Tran. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mechanics, **44** (7), 41–49 (2024);
https://doi.org/10.30546/2706-7734.44.7.2024.45

Lamb's problem for a non-homogeneous half-plane

Nazile B. Rassoulova · Tahmine M. Mahmudzade

Received: 10.03.2024 / Revised:12.05.2024 / Accepted: 27.06.2024

Abstract. *The Lamb problem is solved for a half-plane with an upper layer made of another elastic material. The problem is investigated analytically, a solution is found that describes the initial stage of the process, which corresponds to the time of action of the impact load. A new wave is discovered, which represents the propagation velocity of the main energy in the elastic layers.*

Keywords. Lamb's problem, layer, half-plane, Laplace transform

Mathematics Subject Classification (2010): 74J05, 74B05

1 Introduction

The Lamb problem, posed in 1904, was intended to study and model the wave processes accompanying earthquakes. It is known - and every geologist can confirm - that the upper layer of the earth's crust, in any parameters and properties, differs sharply from the lower layers. Therefore, for more accurate modelling, it is necessary to study wave processes during an earthquake in layered media, rather than in homogeneous ones.

For this purpose, a similar Lamb problem is considered here for a half-plane on which a layer of another elastic material lies. For a homogeneous half-plane, the problem was solved by V.V. Poruchikov, in a rather complex way [2]. Then, in [3], after the discovery of a new homogeneous solution of the wave equation, a compact solution of the same problem is given.

2 Statement and solution of the problem

Let a concentrated impulse force act on the boundary of an elastic layer of thickness b, lying on the surface of a half-plane made of another elastic material, which is at rest at $t < 0$, at

N.B. Rassoulova
Institute of Mathematics and Mechanics
E-mail: rasulova@gmail.com,

T.M. Mahmudzade
Baku State University,Baku, Azerbaijan
E-mail: tehminemahmudzade1996@gmail.com

$t = 0$:

$$\sigma_{yy}\left(t,x,y\right) = -I\delta\left(x\right)\delta\left(t\right), \quad y = 0 \quad \left(I = const\right) \tag{2.1}$$

The system of equations of the boundary and initial conditions of the motion of the medium has the following form in potentials Φ and Ψ:

$$\begin{gathered} \rho\frac{\partial U}{\partial t^2} = \left(\lambda+\mu\right)\mathrm{graddiv}\overrightarrow{U} \;\; + \mu\Delta\overrightarrow{U} \\ \overrightarrow{U} = grad\Phi + rot\left(\Psi\overline{e_3}\right) \\ \overrightarrow{U} = \overrightarrow{U}\left(u_1,u_2\right) \end{gathered} \tag{2.2}$$

We can write the expression of potentials for each medium as follows:

$$\begin{gathered} \Phi^{*(1)} = A_1 e^{y\upsilon_1^{(1)}} + D_1 e^{-y\upsilon_1^{(1)}} \\ \Psi^{*(1)} = B_1 e^{y\upsilon_2^{(1)}} + E_1 e^{-y\upsilon_2^{(1)}} \\ \Phi^{*(2)} = A_2 e^{y?_1^{(2)}} \\ \Psi^{*(2)} = B_2 e^{y?_2^{(2)}} \end{gathered} \tag{2.3}$$

where $\upsilon_i^{(j)} = \frac{\sqrt{p^2+c_i^{(j)2}q^2}}{c_i^{(j)2}}$. Accordingly, the boundary conditions will be as follows:

$$\begin{gathered} u_1 = u_2\,;\sigma_{yy}^{(1)} = \sigma_{yy}^{(2)} \text{ for } y=-b \\ v_1 = v_2;\; \sigma_{xy}^{(1)} = \sigma_{xy}^{(2)} \\ \sigma_{yy}^{(1)} = -I\delta(x)\delta(t); \quad \sigma_{xy}^{(1)} = 0 \text{ for } y=0 \end{gathered} \tag{2.4}$$

where $I-$ is the impulse force. Let us note the formulas for the components of the displacement vector associated with the potentials Φ and Ψ:

$$u = \frac{\partial\Phi}{\partial x} + \frac{\partial\Psi}{\partial y}\;;\; v = \frac{\partial\Phi}{\partial y} - \frac{\partial\Psi}{\partial x} \tag{2.5}$$

Applying the Laplace and Fourier integral transforms to these formulas, we can write for each medium:

$$\begin{gathered} \tilde{u}_1 = -q\Phi^{*(1)} + \frac{d\Psi^{*(1)}}{dy}; \quad \tilde{v}_1 = \frac{d\Phi^{*(1)}}{dy} - q\Psi^{*(1)} \\ \tilde{u}_2 = -q\Phi^{*(2)} + \frac{d\Psi^{*(2)}}{dy}; \quad \tilde{v}_2 = \frac{d\Phi^{*(2)}}{dy} - q\Psi^{*(2)} \end{gathered} \tag{2.6}$$

Let us also note the formulas for the relationship between stresses and displacements as follows:

$$\begin{gathered} \sigma_{yy}^{(1)} = \left(\lambda_1+2\mu_1\right)\frac{\partial v_1}{\partial y} + \lambda_1\frac{\partial u_1}{\partial x}; \quad \sigma_{xy}^{(1)} = \mu_1\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right) \\ \sigma_{yy}^{(2)} = \left(\lambda_2+2\mu_2\right)\frac{\partial v_2}{\partial y} + \lambda_2\frac{\partial u_2}{\partial x}; \quad \sigma_{xy}^{(2)} = \mu_2\left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}\right) \end{gathered} \tag{2.7}$$

Applying integral transformations to these expressions, we obtain:

$$\sigma_{yy}^{*(1)} = \left(\lambda_1+2\mu_1\right)\frac{\partial\overline{v_1}}{\partial y} + \lambda_1 q\overline{u_1}; \quad \sigma_{xy}^{*(1)} = \mu_1\left(\frac{\partial\overline{u_1}}{\partial y} - q\overline{v_1}\right)$$

$$\sigma_{yy}^{*(2)} = (\lambda_2 + 2\mu_2)\frac{\partial \overline{\upsilon_2}}{\partial y} + \mu_2 q\overline{u_2}; \quad \sigma_{xy}^{*(2)} = \mu_2\left(\frac{\partial \overline{u_2}}{\partial y} - q\overline{\upsilon_2}\right) \tag{2.8}$$

$$\begin{gathered} u_1 = -qA_1 e^{y\upsilon_1^{(1)}} - qD_1 e^{-y\upsilon_1^{(1)}} + \upsilon_2^{(1)} B_1 e^{y\upsilon_2^{(1)}} - \upsilon_2^{(1)} E_1 e^{-y\upsilon_2^{(1)}} \\ \upsilon_1 = \upsilon_1^{(1)} A_1 e^{y\upsilon_1^{(1)}} - \upsilon_1^{(1)} D_1 e^{-y\upsilon_1^{(1)}} - qB_1 e^{y\upsilon_2^{(1)}} - qE_1 e^{-y\upsilon_2^{(1)}} \\ u_2 = -qA_2 e^{y\upsilon_1^{(2)}} + \upsilon_2^{(2)} B_2 e^{y\upsilon_2^{(2)}} \\ \upsilon_2 = \upsilon_1^{(2)} A_2 e^{y\upsilon_1^{(2)}} - qB_2 e^{y\upsilon_2^{(2)}} \end{gathered} \tag{2.9}$$

Substituting the obtained expressions into the corresponding boundary conditions (2.4), after possing this equations to images we obtain a system of linear equations with respect to six constants: $A_1,\ B_1 \ldots\ldots\ B_2$; ;

$$\left\{\begin{matrix} A_1 \\ D_1 \\ B_1 \\ E_1 \\ A_2 \\ B_2 \end{matrix}\right\} \times [[a_{ik}]] = \left\{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ -I \\ 0 \end{matrix}\right\} \tag{2.10}$$

$$\begin{gathered} a_{11} = -qe^{-b\upsilon_1^{(1)}} \\ a_{12} = -qe^{b\upsilon_1^{(1)}} \\ a_{13} = \upsilon_2^{(1)} e^{-b\upsilon_2^{(1)}} \\ a_{14} = -\upsilon_2^{(1)} e^{b\upsilon_1^{(1)}} \\ a_{15} = qe^{-b\upsilon_1^{(2)}} \\ a_{16} = -\upsilon_2^{(2)} e^{-b\upsilon_2^{(2)}} \end{gathered}$$

$$\begin{gathered} a_{21} = \upsilon_1^{(1)} e^{-b\upsilon_1^{(1)}} \\ a_{22} = -\upsilon_1^{(1)} e^{b\upsilon_1^{(1)}} \\ a_{23} = -qe^{-b\upsilon_2^{(1)}} \\ a_{24} = -qe^{b\upsilon_2^{(1)}} \\ a_{25} = -\upsilon_1^{(2)} e^{-b\upsilon_1^{(2)}} \\ a_{26} = qe^{-b\upsilon_2^{(2)}} \end{gathered}$$

$$\begin{gathered} a_{31} = \left[(\lambda_1 + 2\mu_1)\,{\upsilon_1^{(1)}}^2 - \lambda_1 q^2\right] e^{-b\upsilon_1^{(1)}} \\ a_{32} = \left[(\lambda_1 + 2\mu_1)\,{\upsilon_1^{(1)}}^2 - \lambda_1 q^2\right] e^{b\upsilon_1^{(1)}} \\ a_{33} = -2\mu_1 q\upsilon_2^{(1)} e^{-b\upsilon_2^{(1)}} \\ a_{34} = -2\mu_1 q\upsilon_2^{(1)} e^{b\upsilon_2^{(1)}} \end{gathered}$$

$$a_{35} = -\left[\left(\lambda_2 + 2\mu_2\right) \upsilon_1^{(2)^2} - \lambda_2 q^2\right] e^{-b\upsilon_1^{(2)}}$$

$$a_{36} = 2\mu_2 q \upsilon_2^{(2)} e^{-b\upsilon_2^{(2)}}$$

$$a_{41} = -2\mu_1 q \upsilon_1^{(1)} e^{-b\upsilon_1^{(1)}}$$

$$a_{42} = 2\mu_1 q \upsilon_1^{(1)} e^{b\upsilon_1^{(1)}}$$

$$a_{43} = \mu_1 \left(q^2 + \upsilon_2^{(1)^2}\right) e^{-b\upsilon_2^{(1)}}$$

$$a_{44} = \mu_1 \left(q^2 + \upsilon_2^{(1)^2}\right) e^{b\upsilon_2^{(1)}}$$

$$a_{45} = 2\mu_2 q \upsilon_1^{(2)} e^{-b\upsilon_1^{(2)}}$$

$$a_{46} = -\mu_2 \left(q^2 + \upsilon_2^{(2)^2}\right) e^{-b\upsilon_2^{(2)}}$$

$$a_{51} = \left[\left(\lambda_1 + 2\mu_1\right) \upsilon_1^{(1)^2} - \lambda_1 q^2\right]$$

$$a_{52} = \left[\left(\lambda_1 + 2\mu_1\right) \upsilon_1^{(1)^2} - \lambda_1 q^2\right]$$

$$a_{53} = -2\mu_1 q \upsilon_2^{(1)}$$

$$a_{54} = 2\mu_1 q \upsilon_2^{(1)}$$

$$a_{55} = 0$$

$$a_{56} = 0$$

$$a_{61} = -2\mu_1 q \upsilon_1^{(1)}$$

$$a_{62} = 2\mu_1 q \upsilon_1^{(1)}$$

$$a_{63} = \mu_1 \left(q^2 + \upsilon_2^{(1)^2}\right)$$

$$a_{64} = \mu_1 \left(q^2 + \upsilon_2^{(1)^2}\right)$$

$$a_{65} = 0$$

$$a_{66} = 0$$

Thus, we obtain a highly complex problem for determining the originals of the sought quantities; since their double transformations will now be presented through 6th-rank determinants.

Here it is appropriate to apply the method that was first used in [4], namely, the method that, in the case when images are presented through high-rank determinants, allows us to more accurately separate from the general solution that part that corresponds to the initial stage of the process. In that work, the solution found exactly coincided with the known data observed in nature during earthquakes.

By skipping intermediate calculations, we can provide ready-made expressions for double transformations for the sought coefficients, and, subsequently, for the layer displacement functions.

$$A_1 = \frac{F_1}{F} = \frac{I \cdot a_{63}}{[-a_{51}a_{63} + a_{61}a_{53}]}$$

$$B_1 = \frac{F_3}{F} = \frac{-I \cdot a_{61}}{[-a_{51}a_{63} + a_{61}a_{53}]}$$

These constants in formulas (2.9) determine the field of longitudinal and transverse waves running from the source in the layer.

The values of the remaining four constants are given below; they have a more complex form:

$$D_1 = \frac{F_2}{F} = \frac{-a_{63}\left[a_{31}a_{25} - a_{35}a_{21}\right] e^{-2v_1^{(1)}b}}{[-a_{51}a_{63} + a_{61}a_{53}]\left[a_{35}a_{22} - a_{32}a_{25}\right]}$$

$$E_1 = \frac{F_4}{F} = \frac{a_{63}\left[a_{35}a_{22} - a_{32}a_{25}\right]\left[a_{11}a_{46} - a_{16}a_{41}\right] + \left[a_{31}a_{22} - a_{32}a_{21}\right] \times}{[-a_{51}a_{63} + a_{61}a_{53}]\left[a_{35}a_{22} - a_{32}a_{25}\right]\left[a_{44}a_{16} - a_{14}a_{46}\right]}$$

$$\frac{\times\left[a_{45}a_{16} - a_{15}a_{46}\right] + \left[a_{31}a_{25} - a_{35}a_{21}\right]\left[a_{12}a_{46} - a_{16}a_{42}\right]}{\cdots} e^{-2v_2^{(1)}b}$$

$$A_2 = \frac{F_5}{F} = \frac{a_{63}\left[a_{31}a_{22} - a_{32}a_{21}\right] e^{-v_1^{(1)}b}}{[-a_{51}a_{63} + a_{61}a_{53}]\left[a_{35}a_{22} - a_{32}a_{25}\right]}$$

$$B_2 = \frac{F_6}{F} = \frac{a_{63}\left[a_{32}a_{25} - a_{35}a_{22}\right]\left[a_{11}a_{44} - a_{14}a_{41}\right] + \left[a_{31}a_{22} - a_{32}a_{21}\right] \times}{[-a_{51}a_{63} + a_{61}a_{53}]\left[a_{35}a_{22} - a_{32}a_{25}\right]\left[a_{44}a_{16} - a_{14}a_{46}\right]}$$

$$\frac{\times\left[a_{44}a_{15} - a_{14}a_{45}\right] + \left[a_{31}a_{25} - a_{35}a_{21}\right]\left[a_{14}a_{42} - a_{44}a_{12}\right]}{\cdots} e^{-v_1^{(1)}b}$$

Now we will plot the distribution graphs of normal displacements on the layer surface for successive time values.

First, we will define the dimensionless displacement field for a longitudinal traveling wave in the transformations:

$$\tilde{v}^{(1)} = -I \cdot \frac{q}{\left[(\lambda_1 + 2\mu_1)\, v_1^{(1)^2} - \lambda_1 q^2\right]} = -I \cdot \frac{\left(c_1^{(1)^2} - 2c_2^{(1)^2}\right) q}{\lambda_1 \left(p^2 + 2c_2^{(1)^2} q^2\right)} \tag{2.11}$$

Calculations will now and in the future be performed for the following values of quantities included in the formulas of all sought quantities:

$$v_1 = v_2 = \frac{1}{3}\ , \quad \frac{E_1}{E_2} = 1,8\ , \quad \frac{\rho_1}{\rho_2} = 1,5$$

$$v^{(1)} = -I \frac{c_1}{\lambda_1 \cdot b} \int_0^{100} \left(q \cdot \cos\left(q \cdot z\right)\right) \int_y^t 0.18 \cdot \left(\sin\left(0.7q\left(t - t\right)\right)\right) J_0\left(q\sqrt{t^2 - y^2}\right) dtdq$$

Note that to derive this formula, well-known methods of inversion for the Laplace transform, tables given in [1] were used, and graphs will be given everywhere for dimensionless quantities, in particular for displacements $\overline{v} = v^{(1)} \Big/ \frac{Ic_1}{\lambda_1 b}\ , \quad \bar{t} = t \cdot \frac{c_1^{(1)}}{b}$

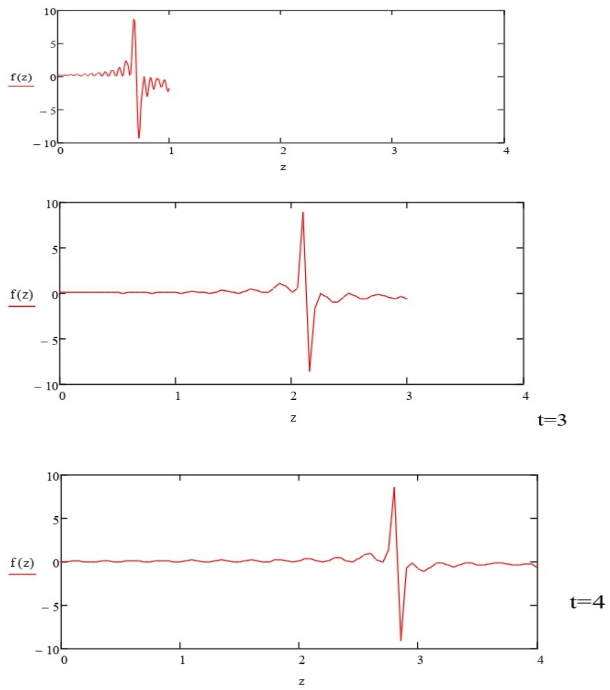


Fig. 1. Graph of the distribution of normal displacements of a longitudinal wave for successive time values.

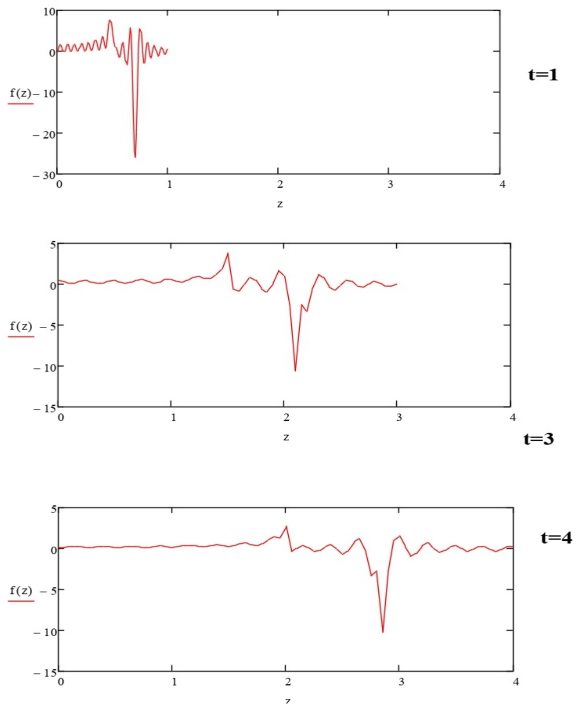


Fig. 2. Graphs of the distribution of normal displacements of a transverse wave for successive time values.

Similarly, formulas for a transverse wave are derived:

$$v^{(2)} = \int_0^{100} (q \cdot \cos(q \cdot z)) \int_{2y}^{t} 0.268 \cdot \sin(0.7q(t-t)) \, J_0 \left(0.5 \times \ q\sqrt{t^2 - (2y)^2}\right) dtdq \tag{2.12}$$

The same picture is observed for longitudinal displacements.

Judging by these graphs, we can say that a very interesting result has been obtained: the speed of propagation of the solution discontinuity is the same for both waves, i.e., and it can represent nothing other than the speed of propagation of the main energy in the elastic layers. Judging by formulas (2.11), (2.12), the speed of this wave is $c_{raz.} = \sqrt{2}c_2^1$.

In Lamb's own problem, where a half-plane is considered, the discontinuities of both waves propagate with the speed of each wave separately [2,3]. It should be noted that we also encountered a wave of the same speed $\sqrt{2} \cdot C_2^{(1)}$ in works [4,5], where the object of study was also an elastic layer but subjected to longitudinal impacts on its end. The present work was set up precisely to once again verify the existence of this wave. Now we can confidently state that a new wave with a velocity of $\sqrt{\frac{2\mu}{\rho}}$ has been discovered, representing the propagation velocity of the main energy in elastic layers, as well as a wave with a velocity of $\sqrt{\frac{E}{\rho}}$ in rods, with the same property.

At the end, we present the total value of the normal velocity on the surface of the layer of all four waves, to confirm the presence of the Rayleigh wave.

$$\alpha_1 = -0.015, \alpha_2 = 0.006, \ \alpha_3 = 0.089, \ \alpha_4 = 0.081, \ \alpha_5 = 0.043,$$

$$\alpha_6 = -0.188, \ \beta_1 = 0.7, \ \beta_1 = 0.485, \ \beta_3 = 0.679, \ \beta_4 = 0.97, \ \beta_5 = 0.7, \ \beta_6 = 0.7$$

$$\gamma = 1.33, \ \eta = 1.37, \ \gamma_0 = 0.8, \ \eta_0 = 0.58$$

$$f(z) = \int_0^{100} (q \cdot \cos(q \cdot z)) \int_{2\cdot(2-y)}^{t} \left(\sum_{k=1}^{6} (\alpha_k \cdot \sin(\beta_k \cdot q(t-t))) \right) \cdot J_0 (0.5 \times$$

$$\times \ q\sqrt{t^2 - (2(2-y))^2}\Big) dtdq$$

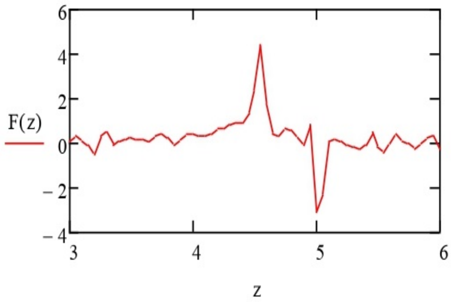
$$g(z) = \int_0^{100} (q \cdot \cos(q \cdot z)) \int_{2y}^{t} 0.268 \cdot \sin(0.7q(t-t)) \, J_0 \left(0.5 \times \ q\sqrt{t^2 - (2y)^2}\right) dtdq$$

$$h(z) = 0.125 \cdot \int_0^{100} (q \cdot \cos(q \cdot z)) \int_{(2-y)}^{t} \left(\frac{\left(\gamma_0^2 - \frac{\gamma^2}{\eta^2} \cdot \eta_0^2\right)\left(4 - 2 \cdot \frac{\gamma^2}{\eta^2}\right)}{\sqrt{2}\gamma\eta\left(1 - \frac{\gamma^2}{\eta^2}\right)} \times \right.$$

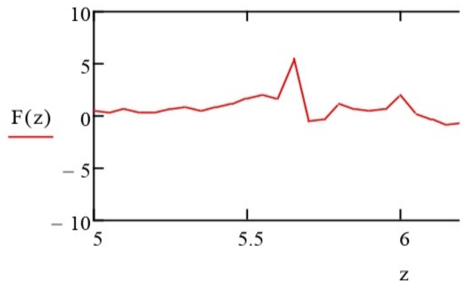
$$\left. \times \sin\left(0.7q\frac{\gamma}{\eta}(t-\tau)\right) + \frac{2\left(\gamma_0^2 - 1\right)\sin(0.7q(t-\tau))}{\sqrt{2(\eta^2 - \gamma^2)}} \right) J_0\left(q\sqrt{\tau^2 - (2-y)^2}\right)\Big) d\tau dq$$

$$k(z) = 0.18 \int_0^{100} (q \cdot \cos(q \cdot z)) \int_{y}^{t} (\sin(0.7q(t-\tau)))J_0\left(q\sqrt{\tau^2 - y^2}\right)\Big) d\tau dq$$

$$F(z) = -f(z) + g(z) - h(z) + k(z)$$



t=10



t=12

Fig. 3. Rayleigh wave, propagating behind the front of a transverse wave

The graphs clearly show the Rayleigh wave formed on the layer surface, but only in the transverse displacement components. Undoubtedly, considering the longitudinal displacement will make the appearance of this wave smoother. Note that the speed of this wave exactly corresponds to the speed of the Rayleigh wave calculated using the practical Bergman-Viktorov formula:

$$c_R = \frac{0.87 + 1.12\upsilon}{1+\upsilon} c_2^{(1)}.$$

3 CONCLUSION

The most complex Lamb problem for the "half-plane + layer" system with different elastic materials was studied analytically.

A solution was obtained in formulas for the initial stage of the process, which is valid during the period of action of the shock pulse. Graphs of the main characteristic quantities for successive time values are given.

A completely new wave was discovered, which represents the speed of propagation of the main energy in elastic layers.

References

1. Ditkin V.A., Prudnikov A.P. *Integral Transformations and Operational Calculus 2nd edition, suppl*. 1974. 544 p.
2. Poruchikov, V. B. *Methods of Dynamic Theory of Elasticity,* Monograph Poruchikov VB -M .: Nauka, 1986. –328 p.
3. Rasulova N.B., Rasulov M.B., *New Class of Homogeneous Solutions to Plane Problems of Elastodynamics* PMM. 2020. Vol. 84. Issue 1. Pp. 85–90.

4. Rassoulova Nazila B., Effect of the Features of the Earth's Crust on Wave Processes during Earthquake, *Tran. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mechanics,* 41 (7), 7–13 (2021).
5. Rassoulova Nazila B., On Dynamics of Bar of Rectangular Cross Section, J. Appl. Mech. Jul 2001, 68(4): p.662-666