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Pulsating fluid flow in a finite length pipe with a narrowing effect

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Abstract. *In the paper, hydroelasticity of a viscous-elastic fluid with compression effect in a finite length pipe is studied based on linearly-averaged linear equations.*

Keywords. visco-elasticity · fluid · pipe · compression · pulsating flow · wave

Mathematics Subject Classification (2010): 76T10

1 Introduction.

The problem of studying waves in deformable shells with flowing fluid is of interest in many aspects. In theoretical aspect this is a problem of mathematical physics, in applied - one it is a necessary stage of calculation of the system subjected to dynamical action. When solving such type of problems, it is necessary to consider the equations of motion of a shell with regard to influence of liquid moving in the cavity on the shell dynamics. Today the totality of such problems compose widely developed field of hydroelasticity [1, 3].

However a number of properties related to simultaneous accounting of rheology of fluid and shell material, its contraction and multilayering generates significant theoretical difficulties related in the first turn to integration of boundary value problems with variable coefficients.

Assume that we are given of length l. It is accepted that it's cross-section area S depends on longitudinal coordinate $x$and it is rigidly fastened to the surrounding medium, and therefore the pipe has no displacement in axial direction. The liquid is assumed to be homogeneous with density $\rho$and dynamical viscosity coefficient μ [2, 3, 6]. The liquid motion may be represented by means of axial velocity component $u(x,t)$, where $t$is time. In one-dimensional statement, it is considered that pressure $p = p(x,t)$, radial displacement of the pipe $W = W(x,t)$. For the accepted mathematical model of pipe-fluid, we write the closed system of equations of the form:

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$$\frac{\partial}{\partial x}(Su) + L\frac{\partial W}{\partial t} = 0, \tag{1.1}$$

$$\frac{S(x)}{\rho}\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial t} + \frac{8\mu}{\rho R^2(x)}Q = 0, \tag{1.2}$$

$$p = \frac{LE}{R^2(x)}\left\{W - \int\limits_{-\infty}^{t} \mathit{\Gamma}(t-\tau)W(x,\tau)d\tau\right\} + \rho_* h\frac{\partial^2 W}{\partial t^2}. \tag{1.3}$$

In the above equations $Q(x,t)$ is fluid flow, h is pipe's thickness, $R(x)$ is its radius, $L(x)$ is a cross section, parameter, ρ_* is the density of the wall's material E is an instant modulus of elasticity, $\mathit{\Gamma}(t-\tau)$ difference core of relaxation [1, 4, 5].

We write the function $R(x)$ as $R(x) = R_0 g(x)$, $g(x)$ where is a positive twice differentiable monotonically decreasing function $\forall x \in [0,l]$,moreover $g(0) = 1$. Then

$$\frac{\pi R_0^2}{\rho}g^2(x)\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial t} + \frac{8\mu}{\rho R_0^2(x)g^2(x)}Q = 0,$$

$$\frac{\partial Q}{\partial x} + 2\pi R_0^2 g(x)\frac{\partial W}{\partial t} = 0, \tag{1.4}$$

$$p = \frac{LE}{R_0^2 g^2(x)}\left\{W - \int\limits_{-\infty}^{t} \mathit{\Gamma}(t-\tau)W(x,\tau)d\tau\right\} + \rho_* h\frac{\partial^2 W}{\partial t^2}.$$

We reduce system (1.4) to the solution of ordinary differential equations assuming that all variables are proportional to time factor $\exp(i\omega t)$, where ω is a given real value of the frequency

$$p = p_1(x)\exp(i\omega t), \;\; Q = Q_1(x)\exp(i\omega t), \;\; W = W_1(x)\exp(i\omega t). \tag{1.5}$$

Taking into account representation (1.5) and introduction for brevity of the following denotations:

$$\alpha = \int\limits_0^\infty \mathit{\Gamma}(\theta)e^{-i\omega\theta}d\theta \;\; (\theta = t-\tau),$$

$$\xi(x) = \frac{Eh}{R_0^2 g^2(x)}(1-\alpha) - \rho_* h\omega^2, \tag{1.6}$$

$$\eta(x) = i\omega + \frac{8\mu}{\rho R_0^2 g^2(x)},$$

we have :

$$p_1 = W_1\xi(x), \;\; \frac{\pi R_0^2 g^2(x)}{\rho}p_1' + r(x)Q_1 = 0, \;\; Q' + 2\pi i\omega R_0 g(x)W_1 = 0.$$

Combining these equations with respect to the function p_1 we get:

$$p_1'' + \mu_1(x)p_1' + \mu_2(x)p_1 = 0, \tag{1.7}$$

where

$$\mu_1(x) = 2\frac{g'(x)}{g(x)} - \frac{\eta'(x)}{\eta(x)}, \;\; \mu_2(x) = -2i\frac{\rho\omega}{R_\infty g(x)}\frac{\eta(x)}{\xi(x)}. \tag{1.8}$$

The Liouville substitution $y(x) = p_1 \exp \frac{1}{2} \int \mu_1(x)dx = p_1\lambda(x)$ reduces (1.7) to the form

$$y'' + I(x)y = 0 \tag{1.9}$$

for the invariant

$$I(x) = \mu_2(x) - \frac{1}{4}\{\mu_1(x)\}^2 - \frac{1}{2}\mu_1'(x). \tag{1.10}$$

We modify equation (1.9), having written it in the following way:

$$y'' + \delta^2 y = q(x)y. \tag{1.11}$$

Here $q(x) = \delta^2 - I(x),$ where δ is a wave number, for $R = R_0$ written by the formula

$$\delta^2 = -2i\frac{\rho\omega}{R_0}\left\{i\omega + \frac{8\mu}{\rho R_0^2}\right\}\left\{\frac{LE(1-\alpha)}{R_0^2} - \rho_* h\omega^2\right\}^{-1}, \tag{1.12}$$

On the function $q(x)$ we impose the integrability condition

$$\int_0^l |q(x)|\, dx < +\infty. \tag{1.13}$$

Formulate the boundary conditions of the problem. Let for $x = 0$ the pressure change by the law $p(0,t) = \eta_0 \exp(i\omega t)$, and for $x = l$ be equal to $p(l,t) = \eta \exp(i\omega t)$, where η_0 and ηl are the given empirical values. Then we immediately write $y(0) = \eta_0\lambda(0),\ y(l) = \eta_l\lambda(l)$. Thus we could reduce the solution of the stated problem to the solution of Sturm-Liouville type regular boundary value problem provided condition (1.13)

$$y'' + \delta^2 y = q(x)y, \tag{1.14}$$

$$y(0) = \eta_0\lambda(0),\ y(l) = \eta_l\lambda(l). \tag{1.15}$$

The solution of equation (1.14) is reduced to the equivalent integral equation

$$y(x,\delta) = \alpha_1 e^{i\delta x} + \alpha_2 e^{-i\delta x} + \frac{1}{\delta}\int_x^l \sin\delta(m-x)q(m)y(m,\delta)dm, \tag{1.16}$$

where α_1 and α_2 are integration constants to be determined from boundary conditions (1.15). We can solve equation (1.16) by the method of successive approximations. We assume

$$y_0(x,\delta) = \alpha_1 e^{i\delta x} + \alpha_2 e^{-i\delta x}$$

and let for $n >> 1$

$$y_n(x,\delta) = y_0(x,\delta) + \frac{1}{\delta}\int_x^l \sin\delta(m-x)q(m)y_{n-1}(m,\delta)dm.$$

From inequality (1.13) by Weierstrass test from uniform convergence of successive approximations it follows that a unique solution of integral equation (1.16) denoted by $y(x,\delta)$, is determined by means of the series

$$y_n(x,\delta) = y_0(x,\delta) + \sum_{n=1}^{\infty}\{y_n(x,\delta) - y_{n-1}(x,\delta)\}. \tag{1.17}$$

Having denoted $y_n(x,\delta) - y_{n-1}(x,\delta) = (1 / \delta^n)\varphi_n(x,\delta)$, we give the following representation of the series (1.17):

$$y(x,\delta) = \alpha_1 e^{i\delta x} + \alpha_2 e^{-i\delta x} + \sum_{n=1}^{\infty} \frac{1}{\delta^n}\varphi_n(x,\delta). \tag{1.18}$$

Here we have the totality of the following recurrent relations :

$$\begin{aligned} \varphi_1(x,\delta) &= \int_x^l \sin\delta(m-x)q(m)y_0(m,\delta)dm, ... \\ \varphi_n(x,\delta) &= \int_x^l \sin\delta(m-x)q(m)y_{n-1}(m,\delta)dm. \end{aligned} \tag{1.19}$$

Based on what has been said above, we can affirm that all the solutions of equation (1.14) for any α_1 and α_2 satisfy equation (1.16). By direct verification we can prove the inverse.

Proceeding from boundary conditions (1.15), we have:

$$\alpha_1 = \frac{ae^{-i\delta l} - b}{e^{-i\delta l} - e^{i\delta l}}, \alpha_2 = \frac{b}{e^{-i\delta l} - e^{i\delta l}}.$$

Here $a = \eta_0\lambda(0) - \sum\limits_{n=1}^{\infty} \frac{1}{\delta^n}\varphi_n(0,\delta)$, and $b = \eta_l\lambda(l) - \sum\limits_{n=1}^{\infty} \frac{1}{\delta^n}\varphi_n(0,\delta)$. In what follows, we use the Euler formula and from (1.18) get:

$$y(x,\delta) = \frac{a\sin\delta(l-x) + b\sin\delta x}{\sin\delta l} + \sum_{n=1}^{\infty} \frac{1}{\delta^n}\varphi_n(x,\delta).$$

From this expression, following formulas (1.4), we can finally determine the sought-for functions $p(x,t),\ W(x,t)$ and $Q(x,t)$. Note that real sides of the obtained solutions are physical quantities.

Leaving aside the influence of such facts as viscosity of the material of the pipe of liquid, and also neglecting the wall inertia, in order assess the contribution arising when taking into account contracting effect, we give the result of calculations for wave velocity. In this approximation, keeping the previous denotation, from (1.6) we have :

$$\xi(x) = \frac{Eh}{R_0^2 g^2(x)}, r = i\omega,$$

Whence, following (1.8), (1.10) and (1.12), we write :

$$\mu_1(x) = z\frac{g'(x)}{g(x)},\ \ \mu_2(x) = \delta^2 g(x),$$

where

$$\delta^2 = \frac{\omega^2}{c_0^2}, \left\{ c_0 = \frac{Eh}{2\rho R_0} \right\},$$

and

$$I(x) = \delta^2 g(x) - \left\{g'(x)/g(x)\right\}^2 - \left\{g'(x)/g(x)\right\}'.$$

We now specify the form of the function $g(x)$, assuming $g(x) = e^{-\beta x}$ $(0 < \beta < 1))$, where β is a dimensional parameter characterizing the cone-shaped contraction of the pipe.

Then we will write the formula for velocity in the form $c_0/c = \delta/\left(\delta^2 e^{-\beta x} - \beta^2\right)^{-1} \times \times \left(\delta^2 e^{-\beta x} - \beta^2\right)^{-1}$, from which we can conclude that for fixed $\delta << 1$ with increasing β the wave velocity increases. This time in order to realize the wave process the nonlinear inequality $\beta^2 < \delta^2 e^{-\beta x}$ should be fulfilled. The last one implies sufficient principal contraction of the pipe for which $\beta << 1$. Fig. 1 depicts the graph of dependence of c/c_0 on β for the chosen parameters: $E = 4 \cdot 10^6 \frac{dH}{cm^2}, R_0 = 2\,cm,\ h = 0,2\,cm,\ \rho = 1\frac{\Gamma}{cm^3}$, $\omega = 10\ sec^{-1}$.

Hence it follows that c/c_0 increases with increasing β, and very significantly .

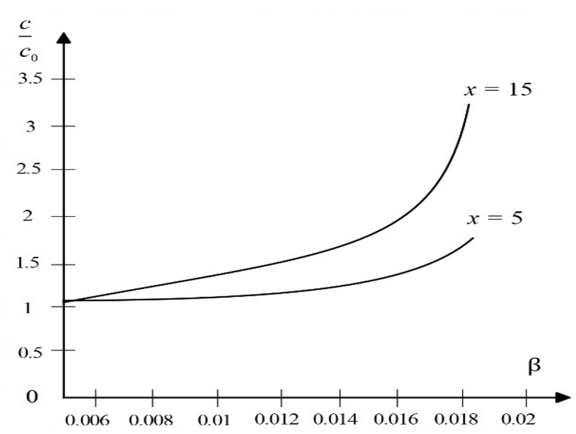

Fig 1. Dependence of c/c_0 on β for the chosen parameters

In conclusion, we note that for $\beta = 0,018$ the ratio of radii of the cross-section of the pipe $R(l)/R_0$ changes from $0,917$ for $x = 5$ cm to $0,763$ for $x = 15$ cm.

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