Transactions of NAS of Azerbaijan, Issue Mechanics, 40 (8), 14–21 (2020).
Series of Physical-Technical and Mathematical Sciences.

On the frequency response of the interface normal stress in the bi-layered hollow cylinder in the 3D dynamic state

Mahir A. Mehdiyev

Received: 25.09.2020 / Revised: 25.11.2020 / Accepted: 15.12.2020

Abstract. *The paper studies frequency response of the interface normal stress in the bi-layered hollow cylinder under time-harmonic loading of that acting on the interior of the bi-layered hollow cylinder in the 3D dynamic state with utilizing the exact equations and relations of the elastodynamics. It is assumed that the mentioned time-harmonic loading is the point-located one with respect to the cylinder axis, and the distribution of that is non-axisymmetric and is located within a certain central angle. The corresponding mathematical problem is solved by employing the Fourier transform with respect to the axial coordinate and these transforms are presented through the Fourer series and each coefficients in these series are determined analytically as a result of the soluton to the corresponding boundary-value problems. The inverse of the transforms are found numerically. In the paper, numerical results related to the frequency response of the interface normal stress and to the influence of the problem parameters on this response are presented and discussed. In particular, it is established that after a certain frequency of the external forces the jumping in the values of the stress under consideration appears and the density of this jumping depends on the problem parameters.*

Keywords. bi-layered hollow cylinder · forced vibration · frequency response · interface normal stress · Fourier transform · Fourier series

Mathematics Subject Classification (2010): 74A60, 74E15

1 Introduction

The investigations of the related 3D dynamic problems for the layered cylindrical systems has a great significance not only in the theoretical sense but also in the application sense. The first attempts in this field were made in the papers [1, 2] in which it was considered the system consisting of the hollow cylinder and surrounding an infinite elastic medium. At the same time, the corresponding axisymmetric problems were considered in the works [3, 4] and others listed therein. In the present work we attempt to consider of the investigation carried out in the paper [2] for the case where the thickness of the surrounding medium is finite, i.e. for the bi-layered hollow cylinder case. This consideration is made within the scope of the 3D exact equations and relations of the elastodynamics.

M.A. Mehdiyev
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku
Azerbaijan State University of Economics,(UNEC), Baku, Azerbaijan
E-mail: mahirmehdiyev@mail.ru

2 Formulation of the problem and solution method

Consider the aforementioned bi-layered hollow cylinder the sketch of which is illustrated in Fig. 1 and assume that the thicknesses of the walls of the inner and outer cylinders are $h^{(2.2)}$ and $h^{(2.1)}$respectively, and the external radius of the cross section of the inner cylinder is R. We associate the cylindrical system of coordinates $Orz\theta$ with the axis of the cylinder and the quantities related to the inner (outer) cylinder we denote by the upper index (2.2) (by the upper index (1)).

Assume that in the interior of the inner hollow cylinder point located with respect to the cylinder axis and that non-uniformly distributed in the circumferential direction time-harmonic normal forces act (Fig. 1). In the present paper, within this framework we attempt to investigate the non-axisymmetric frequency response of the bi-layered hollow cylinder to these time-harmonic forces and analyze the amplitude of the interface radial normal stress.

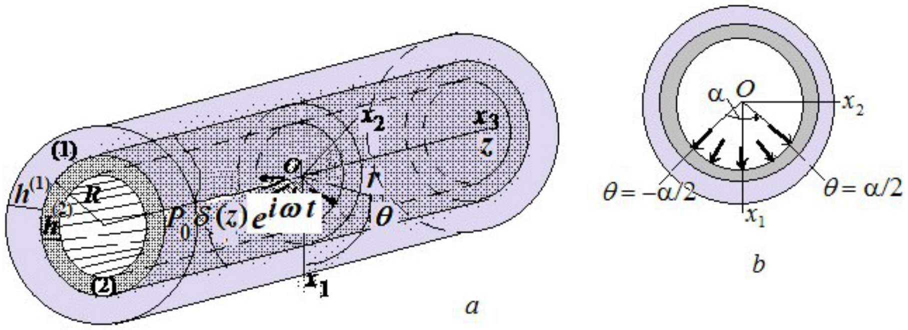

Fig. 1. The sketch of the considered system (a) and the sketch of the distribution of the non-axisymmetric normal forces (b)

This investigation we make within the scope of the following complete system of field equations of the 3D elastodynamics, as well as within the scoop the corresponding boundary and contact conditions.

Equations of motion:

$$\frac{\partial \sigma_{rr}^{(m)}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}^{(m)}}{\partial \theta}+\frac{\partial \sigma_{rz}^{(m)}}{\partial z}+\frac{1}{r}(\sigma_{rr}^{(m)}-\sigma_{\theta\theta}^{(m)})=\rho^{(m)}\frac{\partial^2 u_r^{(m)}}{\partial t^2}$$

$$\frac{\partial \sigma_{r\theta}^{(m)}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}^{(m)}}{\partial \theta}+\frac{\partial \sigma_{z\theta}^{(m)}}{\partial z}+\frac{2}{r}\sigma_{r\theta}^{(m)}=\rho^{(m)}\frac{\partial^2 u_\theta^{(m)}}{\partial t^2}$$

$$\frac{\partial \sigma_{rz}^{(m)}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{z\theta}^{(m)}}{\partial \theta}+\frac{\partial \sigma_{zz}^{(m)}}{\partial z}+\frac{1}{r}\sigma_{rz}^{(m)}=\rho^{(m)}\frac{\partial^2 u_z^{(m)}}{\partial t^2}. \quad (2.1)$$

Elasticity relations:

$$\sigma_{rr}^{(m)}=(\lambda^{(m)}+2\mu^{(m)})\frac{\partial u_r^{(m)}}{\partial r}+\lambda^{(m)}\frac{1}{r}(\frac{\partial u_\theta^{(m)}}{\partial r}+u_r^{(m)})+\lambda^{(m)}\frac{\partial u_z^{(m)}}{\partial z},$$

$$\sigma_{\theta\theta}^{(m)}=\lambda^{(m)}\frac{\partial u_r^{(m)}}{\partial r}+(\lambda^{(m)}+2\mu^{(m)})\frac{1}{r}(\frac{\partial u_\theta^{(m)}}{\partial r}+u_r^{(m)})+\lambda^{(m)}\frac{\partial u_z^{(m)}}{\partial z},$$

$$\sigma_{zz}^{(m)}=\lambda^{(m)}\frac{\partial u_r^{(m)}}{\partial r}+\lambda^{(m)}\frac{1}{r}(\frac{\partial u_\theta^{(m)}}{\partial r}+u_r^{(m)})+(\lambda^{(m)}+2\mu^{(m)})\frac{\partial u_z^{(m)}}{\partial z},\sigma_{r\theta}^{(m)}=$$

$$= \mu^{(m)} \frac{\partial u_\theta^{(m)}}{\partial r} + \mu^{(m)} (\frac{1}{r} \frac{\partial u_r^{(m)}}{\partial \theta} - \frac{1}{r} u_\theta^{(m)}),$$

$$\sigma_{z\theta}^{(m)} = \mu^{(m)} \frac{\partial u_\theta^{(m)}}{\partial z} + \mu^{(k)} \frac{\partial u_z^{(m)}}{r \partial \theta}, \sigma_{zr}^{(k)} = \mu^{(k)} \frac{\partial u_r^{(k)}}{\partial z} + \mu^{(k)} \frac{\partial u_z^{(k)}}{\partial r}. \quad (2.2)$$

In equations (2.1) and (2.2) the conventional notation of the theory of elasticity is used. Consider also formulation of the corresponding boundary and contact conditions which can be written as follows.

$$\sigma_{rr}^{(2)}\Big|_{r=R-h^{(2)}} = \begin{cases} -P_\alpha \delta(z) e^{i\omega t} \text{ for } -\alpha/2 \le \theta \le \alpha/2 \\ 0 \quad \text{for} \quad \theta \in ([-\pi, +\pi] - [-\alpha/2, \alpha/2]) \end{cases},$$

$$\sigma_{r\theta}^{(2)}\Big|_{r=R-h^{(2)}} = 0, \, \sigma_{rz}^{(2)}\Big|_{r=R-h^{(2)}} = 0,$$

$$\sigma_{rr}^{(1)}\Big|_{r=R+h^{(1)}} = 0, \, \sigma_{r\theta}^{(1)}\Big|_{r=R+h^{(1)}} = 0, \, \sigma_{rz}^{(1)}\Big|_{r=R+h^{(1)}} = 0, \quad (2.3)$$

$$\sigma_{rr}^{(1)}\Big|_{r=R} = \sigma_{rr}^{(2)}\Big|_{r=R}, \, \sigma_{r\theta}^{(1)}\Big|_{r=R} = \sigma_{r\theta}^{(2)}\Big|_{r=R}, \, \sigma_{rz}^{(1)}\Big|_{r=R} = \sigma_{rz}^{(2)}\Big|_{r=R},$$

$$u_r^{(1)}\Big|_{r=R} = u_r^{(2)}\Big|_{r=R} \, u_\theta^{(1)}\Big|_{r=R} = u_\theta^{(2)}\Big|_{r=R} \, u_z^{(1)}\Big|_{r=R} = u_z^{(2)}\Big|_{r=R}, \quad (2.4)$$

$$\left|\sigma_{rr}^{(k)}\right|; \left|\sigma_{\theta\theta}^{(k)}\right|; \left|\sigma_{zz}^{(k)}\right|; \left|\sigma_{r\theta}^{(k)}\right|; \left|\sigma_{rz}^{(k)}\right|; \left|\sigma_{\theta z}^{(k)}\right|; \left|u_r^{(k)}\right|; \left|u_\theta^{(k)}\right|; \left|u_z^{(k)}\right| \to 0,$$

$$k = 1, 2 \text{as} \, |z| \to +\infty, \quad (2.5)$$

where in (2.3) P_α is determined from the following relation

$$\int\limits_{-\alpha/2}^{+\alpha/2} P_\alpha (R-h) \cos\theta \mathrm{d}\theta = (R-h) P_0 \, = const \Rightarrow \; P_\alpha = P_0 / (2 \sin(\alpha/2)) \quad (2.6)$$

Thus, the investigation of the problem is reduced to the boundary-contact problem (2.1) – (2.6) for solution to which the method developed in the papers [1, 2] is employed. Now we consider some fragments of the application of this method for the problem under consideration.

3 Method of solution

As in the papers [1, 2] for solution to the foregoing mathematical problem, according to [5], we use the following representation:

$$u_r^{(m)} = \frac{1}{r} \frac{\partial}{\partial \theta} \Psi^{(m)} - \frac{\partial^2}{\partial r \partial z} \mathrm{X}^{(m)},$$

$$u_\theta^{(m)} = -\frac{\partial}{\partial r} \Psi^{(m)} - \frac{1}{r} \frac{\partial^2}{\partial \theta \partial z} \mathrm{X}^{(m)},$$

$$u_z^{(m)} = (\lambda^{(m)} + \mu^{(m)})^{-1} \left((\lambda^{(m)} + 2\mu^{(m)}) \Delta_1 + \, \mu^{(m)} \frac{\partial^2}{\partial z^2} - \rho^{(m)} \frac{\partial^2}{\partial t^2} \right) \mathrm{X}^{(m)},$$

$$\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, \, m = 1, 2 \quad (3.1)$$

In (3.1) the functions $\Psi^{(m)}$ and $X^{(m)}$ are the solutions of the equations

$$\left(\Delta_1+\frac{\partial^2}{\partial z^2}-\frac{\rho^{(k)}}{\mu^{(k)}}\frac{\partial^2}{\partial t^2}\right)\Psi^{(m)}=0,$$

$$\left[\left(\Delta_1+\frac{\partial^2}{\partial z^2}\right)\left(\Delta_1+\frac{\partial^2}{\partial z^2}\right)-\rho^{(m)}\frac{\lambda^{(m)}+3\mu^{(m)}}{\mu^{(m)}(\lambda^{(m)}+2\mu^{(m)})}\times\right.$$

$$\left.(\Delta_1+\frac{\partial^2}{\partial z^2})\right)\frac{\partial^2}{\partial t^2}+\frac{(\rho^{(m)})^2}{\mu^{(m)}(\lambda^{(m)}+2\mu^{(m)})}\frac{\partial^4}{\partial t^4}\Bigg]X^{(m)}=0. \tag{3.2}$$

We represent all the sought quantities with mutiplying $e^{i\omega t}$, according to which the operators $\partial^2/\,\partial t^2$ and $\partial^4/\,\partial t^4$in the foregoing equations are replaced with the constants ω^2 and ω^4, respectively, and in this way, it is obtained the corresponding field equations for the amplitudes of the sought values. After this replacing, the exponential Fourier transform $f_F=\int\limits_{-\infty}^{+\infty}f(z)e^{isz}dz$ with respect to the coordinate z (where s is a transformation parameter) is applied to all the foregoing equations and relations rewritten for the amplitudes.

Note that below we will use the same symbols for indicated amplitudes of the corresponding quantity.

Thus, according to the problem statement the originals of the sought values can be presented through their Fourier transforms by the following relations.

$$\left\{\sigma_{rr}^{(m)};\ \sigma_{\theta\theta}^{(m)};\sigma_{zz}^{(m)};\sigma_{r\theta}^{(m)};u_r^{(m)};u_\theta^{(m)};\Psi^{(m)}\right\}=$$

$$=\frac{1}{\pi}\int\limits_0^{+\infty}\left\{\sigma_{rrF}^{(m)};\ \sigma_{\theta\theta F}^{(m)};\sigma_{zzF}^{(m)};\sigma_{r\theta F}^{(m)};u_{rF}^{(m)};u_{\theta F}^{(m)};\Psi_F^{(m)}\right\}\cos(sz)ds,$$

$$\left\{\sigma_{\theta z}^{(m)};\ \sigma_{rz}^{(m)};u_z^{(m)};\ X^{(m)}\right\}=\frac{1}{\pi}\int\limits_0^{+\infty}\left\{\sigma_{\theta zF}^{(m)};\ \sigma_{rzF}^{(m)};u_{zF}^{(m)};\ X_F^{(m)}\right\}\sin(sz)ds. \tag{3.3}$$

We use the dimensionless coordinates $r'=r/h$ and $z'=z/h$ (the upper prime will be omitted below)

and introduce the notation

$$\Omega=\frac{\omega h^{(2)}}{c_2^{(2)}} \tag{3.4}$$

where $c_2^{(2.2)}=\sqrt{\mu^{(2.2)}/\rho^{(2.2)}}$and call it the dimensionless frequency.

Thus, substituting the expressions in (3.3) into the foregoing equations and relations, and taking the notation (3.4) into consideration we obtain the following equations for the functions $\Psi_F^{(m)}$ and $X_F^{(m)}$:

$$\left(\Delta_1-\left(s^2-\frac{\Omega^2(c_2^{(2)})^2}{(c_2^{(m)})^2}\right)\right)\Psi_F^{(m)}=0,\left[\left(\Delta_1-s^2\right)\left(\Delta_1-s^2\right)+\right.$$

$$\left.+\frac{\lambda^{(m)}+3\mu^{(m)}}{\lambda^{(m)}+2\mu^{(m)}}(\Delta_1-s^2))\right)\frac{\Omega^2(c_2^{(2)})^2}{(c_2^{(m)})^2}+\frac{1}{(\lambda^{(m)}/\mu^{(m)}+2)}\frac{\Omega^4(c_2^{(2)})^4}{(c_2^{(m)})^4}\Bigg]\mathrm{X}_F^{(m)}=0. \tag{3.5}$$

where $c_2^{(m)} = \sqrt{\mu^{(m)}/\rho^{(m)}}$.

According to the paper [2], the Fourier transform of the functions $\Psi_F^{(m)}$ and $X_F^{(m)}$ can be presented in the Fourier series form as follows.

$$\Psi_F^{(m)}(r,s,\theta) = \sum_{n=1}^{\infty} \Psi_{Fn}^{(m)}(r,s)\sin n\theta, X_F^{(m)}(r,s,\theta) =$$

$$= \frac{1}{2}X_{F0}^{(m)}(r,s) + \sum_{n=1}^{\infty} X_{Fn}^{(m)}(r,s)\cos n\theta. \tag{3.6}$$

Substituting expressions in (3.6) into equation (3.5), we obtain:

$$\left(\Delta_{1n} - (\zeta_1^{(m)})^2\right)\psi_{Fn}^{(m)} = 0, \left(\Delta_{1n} - (\zeta_2^{(m)})^2\right)\left(\Delta_{1n} - (\zeta_3^{(m)})^2\right)X_{Fn}^{(m)} = 0,$$

$$\Delta_{1n} = \frac{d^2}{dr^2} + \frac{d}{rdr} - \frac{n^2}{r^2}, \tag{3.7}$$

where

$$(\zeta_1^{(m)})^2 = \left(s^2 - \frac{\Omega^2 (c_2^{(2)})^2}{(c_2^{(m)})^2}\right). \tag{3.8}$$

In (3.7) $(\zeta_2^{(m)})^2$ and $(\zeta_2^{(m)})^2$ are determined as solutions of the following equation.

$$(\zeta^{(m)})^4 - (\zeta^{(m)})^2\left[-\frac{\Omega^2 (c_2^{(2)})^2}{(c_2^{(m)})^2} - s^2(\lambda^{(m)}/\mu^{(m)} + 2) + \right.$$

$$\left. + \frac{\mu^{(m)}}{\lambda^{(m)} + 2\mu^{(m)}}\left(-\frac{\Omega^2 (c_2^{(2)})^2}{(c_2^{(m)})^2} - s^2\right) + s^2\frac{(\lambda^{(m)} + \mu^{(m)})^2}{\mu^{(m)}(\lambda^{(m)} + 2\mu^{(m)})}\right] +$$

$$s^2\left(\frac{-1}{\lambda^{(m)}/\mu^{(m)} + 2}\frac{\Omega^2 (c_2^{(2)})^2}{(c_2^{(m)})^2} - 1\right)\left(-\frac{\Omega^2 (c_2^{(2)})^2}{(c_2^{(m)})^2} - s^2\right) = 0. \tag{3.9}$$

Thus, the solutions to equations in (3.7) are determined as follows:

$$\psi_{Fn}^{(m)} = A_{1n}^{(m)} H_n^{(1)}(\zeta_1^{(m)} r) + B_{1n}^{(m)} H_n^{(2)}(\zeta_1^{(m)} r), \chi_{Fn}^{(m)} =$$

$$= A_{2n}^{(m)} H_n^{(1)}(\zeta_2^{(m)} r) + A_{3n}^{(m)} H_n^{(1)}(\zeta_3^{(m)} r) +$$

$$B_{2n}^{(m)} H_n^{(1)}(\zeta_2^{(m)} r) + B_{3n}^{(m)} H_n^{(2)}(\zeta_3^{(m)} r), m = 1,2. \tag{3.10}$$

where $H_n^1(x)$ and $H_n^2(x)$ are the Hankel functions of the $n-th$ order of the first and second kinds, respectively, and the $B_{1n}^{(m)}$, $B_{2n}^{(m)}$, $B_{3n}^{(m)}$ $A_{1n}^{(m)}$, $A_{2n}^{(m)}$ and $A_{3n}^{(m)}$ in (3.10) are unknown constants which will be determined from the boundary (2.3) and contact (2.4) conditions.

In this way we determine completely the Fourier transforms of the sought values the originals of which are found numerically with employing the algorithm described in the paper [2].

This completes the consideration of the solution method more detail version of which is given in the papers [1, 2].

4 Numerical results

As noted above, the calculation algorithm used under obtaining the numerical results which will be discussed below, are detailed in the works [1 - 4, 6] and therefore here we do not consider this algorithm again. Nevertheless, we note that under obtaining numerical results we take twenty terms in the series in (3.6) and these terms are enough for obtaining convergence numerical results.

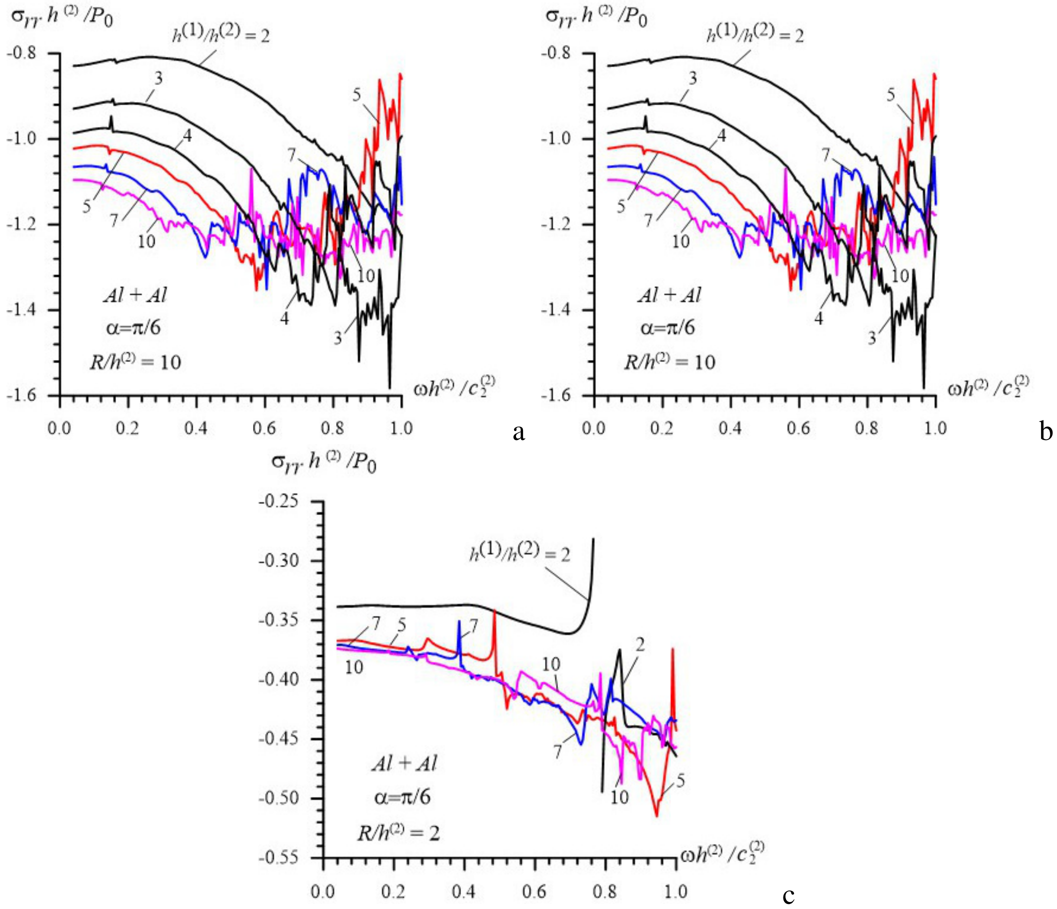

Fig. 2. The graphs of the frequency responses of the normal stress in the $Al + Al$ case constructed under $R/h^2 = 10$(a), 5 (b) and 2 (c) for various values of the ratio h^1/h^2

Assume that the material of the inner cylinder is steel (St), however, the material of the outer cylinder is aluminium (Al). According to the monograph by Guz [7], the material density, modulus of elasticity, Poisson's coefficients and shear wave propagation velocity of the St (Al) we select as $\rho_{St} = 7795\, kg/m^3$, $E_{St} = 19.6\, GPa$, $\nu_{St} = 0.27$ and $c_{2St} = 3152 m/s$ ($\rho_{Al} = 2770\, kg/m^3$,$E_{Al} = 7.28\, GPa \nu_{Al} = 0.30$ and $c_{2Al} = 3179 m/s$), respectively.

Assume that $\theta = 0$, $z/h = 0$ and $\alpha = \pi/30$, and consider frequency response of the interface normal stress $\sigma_{rr}(R) = \sigma^2_{rr,St}(R) = \sigma^1_{rr,Al}(R)$ obtained for various values of the ratios R/h^2 and $h^1/h^1h^2h^2$. Note that the main parameter which characterizes the difference of the present results from the corresponding ones obtained in the paper [2] is the ratio $h^1/h^1h^2h^2$, therefore, in the present analyses of the numerical results, the attention is focused on the influence of this ration on the mentioned frequency response.

First, we assume that the materials of the cylindrical layers are the same and these materials are aluminium. Consider the graphs given in Fig. 2 which illustrate the frequency response of the interface normal stress in the cases where $R/h^2 = 10$(Fig. 2a), 5 (Fig. 2b) and 2 (Fig. 2c) under various values of the ratio h^1/h^2. It follows from these graphs that a decrease in the values of the ratio R/h^2causes to decrease in the absolute values of the normal stress under consideration. As well as it follows from the results that an increase in the values of the ratio h^1/h^2causes to increase of the absolute values of the mentioned stress. In all the considered cases the numerical results approach to each other with increasing the raio h^1/h^2 and this situation agrees with the well-known physico-mechanical considerations.

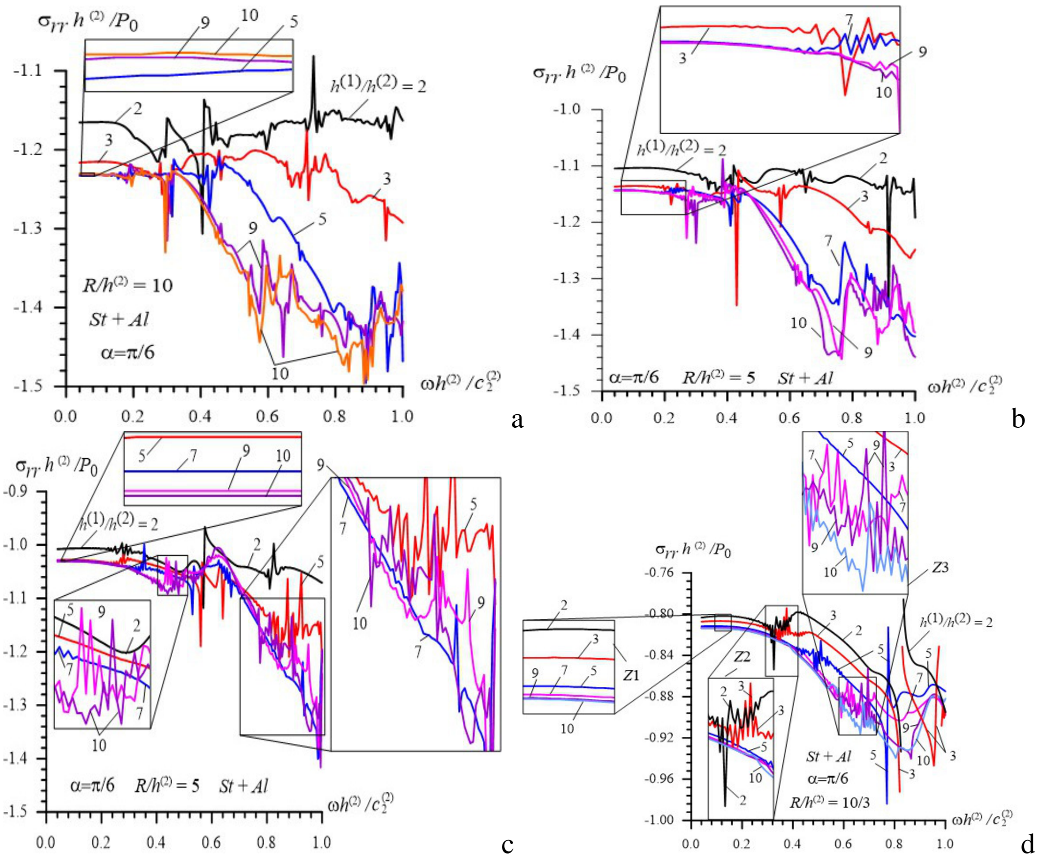

Fig. 3. The graphs of the frequency responses of the normal stress in the *St+Al* case constructed under $R/h^2 = 10$(a), 7 (b), 5(c) and 10/3 (d) for various values of the ratio h^1/h^2

Now we consider the numerical results obtained for the bi-layered $St + Al$ cylinder, i.e. consider the case where the material of the inner layer of the cylinder is St, however the material of the outer layer is Al. Analyse the graphs given in Fig. 3 which illustrate the frequency response of the radial normal stress acting on the interface surface between the layers in the cases where $R/h^2 = 10$(Fig. 3a), 7 (Fig. 3b), 5(Fig. 3c) and 10/3 (Fig. 3d). These results show that an increase in the values of the ratio R/h^2, in general, causes fo increase the absolute values of the stress under consideration. As well as, these results show that an increase in the values of the ratio h^1/h^1 causes also, in generally, to increase the absolute values of the stress under consideration.

Note that in all the foregoing results it is observed jumping in the values of the stress and the “density” of these jumps increase with the frequency of the external forces. It is known that these jumping is characteristic one for the considered type problems and is caused by the reflection of the waves from the interface and from the outer and inner free surfaces of the cylinder. Moreover, these jumps can also correspond to the resonance cases of the considered bi-layered cylinder.

5 Conclusions

Thus, in the present paper, the 3D dynamic problem forced vibration of the bi-layered hollow cylinder caused by the time-harmonic load acting in the interior of this cylinder is studied with employing 3D exact equations of elastodynamics. It is assumed that the forces acting in the interior of the inner layer of the cylinder is point located with respect to the axial coordinate and is distributed along a certain arc within the corresponding central angle. The corresponding mathematical problem is solved by employing the Fourier transform with respect to the axial coordinate and by employing the Fourier series presentation with respect to the circumferential coordinate of the Fourier transforms of the sought values. The coefficients of these series are unknown functions with respect to the radial coordinates the analytical expressions for which are determined through the solution of the corresponding equations. The originals of the Fourier transforms are found numerically. Numerical results on the frequency response of the interface radial normal stress are presented and discussed. It is established that a decrease of the thickness of the external layer of the cylinder causes to decrease of the absolute values of the mensioned stress. Moreover, it is established that the magnitude of the stress become more considerable with an increase of the external radius of the inner layer cross section.

References

1. Akbarov, S. D., Mehdiyev, M.A. and Ozisik, M.: Three-dimensional dynamics of the moving load acting on the interior of the hollow cylinder surrounded by the elastic medium. *Structural Engineering and Mechanics,* 67(2), (2018), 185-206
2. Akbarov S.D. and Mehdiyev M.A.: The interface stress field in the elastic system consisting of the hollow cylinder and surrounding elastic medium under 3D non-axisymmetric forced vibration, *CMC: Computers, Materials & Continua* 54(1),(2018), 61 81.
3. Ozisik M., Mehdiyev M.A. and Akbarov S.D.: The influence of the imperfectness of contact conditions on the critical velocity of the moving load acting in the interior of the cylinder surrounded with elastic medium, *CMC: Computers, Materials & Continua* 54(2), (2018), 103-136,
4. Akbarov S.D. and Mehdiyev M.A. *Influence of initial stresses on the critical velocity of the moving load acting in the interior of the hollow cylinder surrounded by an infinite elastic medium*, Struct Eng Mech 66(1), 45-59, (2018).
5. Guz A.N. Fundamentals of the three-dimensional theory of stability of deformable bodies, *Springer, Berlin,* (1999).
6. Akbarov S.D. Dynamics of pre-strained bi-material elastic systems: Linearized three-dimensional approach. Springer. (2015)
7. Guz, A.N. Elastic waves in bodies with initial (residual) stresses. A.C.K., Kiev.(2004)