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Functionally invariant method in solving three-dimensional problems of elastodynamics

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Abstract. *The method of solving multidimensional problems of mathematical physics allowing overcoming the difficulties connected with the splitting of the equations of boundary conditions containing derivatives of unknown unknown functions is proposed in the article. As an example, we consider the nonstationary problem of elastodynamics.*

A method is proposed for solving linear, multidimensional partial differential equations in integral form with a kernel satisfying a wave equation that remains unchanged (up to a complex variable and a multiplier) after Laplace-Fourier transforms, which has interchangeable derivatives with respect to all variables, and thus allows us to reduce the number of independent variables in the auxiliary equation.

Keywords. nonstationary · Lamb's problems · elasticity theory · wave equation · Laplace-Fourier transform · analytic function.

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1 Introduction

Linear dynamic equilibrium equations using stress-strain relationships

$$\sigma_{i,j} = \lambda \delta_{ij} \theta + \mu \left(u_{i,j} + u_{j,i} \right), \tag{1.1}$$

where λ, μ- elastic medium constants, δ_{ij}- Kronecker symbol, θ- volume expansion

$$\theta = div\, u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

expressed through displacement

$$\begin{array}{l} \mu\, \Delta u_1 + (\lambda + \mu) \frac{\partial \theta}{\partial x_1} = \rho \frac{\partial^2 u_1}{\partial t^2} \\ \mu\, \Delta u_2 + (\lambda + \mu) \frac{\partial \theta}{\partial x_2} = \rho \frac{\partial^2 u_2}{\partial t^2}\,, \\ \mu\, \Delta u_3 + (\lambda + \mu) \frac{\partial \theta}{\partial x_3} = \rho \frac{\partial^2 u_3}{\partial t^2} \end{array} \tag{1.2}$$

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where $u_1,\, u_2,\, u_3$- displacement vector projection, ρ- density, $\omega_1,\, \omega_2,\, \omega_3$- rotation vector projection

$$\bar{\omega} = \frac{1}{2} rot\, u$$

will be

$$\omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right), \omega_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right), \omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

differentiating θ by $x_1,\, x_2,\, x_3$ obtained three equations

$$\begin{aligned} \Delta u_1 &= \frac{\partial \theta}{\partial x_1} - 2\frac{\partial \omega_3}{\partial x_2} + 2\frac{\partial \omega_2}{\partial x_3} \\ \Delta u_2 &= \frac{\partial \theta}{\partial x_2} - 2\frac{\partial \omega_1}{\partial x_3} + 2\frac{\partial \omega_3}{\partial x_1} \\ \Delta u_3 &= \frac{\partial \theta}{\partial x_3} - 2\frac{\partial \omega_2}{\partial x_1} + 2\frac{\partial \omega_1}{\partial x_2} \end{aligned} \tag{1.3}$$

Substituting (1.3) in (1.2), we obtain

$$\begin{aligned} \frac{\partial^2 u_1}{\partial t^2} &= c_1^2 \frac{\partial \theta}{\partial x_1} - 2c_2^2 \frac{\partial \omega_3}{\partial x_2} + 2c_2^2 \frac{\partial \omega_2}{\partial x_3} \\ \frac{\partial^2 u_2}{\partial t^2} &= c_1^2 \frac{\partial \theta}{\partial x_2} - 2c_2^2 \frac{\partial \omega_1}{\partial x_3} + 2c_2^2 \frac{\partial \omega_3}{\partial x_1} \\ \frac{\partial^2 u_3}{\partial t^2} &= c_1^2 \frac{\partial \theta}{\partial x_3} - 2c_2^2 \frac{\partial \omega_2}{\partial x_1} + 2c_2^2 \frac{\partial \omega_1}{\partial x_2}, \\ c_1^2 &= \frac{\lambda + 2\mu}{\rho},\ c_2^2 = \frac{\mu}{\rho}\,. \end{aligned} \tag{1.4}$$

Introducing the antisymmetric tensor

$$\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \quad i, j = 1, 2, 3\,.$$

And knowing the relation

$$\begin{aligned} \omega_1 &= \omega_{32} = -\omega_{23} \\ \omega_2 &= \omega_{13} = -\omega_{31} \\ \omega_3 &= \omega_{21} = -\omega_{12} \end{aligned}$$

we rewrite expressions (1.4) in the form

$$u_{i,tt} = c_1^2 \theta_{,i} + 2c_2^2 \omega_{ij,j} \tag{1.4 a}$$

Acting on (1.3) and (1.4) operators $\frac{\partial^2}{\partial t^2}$ and Δ, subtracting, we see that θ and ω_i satisfy the wave equations

$$L_i = \Delta - \frac{1}{c_i^2}\frac{\partial^2}{\partial t^2}; \qquad L_i u = 0. \tag{1.5}$$

Using (1.4 a) in (1.1) we obtain a relation for the normal and shear stresses

$$\sigma_{ii,tt} = \lambda \theta_{,tt} + 2\mu c_1^2 \theta_{,ii} + 4\mu c_2^2 \omega_{ij,ij} + 4\mu c_2^2 \omega_{ik,ik}$$

(do not summarize)

$$\sigma_{ij,tt} = 2\mu c_1^2\theta_{,ij} - 2\mu\omega_{ij,tt} + 4\mu c_2^2\omega_{ij,jj} + 4\mu c_2^2\omega_{ik,jk} \qquad i \neq j \neq k, \tag{1.6}$$

where it is considered: $div\,\bar{\omega} = 0$, and (1.5)

The further course of the solution involves [4] difficulty splitting the equations (1.6). This requires a special form of representation of solutions (1.5).
In this paper, we will use (see the appendix) the following representation of solutions (1.5)

$$F_i^{\pm}(x,y,z,t) = \int\limits_0^t \int\limits_r^{\sqrt{\tau_2^2 c_i - z^2}} \frac{c_i F_{0i}^{\pm}\,(c_i(t-\tau_2))\, H(\tau_2 c_i - R)}{\sqrt{c_i^2\tau_2^2 - z^2 - \tau_1^2}\sqrt{\tau_1^2 - x^2 - y^2}} d\tau_1 d\tau_2, \quad i = 1,2, \tag{1.7}$$

where $R = \left(x^2+y^2+z^2\right)^{\frac{1}{2}}$; $r = \left(x^2+y^2\right)^{\frac{1}{2}}$ $H(\tau_2 c_i - R)$ Heaviside function having the following expressions for derivatives

$$\begin{aligned} \frac{\partial^n F_i^{\pm}}{\partial x^n} &= c_i^{-n}\left(F_{oi,tt}^{\pm}\left(i\,shp^{\pm}chq_i\right)^n\right) \\ \frac{\partial^n F_i^{\pm}}{\partial y^n} &= c_i^{-n}\left(F_{oi,tt}^{\pm}\left(shp^{\pm}shq_i\right)^n\right) \\ \frac{\partial^n F_i^{\pm}}{\partial z^n} &= c_i^{-n}\left(F_{oi,tt}^{\pm}\left(-chq_i\right)^n\right), \end{aligned} \tag{1.8}$$

where the notation is introduced

$$c_i^{-n}\left(F_{oi,tt}^{\pm}\left(i\,shp^{\pm}chq\,i\right)^n\right) =$$

$$= \bar{c}_i^n \int\limits_o^t \int\limits_r^{\sqrt{\tau_2^2 c_i - z^2}} \frac{\partial^n F_{0i}^{\pm}}{\partial t^n} \frac{c_i\left(i\,shp^{\pm}\,chq_i\right)^n}{\sqrt{c_i^2\tau_2^2 - z^2 - \tau_1^2}\sqrt{\tau_1^2 - x^2 - y^1}} d\tau_1 d\tau_2$$

$$p^{\pm} = \ln\frac{x \pm iy}{\tau_1 - \sqrt{\tau_1^2 - x^2 - y^2}};$$

$$q_i = \ln\frac{z + i\tau_1}{c_i\tau_2 - \sqrt{c_i^2\tau_2^2 - z^2 - \tau_1^2}}; \quad i = 1,2.$$

Formulas (1.8) make it possible to get rid of the derivatives with respect to t by integrating (1.6); however, reducing them to algebraic ones requires the representation of the functions given on the boundary similarly to (1.7)

$$\begin{aligned} \sigma_{ij}(x,y,z,t) = \int\limits_0^t \int\limits_r^{\sqrt{\tau_2^2 - z^2}} \sigma_{ij}^o(t-\tau_2)\left(\tau_2^2 - z^2 - \tau_1^2\right)^{-\frac{1}{2}} \times \\ \times \left(\tau_1^2 - x^2 - y^2\right)^{-\frac{1}{2}} d\tau_1 d\tau_2. \end{aligned} \tag{1.9}$$

If this succeeds, then to find unknowns $\theta$and ω_i, we obtain the algebraic equation

$$\sigma = DF, \tag{1.10}$$

where $\sigma = \left(\sigma^o_{yy}, \sigma^o_{yx}, \sigma^o_{yz}\right)^T$, $F = \left(\theta^+_0, \theta^-_0, \omega^+_{03}, \omega^-_{03}, \omega^+_{01}, \omega^-_{01}\right)^T$

$$D = \begin{bmatrix} a^+_{11}, & a^-_{11}, & a^+_{12}, & a^-_{12}, & a^+_{13}, & a^-_{13} \\ a^+_{21}, & a^-_{21}, & a^+_{22}, & a^-_{22}, & a^+_{23}, & a^-_{23} \\ a^+_{31}, & a^-_{31}, & a^+_{32}, & a^-_{32}, & a^+_{33}, & a^-_{33} \end{bmatrix}$$

$$a^\pm_{11} = \lambda + 2\mu c_1^2 sh^2 p^\pm sh^2 q_1 \quad a^\pm_{12} = 4\mu \left(chp^\pm shp^\pm shq_2\right)$$

$$a^\pm_{13} = 4\mu i \left(sh^2 p^\pm shq_2 chq_2\right) \quad a^\pm_{21} = 2\mu i \left(sh^2 p^\pm shq_1 chq_1\right)$$

$$a^\pm_{22} = 4\mu i \left(shp^\pm chp^\pm chq_2\right) \quad a^\pm_{23} = -2\mu \left(1 + 2sh^2 p^\pm ch^2 q_2\right)$$

$$a^\pm_{31} = -2\mu i \left(shp^\pm chp^\pm shq_1\right)$$
$$a^\pm_{32} = 2\mu \left(1 - 2ch^2 p^\pm\right)$$
$$a^\pm_{33} = -4\mu i \left(shp^\pm chp^\pm chq_2\right) .$$

2 Solution

As an example, consider the three-dimensional Lamb problem. Let the normal stress be given on the boundary of half-space $y = 0$

$$\sigma_{yy}(x, o, z, t) = \sigma^0_0 \delta(x)\, \delta(z)\, \delta(t)\ ; \quad \sigma_{yx} = \sigma_{yz} = 0\,, \tag{2.1}$$

where $\delta(x)$, $\delta(z)$, $\delta(t)$ functions of Darak.

Let us prove that the function

$$\sigma_{yy}(x, y, z, t) = \int_o^t \int_r^{\sqrt{\tau_2^2 - z^2}} \frac{\sigma^o \delta(t - \tau_2)\ shp\, shq}{\sqrt{\tau_2^2 - z^2 - \tau_1^2}\sqrt{\tau_1^2 - x^2 - y^2}} d\tau_1\, d\tau_2, \tag{2.2}$$

where $q = \ln \frac{z + i\tau_1}{\tau_2^2 - \sqrt{\tau_2^2 - z^2 - \tau_1^2}}$ on the border $y = 0$ coincides with (2.1).

Indeed, applying first to (2.2) the Laplace transforms in t and Fourier transforms in z, we obtain

$$\bar{\sigma}_{yy}(x, k_z, y, s) = \sigma^o \int_r^\infty e^{-\tau_1 \sqrt{s^2 + k_z^2}} \frac{-ish\, p\, d\tau_1}{\sqrt{\tau_1^2 - x^2 - y^2}}\,. \tag{2.3}$$

Applying the Fourier transform with respect to (2.3), we obtain (see appendices)

$$\bar{\sigma}_{yy}(k_x, y, k_z, s) = \sigma^o e^{-y\sqrt{s^2 + k_x^2 + k_z^2}}\,. \tag{2.4}$$

Hence it is seen that for $y = 0$ (2.4) coincides with the correspondingly transformed (2.1).

Considering (2.2) in (1.10) and bearing in mind $\theta^-_o = \omega^-_{03} = \omega^-_{01} = 0$, we can define θ^+_o, ω^+_{03} and ω^+_{01}, $F = D^{-1}\sigma$.

Knowing θ, ω_1 and ω_3, you can determine the displacement and stress.

Appendix

Applying the Laplace transform by t to the left, and the inverse Fourier transform by x to the right side of the equality

$$\int_0^{k\left(t^2 - y^2\right)} J_0\left(\sqrt{k^2 t^2 - (\tau + yk)^2}\right) f(\tau)\, d\tau = \int_0^{k\left(t^2 - y^2\right)} J_0\left(\alpha\sqrt{(k - b)^2 - c^2}\right) f(\tau)\, d\tau,$$

Where J_0 - Bessel function, k- parameter Fourier transform, $a = \sqrt{t^2 - y^2}$; $b = \frac{y\tau}{t^2 - y^2}$; $c = \frac{t\tau}{t^2 - y^2}$. We get [1] a relation equivalent to the expression,

$$\int_0^\infty \int_{-\infty}^\infty e^{-st+ikx} \frac{F(\zeta)}{\sqrt{t^2 - y^2 - x^2}} dtdx = \frac{e^{-y\sqrt{s^2+k^2}}}{\sqrt{s^2 + k^2}} F\left(\bar{\zeta}\right), \qquad [A.1]$$

proving the immutability (to a complex variable) of a function

$$F(\zeta) = \int_0^\infty e^{-\tau\zeta} f(\tau)\, d\tau$$

after Laplace-Fourier transforms; where $\zeta = \frac{t\sqrt{t^2-y^2-x^2}+ixy}{t^2-y^2}$,$\bar{\zeta} = \sqrt{\frac{s^2}{k^2} + 1}$

s- Laplace transform parameter.

his result is consistent with the previously known methods [3], [2], but indicates a way to obtain simpler formulas for differentiation. Indeed, knowing that differentiation [A.1] with respect to 1 and 2 can be reduced only to multiplication of the left-hand side

$$-ik = -\frac{s}{\sqrt{1 - \bar{\zeta}^2}} = -s\, sh\bar{p},$$

$$-\sqrt{s^2 + k^2} = +\frac{is\bar{\zeta}}{\sqrt{1 - \bar{\zeta}^2}} = +s\, ish\,\bar{p}, \quad \bar{\zeta} = \sqrt{\frac{s^2}{k^2} + 1} = th\bar{p}.$$

then, considering the properties of the Laplace transform, zero initial conditions, and the invariability of function $F(\zeta)$ gives the right to assert

$$\begin{aligned} \frac{\partial^n \psi}{\partial x^n} &= \frac{\partial^n}{\partial t^n}\left((-chp)^n \psi\right) \\ \frac{\partial^n \psi}{\partial y^n} &= \frac{\partial^n}{\partial t^n}\left((ishp)^n \psi\right) \end{aligned} \qquad [A.2]$$

where

$$\psi = \frac{F(\zeta)}{\sqrt{t^2 - x^2 - y^2}} \zeta = \frac{t\sqrt{t^2 - x^2 - y^2} + ixy}{t^2 - y^2} = thp \qquad [A.3]$$

$$chp = \frac{1}{\sqrt{1 - \zeta^2}} = \left(x^2 + y^2\right)^{-1}\left(tx + iy\sqrt{t^2 - x^2 - y^2}\right) = \frac{1}{2}\left(\frac{z}{R^-} - \frac{R^-}{z}\right)$$

$$shp = \frac{\zeta}{\sqrt{1 - \zeta^2}} = \left(x^2 + y^2\right)^{-1}\left(x\sqrt{t^2 - x^2 - y^2} + iyt\right) = \frac{1}{2}\left(\frac{z}{R^-} + \frac{R^-}{z}\right)$$

$$p = \ln\frac{z}{R}; \quad z = x + iy; \quad R^- = t - \sqrt{t^2 - |z|^2}, \;\; |z| = \left(x^2 + y^2\right)^{\frac{1}{2}}.$$

If $t^2 < x^2 + y^2$ hyperbolic functions are replaced with the corresponding trigonometric ones.

Properties [A.2] can be used to solve partial differential equations.

Consider the integral

$$u(x, y, z) = \int_0^\infty u_0(\tau, z)\, \psi(p(x, y, \tau))\, d\tau \qquad [A.4]$$

where ψ- arbitrary function [A.3], selected in such a way as to satisfy the condition

$$\frac{\partial^n \psi}{\partial \tau^n}\Big|_0^\infty = 0 \quad n = 1, 2, ..., \qquad [A.5]$$

a $u_0(z, \tau)$ is an unknown function to be determined.

Substituting [A.4] in the three-dimensional biharmonic equation

$$\Delta\Delta u = 0$$

considering expression [A.2] and integrating by parts, we obtain.

$$\int \left(U_0 \frac{\partial^4}{\partial \tau^4} \left(\psi (ch^4 p - 2ch^2 p \cdot sh^2 p + sh^4 p) \right) + \right.$$
$$\left. +2 \frac{\partial^2 U_0}{\partial {x_3}^2} \frac{\partial^2}{\partial \tau^2} \left(\psi \left(ch^2 p - sh^2 p \right) \right) + \frac{\partial^4 U_0}{\partial {x_3}^4} \psi \right) d\tau =$$
$$= \int \left(U_0 \frac{\partial^4 \psi}{\partial \tau^4} + 2 \frac{\partial^2 U_0}{\partial {x_3}^2} \frac{\partial^2 \psi}{\partial \tau^2} + \frac{\partial^4 U_0}{\partial {x_3}^4} \psi \right) d\tau =$$
$$= \int \left(\frac{\partial^4 U_0}{\partial \tau^4} + 2 \frac{\partial^4 U_0}{\partial \tau^2 \partial {x_3}^2} + \frac{\partial^4 U_0}{\partial {x_3}^4} \right) \psi d\tau = 0$$

Now consider the function

$$U(x_1, x_2, x_3, x_4) = \int_0^\infty \int_0^\infty U_0(x_4, \tau_2) \cdot \psi_2(p_2(x_3, \tau_1, \tau_2)) \cdot \psi_1(p_1(x_1, x_2, \tau_1)) d\tau_1 d\tau_2 \quad [A.6]$$

where $U_0(x_4, \tau_2)$- unknown, to be determined, function, ψ_1 and - ψ_2-arbitrary, also determined from expression [A.3] corresponding to the change of variables. Substitute [A.6] into the wave equation.

$$L_c U = \Delta U - \frac{1}{c^2} \frac{\partial^2 U}{\partial {x_4}^2} = 0$$

get

$$\iint \left(U_0 \psi_2 \frac{\partial^2 \psi_1}{\partial {\tau_1}^2} + U_0 \frac{\partial^2}{\partial {\tau_2}^2} (\psi_2 (ch^2 p_2)) \psi_1 - \frac{1}{c^2} \frac{\partial^2 U_0}{\partial {x_4}^2} \psi_2 \psi_1 \right) d\tau_2 d\tau_1 =$$
$$= \iint \left(U_0 \frac{\partial^2{}_1}{\partial {\tau_2}^2} \left(\psi_2 \left(-sh^2 p_2 \right) \right) \psi_1 + \right.$$
$$\left. +U_0 \frac{\partial^2}{\partial {\tau_2}^2} \left(\psi_2 \left(ch^2 p_2 \right) \right) \psi_1 - \frac{1}{c^2} \frac{\partial^2 U_0}{\partial {x_4}^2} \psi_2 \psi_1 \right) d\tau_2 d\tau_1 =$$
$$= \iint \left(U_0 \frac{\partial^2 \psi_2}{\partial {\tau_2}^2} \psi_1 - \frac{1}{c^2} \frac{\partial^2 U_0}{\partial {x_4}^2} \psi_2 \psi_1 \right) d\tau_2 d\tau_1 =$$
$$= \iint \left(\frac{\partial^2 U_0}{\partial {\tau_2}^2} - \frac{1}{c^2} \frac{\partial^2 U_0}{\partial {x_4}^2} \right) \psi_2 \psi_1 d\tau_2 d\tau_1 = 0$$

Substituting expression [A.6] into the equation

$$\Delta L_{c_1} L_{c_2} U = 0$$

get

$$\iint \left(\frac{\partial^4 U_0}{\partial {\tau_2}^4} - \left(\frac{1}{c_1^2} + \frac{1}{c_2^2} \right) \frac{\partial^4 U_0}{\partial {\tau_2}^2 \partial {x_4}^2} + \frac{1}{c_1^2} \frac{1}{c_2^2} \frac{\partial^4 U_0}{\partial {x_4}^4} \right) \psi_2 \psi_1 d\tau_2 d\tau_1 = 0.$$

As can be seen from the solutions in all cases, multidimensional problems of mathematical physics are reduced to one-dimensional.

3 Conclusion

A general solution of spatial non-stationary problems of elasticity theory is constructed using analytical functions that remain unchanged (up to a complex variable) after Laplace-Fourier transformations and, as a consequence, acquired the property of images, which allows splitting partial differential equations. As an example, the solution of the three-dimensional Lamb problem is given.

References

1. Bateman G. and Erdein A.: *Tables of integral transformations.* **1** (1969), 112-123.
2. Cagniard L.: *Reflection and refraction des Andes seismigues progressive.* (These). Paris. 1939, Reflection and retraction of progressive waves. New York, 1962.
3. Smirnov V.I, Sobolev S.L.: On the application of the new method to the study of elastic vibrations in space in the presence of axial symmetry. *Tr. seismic. Institute of the Academy of Sciences of the USSR,* **29**, 1933, 43-51.
4. Tikhonov A. N., Samarskii A. A.: *Equation of Mathematical Physics,* 1976, 437-440.