \right.$$

$$+ \left(\frac{2G_0\lambda_0}{G_0+\lambda_0} \sin\beta_{0k} - 2G_0\beta_{0k}\cos\beta_{0k}\right)\sin(\beta_{0k}\rho) +$$

$$+ O(\varepsilon)] \exp\left(\frac{1}{\varepsilon}\left(\beta_{0k} + \varepsilon\beta_{1k} + ...\right)\xi\right).$$

The general solution (3.2) will be the sum of solutions (3.3), (3.6):

$$u_\rho(\rho,\xi) = u_\rho^{(1)} + u_\rho^{(2)}, u_\xi(\rho,\xi) = u_\xi^{(1)} + u_\xi^{(2)}. \tag{3.9}$$

For stress tensor components we have:

$$\sigma_{\rho\rho} = \sigma_{\rho\xi}^{(1)} + \sigma_{\rho\rho}^{(2)}, \sigma_{\rho\xi} = \sigma_{\rho\xi}^{(1)} + \sigma_{\rho\xi}^{(2)}, \sigma_{\varphi\varphi} = \sigma_{\varphi\varphi}^{(1)} + \sigma_{\varphi\varphi}^{(2)}, \sigma_{\xi\xi} = \sigma_{\xi\xi}^{(1)} + \sigma_{\xi\xi}^{(2)}. \tag{3.10}$$

The solutions (3.9) are of boundary layer character and their first term equivalent to the Saint-Venant edge effect of an inhomogeneous isotropic plate [14]. When deleting from the ends of the cylinder inside the domain occupied by the cylinder, the solution (3.9) exponentially decreases.

4 Satisfaction of boundary conditions of the cylinder's ends

To determine the unknown constants $T_k, F_k (k = 1, 2, ...)$, we use the Lagrange variational principle [9]. Since the solutions satisfy the equilibrium equation and boundary conditions on the lateral surface, the variational principle has the following form [11, 12]:

$$\sum_{s=1}^{2} \int_{-1}^{1} \left[\left(\sigma_{\rho\xi} - f_{1s}\right)\delta u_\rho + \left(\sigma_{\xi\xi} - f_{2s}\right)\delta u_\xi \right] \Bigg|_{\xi=\pm l} e^{2\varepsilon\rho} d\rho = 0. \tag{4.1}$$

Substituting (3.9), (3.10) in (4.1) and assuming $\delta T_k, \delta F_k$ as independent variations, from (4.1) we get the following system of linear algebraic equations:

$$\sum_{k=1}^{\infty} M_{jk} T_{k0} = d_{0j}^{'}; (j = \overline{1,\infty}) \tag{4.2}$$

$$\sum_{k=1}^{\infty} Q_{jk} F_{k0} = d_{0j}^{''}; (j = \overline{1,\infty}), \tag{4.3}$$

where

$$M_{jk} = \beta_{0k} \int_{-1}^{1} \langle G_0 \left[\cos\beta_{0k}(\sin(\beta_{0k}\rho) + 2\beta_{0k}\rho\cos(\beta_{0k}\rho)) + \right.$$
$$\left. + \left(2\beta_{0k}\sin\beta_{0k} - \frac{3G_0+\lambda_0}{G_0+\lambda_0}\cos\beta_{0k}\right)\sin(\beta_{0k}\rho)\right] \times$$
$$\times \left[\left(\beta_{0j}\sin\beta_{0j} - \frac{3G_0+\lambda_0}{G_0+\lambda_0}\cos\beta_{0j}\right)\sin(\beta_{0j}\rho) + \right.$$
$$+\beta_{0j}\rho\cos\beta_{0j}\cos(\beta_{0j}\rho)] + \beta_{0j}\left[2G_0\ \beta_{0k}\rho\cos\beta_{0k}\sin(\beta_{0k}\rho) - \right.$$
$$\left. - \left(2G_0\beta_{0k}\sin\beta_{0k} + \frac{2G_0\lambda_0}{G_0+\lambda_0}\cos\beta_{0k}\right)\cos(\beta_{0k}\rho)\right] \times$$
$$\times \left[\rho\cos\beta_{0j}\sin(\beta_{0j}\rho) - \sin\beta_{0j}\cos(\beta_{0j}\rho)\right]\rangle\, d\rho \times$$
$$\times \exp\left(-\frac{(\beta_{0j}+\beta_{0k})l}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{(\beta_{0j}+\beta_{0k})l}{\sqrt{\varepsilon}}\right)$$

$$d_{0j}^{'} = \int_{-1}^{1} \sum_{s=1}^{2} \left\{ f_{1s}(\rho) \left[\left(\beta_{0j}\sin\beta_{0j} - \frac{3G_0+\lambda_0}{G_0+\lambda_0}\cos\beta_{0j}\right)\sin(\beta_{0j}\rho) + \right.\right.$$
$$+\beta_{0j}\rho\cos\beta_{0j}\cos(\beta_{0j}\rho)] + f_{2s}(\rho)\beta_{0j} \times$$
$$\times \left[\rho\cos\beta_{0j}\sin(\beta_{0j}\rho) - \sin\beta_{0j}\cos(\beta_{0j}\rho)\right]\}\, d\rho \exp\left((-1)^s\frac{\beta_{0j}l}{\varepsilon}\right),$$

$$Q_{jk} = \beta_{0k} \int_{-1}^{1} \langle\ G_0 \left[\left(2\beta_{0k}\cos\beta_{0k} + \frac{3G_0+\lambda_0}{G_0+\lambda_0}\sin\beta_{0k}\right)\cos(\beta_{0k}\rho) - \right.$$
$$\left. - \sin\beta_{0k}(\cos(\beta_{0k}\rho) - 2\beta_{0k}\rho\sin(\beta_{0k}\rho))\right] \times$$
$$\times \left[\left(\beta_{0j}\cos\beta_{0j} + \frac{3G_0+\lambda_0}{G_0+\lambda_0}\sin\beta_{0j}\right)\cos(\beta_{0j}\rho) + \beta_{0j}\rho\sin\beta_{0j}\sin(\beta_{0j}\rho)\right] +$$
$$+ \left(2G_0\beta_{0k}\rho\sin\beta_{0k}\cos(\beta_{0k}\rho) - (2G_0\beta_{0k}\cos\beta_{0k} - \frac{2G_0\lambda_0}{G_0+\lambda_0}\sin\beta_{0k}\right) \times$$
$$\times \sin(\beta_{0k}\rho)]\ \beta\ _{0j}\left[\rho\sin\beta_{0j}\cos(\beta_{0j}\rho) - \cos\ \beta_{0j}\sin(\beta_{0j}\rho)\right]\rangle\, d\rho \times$$
$$\times \left(\exp\left(-\frac{(\beta_{0j}+\beta_{0k})l}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{(\beta_{0j}+\beta_{0k})l}{\sqrt{\varepsilon}}\right)\right),$$

$$d_{0j}^{''} = -\int_{-1}^{1} \sum_{s=1}^{2} \left\{ f_{1s}(\rho) \left[\left(\beta_{0j}\cos\beta_{0j} + \frac{3G_0+\lambda_0}{G_0+\lambda_0}\sin\beta_{0j}\right)\ \cos(\beta_{0j}\rho) - \right.\right.$$
$$-\beta_{0j}\rho\sin\beta_{0j}\sin(\beta_{0j}\rho)] + f_{2s}(\rho)\beta_{0j}\left[\rho\sin\beta_{0j}\cos(\beta_{0j}\rho) - \right.$$
$$\left. - \cos\beta_{0j}\sin(\beta_{0j}\rho)\right]\}\, d\rho \exp\left((-1)^s\frac{\beta_{0j}l}{\varepsilon}\right).$$

Definition of the constants $T_{kp}, F_{kp} (p = 1, 2, ...)$ is un variably reduced to the systems whose matrices coincide with the matrices of systems (4.2), (4.3).

The system of infinite linear algebraic equations (4.2), (4.3) is positive definite the energy space and therefore it is always solvable in physically meaningful conditions imposed on the right hand side [4]. Solvability and convergence of the reduction method for (4.2), (4.3) was proved in [13, 14].

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