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Axisymmetrical deformations of non-uniform rotation bodies about an axis

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Abstract. *The article examines the groups of equations related to axisymmetrical deformation of rotational bodies, and as a result, two systems of differential equations are obtained together with the equation of equilibrium and the equation of uniformity of deformations As an example, the elastic modulus of the material and the differential equation of the stress function taking into account the inhomogeneity of the Poisson's coefficient are obtained for the tension of a circular cylinder by the force of a central ball. Using this equation, the expressions of the normal voltages of the circular cylinder were obtained.*

Keywords. rotational bodies · deformation · circular cylinder · the stress function · normal stress.

Mathematics Subject Classification (2010): 74A10, 74B10

1 Introduction.

The calculation of elastic bodies and structures taking into account their non-uniformity properties are one of the important issues solved by the theory of elastic bodies with solid heterogeneity. Studies on the finding of parameters that determine the presence of random heterogeneity and play an important role in the processes of fracture, fatigue breakdown, characterizing the beating of stresses and deformations have a significant impact on the problems of rigidity and reliability. To achieve the solution of the intended problem, let's turn to the basic equations of the theory of elasticity in the system of cylindrical coordinates. The rotational axis "z", $n_0 = 0$ around it is perceived as the r-functions of the Yong module E and Poisson coefficient θ, the mass forces are not affected, they depend on the angle of displacement, deformation and stress θ, from the surface forces.

It is not intended to be observed with torsion because of the issues related to deformation of symmetrical bodies from the axis.

The systems of relevant equations related to the issue are as follows:

Equilibrium equations:

$$\left.\begin{array}{c} \frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r-\sigma_\theta}{r} + \frac{\partial\tau_{rz}}{\partial z} = 0, \\ \frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r-\sigma_\theta}{r} + \frac{\partial\tau_{rz}}{\partial z} = 0, \\ \frac{\partial\tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + \frac{\partial\sigma_z}{\partial z} = 0 \end{array}\right\} \tag{1.1}$$

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Cauchy's dependence

$$\left.\begin{array}{c}\varepsilon_r = \frac{\partial u}{\partial r}, \varepsilon_\theta = \frac{u}{r}, \varepsilon_z = \frac{\partial w}{\partial z}, \\ \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\end{array}\right\} \quad (1.2)$$

Equation of uniformity of deformations

$$\left.\begin{array}{c}\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varepsilon_\theta}{\partial r}\right) - \frac{\partial \varepsilon_r}{\partial r} = 0 \\ \frac{\partial^2 \varepsilon_r}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial r^2} - \frac{\partial^2 \gamma_{rz}}{\partial r \partial z} = 0, \\ \frac{\partial^2 \varepsilon_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial \varepsilon_z}{\partial r} - \frac{1}{r}\frac{\partial \gamma_{rz}}{\partial z} = 0 \\ r\frac{\partial}{\partial z}\left[\varepsilon_r - \frac{\partial (r \cdot \varepsilon_\theta)}{\partial r}\right] = 0.\end{array}\right\} \quad (1.3)$$

Hooke's Law (in linear form)

$$\left.\begin{array}{c}varepsilon_r = \frac{1}{E}\left[\sigma_r - \mu\left(\sigma_\theta + \sigma_z\right)\right] \\ \varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \mu\left(\sigma_z + \sigma_r\right)\right] \\ \varepsilon_z = \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_r + \sigma_\theta\right)\right]\end{array}\right\} \quad (1.4)$$

$$\left.\begin{array}{c}\sigma_r = \lambda\theta + 2G\varepsilon_r, \\ \sigma_\theta = \lambda\theta + 2G\varepsilon_\theta, \\ \sigma_z = \lambda\theta + 2G\varepsilon_z\end{array}\right\} \quad (1.5)$$

(1.3) we integrate the end of equations:

$$r\frac{\partial \varepsilon_\theta}{\partial r} + \varepsilon_\theta - \varepsilon_r = f(r) \quad (1.6)$$

Differentiating the last equation on r, we present it as

$$r\frac{\partial^2 \varepsilon_\theta}{\partial r^2} + 2\frac{\partial \varepsilon_\theta}{\partial r} - \frac{\partial \varepsilon_r}{\partial r} = f'(r) \quad (1.7)$$

From the first equation of (1.3) we obtain:

$$r\frac{\partial^2 \varepsilon_\theta}{\partial r^2} + 2\frac{\partial \varepsilon_\theta}{\partial r} - \frac{\partial \varepsilon_r}{\partial r} = 0 \quad (1.8)$$

We compare equation (1.8) with equation (1.7) and get: $f'(r) = 0$ and $f(r) = c = Const$ and determine that these equations are reduced to one equation:

$$r\frac{\partial \varepsilon_\theta}{\partial r} + \varepsilon_\theta - \varepsilon_r = c$$

multiplying the third equation of (1.3) on r, we pass the first line to the right:

$$\frac{\partial \varepsilon_z}{\partial r} - \frac{\partial \gamma_{rz}}{\partial z} = -r\frac{\partial^2 \varepsilon_\theta}{\partial z^2}$$

Differentiated this equation according to r:

$$\frac{\partial^2 \varepsilon_z}{\partial r^2} - \frac{\partial^2 \gamma_{rz}}{\partial r \partial z} = -\frac{\partial^2 \varepsilon_\theta}{\partial z^2} - r\frac{\partial^3 \varepsilon_\theta}{\partial r \partial z^2} \quad (1.9)$$

We get from the second equation from (1.3):

$$\frac{\partial^2 \varepsilon_z}{\partial r^2} - \frac{\partial^2 \gamma_{rz}}{\partial r \partial z} = -\frac{\partial^2 \varepsilon_r}{\partial z^2} \quad (1.10)$$

We make the right sides equal, since the left sides of the equation (1.9) and (1.10) are the same:

$$\frac{\partial^2 \varepsilon_\theta}{\partial z^2} - \frac{\partial^3 \varepsilon_\theta}{\partial r \partial z^2} = \frac{\partial^2 \varepsilon_r}{\partial z^2},$$

or

$$\frac{\partial^2}{\partial z^2}\left(\varepsilon_\theta - \varepsilon_r + r\frac{\partial \varepsilon_\theta}{\partial r}\right) = 0$$

Thus, equations (1.3) are reduced to the following two equations:

$$\left.\begin{array}{c} \frac{\partial \varepsilon_z}{\partial r} + r\frac{\partial^2 \varepsilon_\theta}{\partial z^2} = \frac{\partial \gamma_{rz}}{\partial z}, \\ r\frac{\partial \varepsilon_\theta}{\partial r} + \varepsilon_\theta - \varepsilon_r = c. \end{array}\right\}, \tag{1.11}$$

Taking into account the Hook's law (1.4), we will obtain an expression of deformations with stresses of the connection conditions from equation (1.11):

$$\left.\begin{array}{c} \frac{\partial}{\partial r}\left\{\frac{1}{E}\left[\sigma_\theta - \mu\left(\sigma_z + \sigma_r\right)\right]\right\} + \frac{1+\mu}{E}\left(\sigma_\theta - \sigma_r\right) = c, \\ \frac{\partial}{\partial r}\left\{\frac{1}{E}\left[\sigma_z - \mu\left(\sigma_r + \sigma_\theta\right)\right]\right\} + r\frac{\partial^2}{\partial z^2}\left\{\frac{1}{E}\left[\sigma_\theta - \mu\left(\sigma_z + \sigma_r\right)\right]\right\} - 2\frac{\partial}{\partial z}\left(\frac{1+\mu}{E}\tau_{rz}\right) = 0 \end{array}\right\}, \tag{1.12}$$

In order to determine the stress $(\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz})$, it is necessary to take the functions of R and z variables that satisfy balance equations (1.1), deformation equations (1.12) and the following boundary conditions:

$$\sigma_r \cdot n_r + \tau_{rz} n_z = q_r, \tau_{rz} n_r + \sigma_z n_z = q_z \tag{1.13}$$

To achieve this, we include the following two functions $\psi(r, z)$ and $\phi(r, z)$:

$$\sigma_r \cdot n_r + \tau_{rz} n_z = q_r, \tau_{rz} n_r + \sigma_z n_z = q_z \tag{1.14}$$

The functions (1.14) satisfy the compatibility equations, and by writing these functions instead of deformations in equation (1.12), we obtain the following equations for the functions ψ and ϕ:

$$\left.\begin{array}{c} \frac{\partial}{\partial r}\left\{\frac{1}{E}\left[\frac{\partial \psi}{\partial r} - \mu\frac{\psi}{r} - r\frac{\partial^2 \phi}{\partial z^2} - \frac{\mu}{r}\cdot\frac{\partial}{\partial r}\left(r\phi\right)\right]\right\} + \\ \frac{(1+\mu)r}{E}\left[\frac{\partial}{\partial r}\left(\frac{\psi}{r}\right) - \frac{\partial^2 \phi}{\partial z^2}\right] = c \\ \frac{\partial}{\partial r}\left\{\frac{1}{E}\left[\frac{\partial}{\partial r}\left(r\phi\right) + \mu r\frac{\partial^2 \phi}{\partial z^2} - \frac{\mu}{r}\cdot\frac{\partial}{\partial r}\left(r\phi\right)\right]\right\} + r\frac{\partial^2}{\partial z^2}\left\{\frac{1}{E}\left[r\frac{\partial^2 \phi}{\partial z^2} + \right.\right. \\ \left.\left.\frac{\mu}{r}\cdot\frac{\partial}{\partial r}\left(r\phi\right) - \frac{\partial \psi}{\partial r} + \mu\frac{\psi}{r}\right]\right\} + 2\frac{\partial}{\partial z}\left[\frac{1+\mu}{E}\frac{\partial \phi}{\partial z}\right] = 0 \end{array}\right\} \tag{1.15}$$

(1.14) the boundary conditions are as follows:

$$\frac{\psi}{r}n_r - \frac{\partial \phi}{\partial z}n_z = q_r; -\frac{\partial \phi}{\partial z}n_r + \frac{1}{r}\frac{\partial}{\partial r}\left(r\phi\right)\cdot n_z = q_z. \tag{1.16}$$

As an example, consider the question of pulling a circular cylinder with a force F, radius a, which is not affected by external forces $(q_r = q_z = 0)$, on the entire side surface of the face. Since the side surface of the cylinder is not affected by external forces, as well as $n_z = 0$, $n_r = 1$, we can write:

$$\sigma_{r|r=a} = 0, \tau_{rz|r=a} = 0 \tag{1.17}$$

In the question under consideration the boundary on the end segments of the shaft can be written as follows:

$$2\pi\int_0^a r\sigma_z dr = F \tag{1.18}$$

Taking the inhomogeneity of the shaft as functions $E(r)$ and $\mu(r)$, we look for the solution of equations (1.16) as the following functions:

$$\psi = \psi(r), \phi = \phi(r) \tag{1.19}$$

given the conditions (1.18) ÷ (1.20), equations (1.16) can be written as:

$$r\left\{\frac{1}{E}\left[\psi' - \mu\frac{\psi}{r} - \frac{\mu}{r}\cdot(r\phi)'\right]\right\}' + \frac{(1+\mu)\,r}{E}\left[\left(\frac{\psi}{r}\right)'\right] = c, \tag{1.20}$$

$$\left\{\frac{1}{E}\left[\frac{1}{r}(r\phi)' - \frac{\mu}{r}(r\psi)'\right]\right\}' = 0 \tag{1.21}$$

The last and subsequent expressions specify the operation of differentiation by r with a stroke. Taking into account (1.15) and (1.18) we can write:

$$\left\{\frac{1}{E}\left[\frac{1}{r}(r\phi)' - \frac{\mu}{r}(r\psi)'\right]\right\}' = 0 \tag{1.22}$$

integrated equation (1.22) we get:

$$\tau_{rz} = 0, \sigma_r = \frac{\psi}{r}, \sigma_\theta = \psi', \sigma_z = \frac{1}{r}(r\phi)' \tag{1.23}$$

here γ is a constant.

We get the differential equation for the stress function by finding $(r\phi)^{'}$ from (1.23) and writing $\psi(r)$ instead of $(r\phi)^{'}$ in (1.21) and (that equation accepts $c = 0$):

$$\psi^{''} + \left[1 + E\left(\frac{1}{E}\right)^{'} r - \frac{2\mu\mu^{'} r}{1-\mu^2}\right]\frac{1}{r}\psi' -$$

$$-\left[1 + E\left(\frac{1}{E}\right)^{'}\frac{\mu(1+\mu)}{1-\mu^2}r + \frac{\mu^{'}(1+2\mu)}{1-\mu^2}r\right]\frac{1}{r^2}\psi = \frac{\gamma E\mu^{'}}{1-\mu^2} \tag{1.24}$$

(1.23) stresses are expressed by the function of $\psi(r)$:

$$\left.\begin{array}{c} \sigma_r = \frac{1}{r}\cdot\psi,\ \sigma_\theta = \psi', \\ \sigma_z = \gamma E + \mu\cdot\left(\psi' + \frac{1}{r}\psi\right), \\ \tau_{rz} = \tau_{r\theta} = \tau_{\theta z} = 0 \end{array}\right\} \tag{1.25}$$

If the Poisson coefficient is not constant, then from the equation (1.25) it turns out that $\psi \neq 0$, and the volume of the tensile state of the cylinder will be (triple).

We can apply the small parameters method to the solution of the issue. It should be noted that the elasticity module and Poisson coefficient are adopted in the form of functions dependent on r:

$$E = E_o[1 + \eta\cdot\lambda(r)], \mu = \mu_o[1 + \eta\cdot\xi(r)] \tag{1.26}$$

As can be seen from the last expression, when $\mu = \mu_o$, $\psi = 0$ and the function $\psi(r)$ will be in the form of the following array:

$$\psi(r) = \eta\cdot\psi_o(r) + \eta^2\cdot\psi_1(r) + ... \tag{1.27}$$

We write (1.26) and (1.27) instead of (1.24) in the equation:

$$\psi_o'' + \frac{1}{r}\psi_o' - \frac{1}{r^2}\psi_o = \frac{\gamma E_o \mu_o'}{1-\mu_o}\xi'(r),$$

from here

$$\psi_o = c_1 r + \frac{c_2}{r} + \frac{\gamma E_o \mu_o}{1-\mu_o^2}\frac{1}{r}\int_0^r r\cdot\xi(r)\,dr$$

In order to be satisfied with the first limit of (1.27), and assuming that $n > 0$ and we define the stress expression. In this case, we can write:

$$\psi = \eta\psi_o = \eta\left(c_1 r + \frac{c_2}{r} + \frac{\gamma E_o \mu_o}{1-\mu_o^2}\cdot\frac{r^{n+1}}{n+2} + 0\left(\eta^2\right)\right)$$

When r=0 σ_r is limited in the value of r=0, in order to overcome uncertainty, we take $c_2 = 0$ in the last expression, while c_1 is set from the first condition (1.17). Then we get:

$$\psi = \frac{\eta\gamma E_o \mu_o}{\left(1-\mu_o^2\right)(n+2)} r\left(r^n - a^n\right) + 0\left(\eta^2\right) \tag{1.28}$$

Equation (1.18) we get from the expression:
$F = 2\pi\int\limits_0^a r\sigma_z dr = 2\pi\gamma\int\limits_0^a rEdr + 0\left(\eta^2\right)$ and from and from this expression we get:

$$\gamma = \frac{F}{\pi a^2 E_o} + 0(\eta) \tag{1.29}$$

Taking into account the function ψ in (1.25) and writing (1.29) instead of (1.28), we get the expressions of the stress with the accuracy of the limits proportional to η:

$$\left.\begin{array}{r} \sigma_r = \frac{\eta F}{\pi a^2}\frac{\mu_o}{\left(1-\mu_o^2\right)}\cdot\frac{1}{n+2}\left(r^n - a^n\right), \\ \sigma_\theta = \frac{\eta F}{\pi a^2}\frac{\mu_o}{\left(1-\mu_o\right)}\cdot\frac{1}{n+2}\left[(n+1)r^n - a^n\right], \\ \sigma_z = \frac{F}{\pi a^2}\left[\frac{E}{E_o} + \frac{\eta\mu_o^2}{\left(1-\mu_o^2\right)}\left[(n+2)r^n - 2a^n\right]\right]. \end{array}\right\} \tag{1.30}$$

The law of stress change (1.30) is shown in Fig. 1 (by taking $\mu_0 = \frac{1}{3}$).

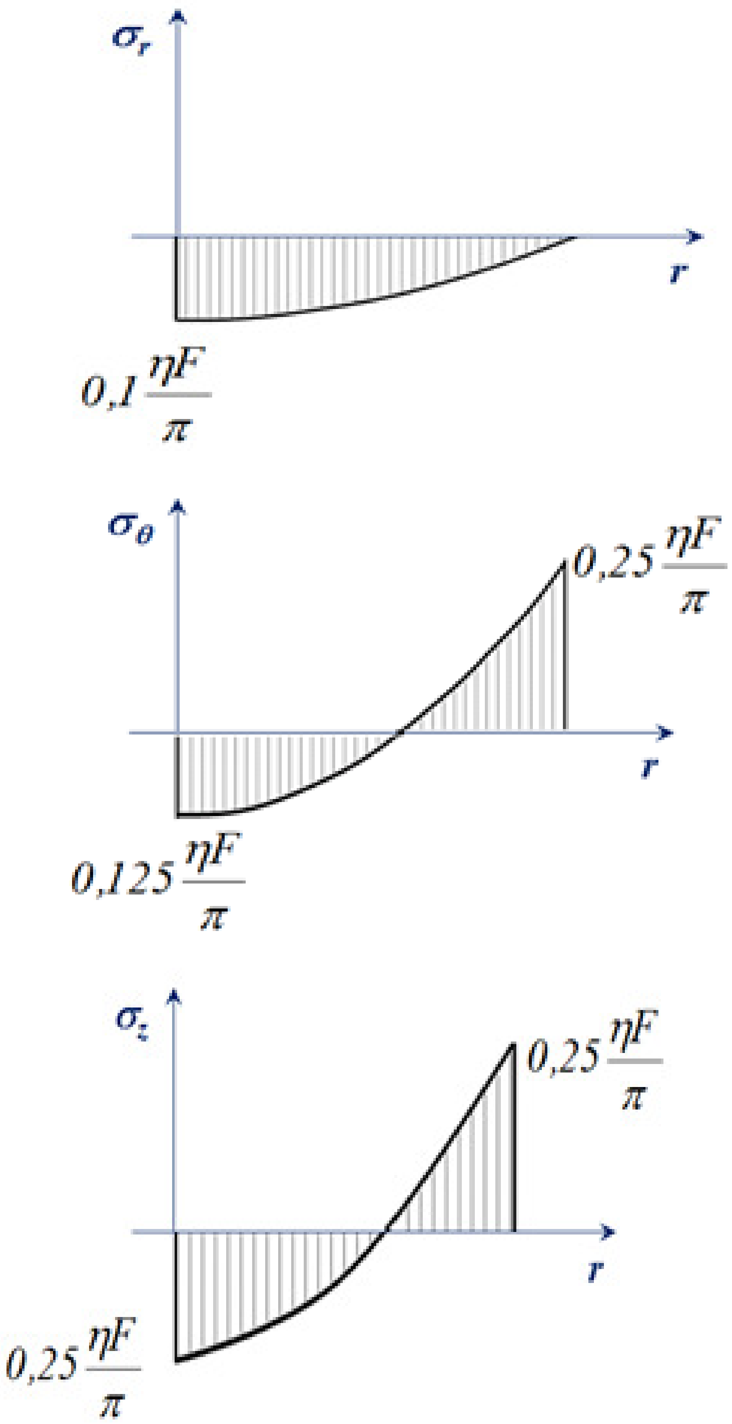

Fig. 1

References

1. Bergman S. Integralniye operatory v teorii linejnyh uravnenij s chastnymi proizvodnymi. M., Mir, 1964.

2. Lekhnickij S.G. Elementarnye resheniya dvuh chastnyh zadach o rav-novesii neodnorodnogo cilindra. Sb. Issledovaniya po uprugosti i plastichnosti, *vyp. 6, Izd–vo Leningr. un–ta,* 1967.
3. Lomakin V.A. *K teorii uprugosti neodnorodnyh tel.*, Izd-vo Mosk. un–ta, 1976, 368 .
4. . Lur'e A.I. *Teoriya uprugosti,* M., Nauka, 1970.
5. Nazarov G.I. *Tochnoe reshenie uravnenij gazovoj dinamiki.* Izv. A. N. SSSR MZhG, 3, 1968.
6. Nazarov G.I., Puchkov A.A. K zadache o kruchenii neodnorodnogo tela vrashcheniya s peremennymi modulyami sdviga. *Prikladnaya ma-tematika i mekhanika,* 9, vyp. 3, 1973.
7. Plevako V.P. K teorii uprugosti neodnorodnyh sred. *Priklad-naya matematika i mekhanika,* 35, vyp. 5, 1971.
8. Rostovcev N.A. K teorii uprugosti neodnorodnoj sredy. *Prikladnaya matematika i mekhanika,* 28, vyp. 4, 1964.
9. Sadigov I.R. Ob odnom metode raschyota ortotropnyh gibkih pologih obolochek s uchyotom fizicheskoj nelinejnosti materiala. *Vestnik Azerbajdzhanskoj Inzhenernoj Akademii,* 2, (3), Baku, 2010.