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Waves in an elastic tube with flowing liquid in the case of recording stiffness environment

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Abstract. *We consider the one-dimensional problem of a pulsating flow of an ideal incompressible fluid in an elastic thin-walled tube, when the interface between the tube and the environment given the elastic friction, depending on the longitudinal coordinate. In the particular case revealed the influence of the dimensionless parameter stiffness on wave characteristics.*

Keywords. external rigidity · a regular problem of Sturm-Liouville · sign Weierstrass · speed of a wave · attenuation.

Mathematics Subject Classification (2010): 76D55

1 Introduction

The problem of wave propagation in deformable tubes with fluid flowing in a cavity is of interest in several aspects. In the theoretical aspect, this is a task of mathematical physics, and in the applied aspect, it is a necessary stage in the calculation of a system subject to dynamic action. To date, the totality of such problems, the specifics of the study of which are laid down in the fundamental works of L. Euler, I. Gromeka, N. Zhukovsky, constitutes a widely developed area of hydroelasticity [1], [2], [3], [4]. However, several features associated with considering the influence of environmental rigidity has not been studied enough. In this regard, the present work is devoted to the formulation and mathematical justification of the one-dimensional problem of the propagation of small-amplitude waves in an ideal fluid flowing in an elastic tube of constant circular cross section, when the rigidity of the external medium is a function of the longitudinal coordinate. The stated problem leads to the solution of the regular Sturm-Liouville boundary value problem. In the special case of constant stiffness, formulas are obtained for determining the wave velocity and attenuation.

1. As in most branches of applied mathematics, before considering the characteristic properties of the system, such as the viscosity of the liquid and the tube, it is necessary to analyze a simple model. Therefore, we establish here the equations describing the flow of an

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ideal incompressible fluid in a thin-walled elastic tube of radius R, length l and h thickness. We write the hydrodynamic equations of motion and continuity in the form [1].

$$\begin{array}{l} \frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \\ \frac{\partial u}{\partial x} + \frac{2}{R}\frac{\partial w}{\partial x} = 0. \end{array} \tag{1.1}$$

Where $p\left(x,t\right)-$ hydrodynamic pressure, $u\left(x,t\right)-$ longitudinal fluid flow velocity, $\rho-$ density, $w\left(x,t\right)-$ radial displacement of the wall, and $x \in \left[0,l\right]-$ longitudinal coordinate.This approach assumes the fulfilment of the primacy $R << 1$. Neglecting the inertia of the wall, we write

$$\sigma = \frac{Eh}{\left(1-\upsilon^2\right)R^2}w$$

Here $E-$ is the modulus of elasticity, $\upsilon-$ Poisson's ratio. Further, we assume that the value of s consists of two types of stresses: hydrodynamic and stress arising assuming that the external environment introduces additional rigidity $-G\frac{\partial w}{\partial t}$. From here $\sigma = p - G\frac{\partial w}{\partial t}$. Taking $G = G\left(x\right) = G_0 g\left(x\right)$, we finally have

$$p = \frac{Eh}{\left(1-\upsilon^2\right)}w + G_0 g\left(x\right)\frac{\partial w}{\partial t} \tag{1.2}$$

Thus, the closed system of hydroelasticity equations is described by relations (1.1) and (1.2). In the following reasoning, we will assume that the functions appearing in them have the smoothness conditions necessary for the legality of carrying out the corresponding operations. Now, combining equations (1.1), (1.2) and introducing the notation

$$c_0^2 = \frac{E}{2\rho}\,,\ \ \eta = \frac{h}{R}\left(1-\upsilon^2\right)^{-1}$$

we arrive at an equation for w:

$$c_0^2\eta\frac{\partial^2 w}{\partial x^2} + \frac{RG_0}{2\rho}g''\left(x\right)\frac{\partial w}{\partial t} + \frac{RG_0}{\rho}g'\left(x\right)\frac{\partial^2 w}{\partial x\partial t} + \frac{RG_0}{2\rho}g\left(x\right)\frac{\partial^3 w}{\partial x^2\partial t} - \frac{\partial^2 w}{\partial t^2} = 0 \tag{1.3}$$

Hence, for $g\left(x\right) = 1$, the case of homogeneous external rigidity is realized.

2. Applying the method of separation of variables, we reduce the solution of equation (1.3) to the solution of an ordinary differential equation. Assuming the function and proportional to the time factor $\exp\left(iwt\right)$ one can write:

$$w = w_1\left(x\right)\exp\left(i\omega t\right) \tag{1.4}$$

where $w-$ is the actual value of the angular frequency, and w_1- is the desired function. Substituting representation (1.4) into (1.3) leads to the equation

$$w_1'' + \mu_1\left(x\right)w_1' + \mu_2\left(x\right)w_1 = 0 \tag{1.5}$$

Here the designations

$$\mu_1\left(x\right) = \frac{2ig_0 g'(x)}{\eta + ig_0 g(x)}$$

$$\mu_2\left(x\right) = \frac{\omega^2/c_0^2 + ig_0 g''(x)}{\eta + ig_0 g(x)}\ ,$$

where $g_0 = \omega R\frac{G_0}{E}-$ dimensionless stiffness parameter. The primes here and below mean differentiation with respect to x using the Liouville change [5]

$$y = w_1 \exp \frac{1}{2} \int \mu_1 (x) \, dx = \lambda (x) \, w_1 \tag{1.6}$$

We obtain the reduced form of the wave equation

$$y'' + I (x) \, y = 0 \tag{1.7}$$

$I (x) -$ with an invariant equal to

$$I (x) = \mu_2 (x) - \frac{1}{4} \mu_1^2 (x) - \frac{1}{2} \mu_1' (x) \tag{1.8}$$

Transforming equation (1.7) using the substitution $q (x) = 1 - \frac{I(x)}{\delta^2}$as the equation of the problem, we obtain

$$y'' + \delta^2 y = \delta^2 q (x) \, y \tag{1.9}$$

We impose the integrability condition on potential (1.4)

$$\int_0^l |q (x) \, dx < +\infty| \tag{1.10}$$

In (1.9), denotes the value at $I (x) -$ Therefore, according to (1.8) $I (x) -$ value at . Therefore, according to (1.8)

$$\delta^2 = \frac{\omega^2}{c_0^2} (\eta + i g_0)^{-1}$$

From here, according to the rule of extracting the square root of a complex number $\delta = \delta_0 - i \delta_1$, one can obtain

$$\begin{aligned} \delta_0 &= \frac{\omega / c_0 \sqrt{\eta}}{\sqrt{\eta^2 + g_0^2}} \sqrt{\frac{\sqrt{1 + g_0^2 / \eta^2} + 1}{2}} \\ \delta_1 &= \frac{\omega / c_0 \sqrt{\eta}}{\sqrt{\eta^2 + g_0^2}} \sqrt{\frac{\sqrt{1 + g_0^2 / \eta^2} - 1}{2}} \end{aligned} \tag{1.11}$$

Equation (1.9) must be supplemented with boundary conditions for $x = 0$ and $x = l$ the formulation of which will be given below. Thus, the solution of the stated problem was reduced to the solution of a regular boundary value problem of the Sturm-Liouville type.

3. Considering (1.9) as an inhomogeneous equation with a known right-hand side and applying the method of variation of arbitrary constants, we reduce equation (1.9) to the solution of the integral equation

$$y = \alpha_1 e^{i \delta x} + \alpha_2 e^{-i \delta x} + \delta \int_x^l \sin \delta (\xi - x) \, q (\xi) \, y (\xi) \, d\xi \tag{1.12}$$

where from $\alpha_1, \alpha_2 -$are constants of integration, which are to be determined based on the corresponding boundary conditions. Equation (1.12) is a Volterra integral equation and is solved by the method of successive approximations. Let's put $y_0 = \alpha_1 e^{i \delta x} + \alpha_2 e^{-i \delta x}$ and let for $n \geq 1$ $y_n = y_0 + \delta \int_x^l \sin \delta (\xi - x) \, q (\xi) \, y_{n-1} (\xi) \, d\xi$. By virtue of inequality (1.10), according to the Weierstrass test [6], [7], it follows from the uniform convergence of successive approximations that the only solution of the integral equation (1.12), which we denote by $y(x, d)$, is determined by the series

$$y (x, \delta) = \alpha_1 e^{i \delta x} + \alpha_2 e^{-i \delta x} + \sum_{n=1}^{\infty} \delta^n f_n (x, \delta) \tag{1.13}$$

In (1.13) we have the set of the following recurrence relations:

$$f_1(x,\delta) = \int_x^l \sin\delta(\xi - x)\, q(\xi)\, y_0(\xi)\, d\xi\,, ...,$$
$$f_n(x,\delta) = \int_x^l \sin\delta(\xi - x)\, q(\xi)\, y_{n-1}(\xi)\, d\xi \tag{1.14}$$

So, based on the above, it can be argued that all solutions of equation (1.9) for any α_1 and α_2 satisfy equation (1.12). A direct verification can show the opposite, namely, that any solution (1.12) with a second-order derivative satisfies equation (1.9) for arbitrary α_1 and α_2. It should be noted that from the structure of the series in (1.13) it follows that the series obtained by its term-by-term differentiation also converge uniformly. Substituting expression (1.13) into (1.6), considering (1.4), leads to the following formula for ??

$$w(x,t) = \frac{1}{\lambda(x)}\left\{\alpha_1 e^{i\delta x} + \alpha_2 e^{-i\delta x} + \sum_{n=1}^{\infty} \delta^n f_n(x,\delta)\right\}\exp(i\omega t)\,. \tag{1.15}$$

Considering this expression in equality (1.2), we find

$$p(x,t) = \frac{E}{R}\{\eta + ig_0 g(x)\}\frac{1}{\lambda(x)}\left\{\alpha_1 e^{i\delta x} + \alpha_2 e^{-i\delta x} + \sum_{n=1}^{\infty} \delta^n f_n(x,\delta)\right\}\exp(i\omega t)\,. \tag{1.16}$$

For the subsequent description of pressure, fluid velocity and displacement, which is related to the definition of α_1 and α_2, one can accept various boundary conditions for $x = 0$,$x = l$. A typical case is the situation in which a pulsating pressure is set at the ends of the tube. Let

$$p(0,t) = p_0 \exp(i\omega t)\ \ ,\ \ p(l,t) = p_l \exp(i\omega t) \tag{1.17}$$

where p_0, p_1—are empirical constants. Now let's move on to the definition of α_1 and α_2. By directly comparing formulas (1.16) and (1.17) with respect to them, we obtain the following system of algebraic equations:

$$\alpha_1 + \alpha_2 = a$$
$$\alpha_1 e^{i\delta l} + \alpha_2 e^{-i\delta l} = b$$

It uses the designations

$$a = \frac{R\lambda(0)}{E(\eta + ig_0 g(0))}p_0 - \sum_{n=1}^{\infty} \delta^n f_n(0)$$
$$b = \frac{R\lambda(0)}{E(\eta + ig_0 g(l))}p_l - \sum_{n=1}^{\infty} \delta^n f_n(l) \tag{1.18}$$

Now, defining α_1 and α_2 using the Euler formula, after rather cumbersome calculations from (1.15) and (1.16) we get:

$$p(x,t) = \frac{E}{R}\{\eta + ig_0 g(x)\}\frac{1}{\lambda(x)}\left\{\frac{a\sin\delta(l-x) + b\sin\delta x}{\sin\delta l} + \sum_{n=1}^{\infty} \delta^n f_n(x,\delta)\right\}\exp(i\omega t) \tag{1.19}$$

$$w(x,t) = \frac{1}{\lambda(x)}\left\{\frac{a\sin\delta(l-x) + b\sin\delta x}{\sin\delta l} + \sum_{n=1}^{\infty} \delta^n f_n(x,\delta)\right\}\exp(i\omega t)\,. \tag{1.20}$$

To determine the flow rate, we will carry out the following reasoning. As above, separating the variables, we write

$$u(x,t) = U(x)\exp(i\omega t)$$

and denote the function $\Phi(x)$

$$\Phi(x)=\{\eta+ig_0g(x)\}\frac{1}{\lambda(x)}\left\{\frac{a\sin\delta(l-x)+b\sin\delta x}{\sin\delta l}+\sum_{n=1}^{\infty}\delta^n f_n(x,\delta)\right\}.$$

Then from the first equation of system (1.1) one can immediately obtain an expression for the velocity distribution. It looks like

$$u(x,t)=2i\frac{c_0^2}{\omega R}\Phi'(x)\exp(i\omega t). \tag{1.21}$$

By formulas (1.19)-(1.21), the problem can be considered solved. Thus, read (1.13) in combination with relations (1.14) gives a constructive representation of the desired solution. Note that the physical quantity is represented by the real parts of expressions (19-21). Let us emphasize that their practical implementation is associated with a specific assignment of the function $g(x)$ and the use of a computer.

4. The solution to the problem when the rigidity of the environment is constant, i.e. $g(x)=1$, and the pressure at the right end of the tube is equal to zero $p_l=0$, due to the obvious equalities $\lambda(x)=1$, $q(x)=0$, under which

$$b=0\,,\ \ a=\frac{p_0}{E}\frac{R}{\eta+ig_0}\,,\ \ \Phi(x)=\frac{p_0}{E}\frac{\sin\delta(l-x)}{\sin\delta l}$$

is greatly simplified and can be written as follows:

$$\begin{array}{l} p(x,t)=p_0\frac{\sin\delta(l-x)}{\sin\delta l}\exp(i\omega t)\ ,\quad w(x,t)=\frac{p_0}{E}\frac{R}{\eta-ig_0}\frac{\sin\delta(l-x)}{\sin\delta l}\exp(i\omega t)\ ,\\ u(x,t)=-2i\frac{p_0}{E}\delta\frac{c_0^2}{\omega}\frac{\cos\delta(l-x)}{\sin\delta l}\exp(i\omega t)\end{array} \tag{1.22}$$

Before considering the influence of taking into account the effect of environmental rigidity on the wave speed $c=\omega/\delta_0$and damping δ_1, we rewrite dependences (1.11) as follows

$$\begin{array}{l} \frac{c}{c_0}=\sqrt{\eta\left(1+\frac{g_0^2}{\eta^2}\right)}\left\{\sqrt{\frac{\sqrt{1+g_0^2/\eta^2}+1}{2}}\right\}^{-1}\\ \delta_1\frac{c_0}{\omega}=\sqrt{\frac{1}{\eta\left(1+g_0^2/\eta^2\right)}}\sqrt{\frac{\sqrt{1+g_0^2/\eta^2}-1}{2}}\end{array} \tag{1.23}$$

Hence, for $g_0/\eta<<1$ we have

$$\frac{c}{c_0}\approx\sqrt{\eta}\ ,\quad \delta_1\frac{c_0}{\omega}\approx 0$$

Taking $g_0/\eta>>1$, we get $\frac{c}{c_0}\approx\sqrt{2g_0}\ ,\quad \delta_1\frac{c_0}{\omega}\sim\frac{1}{\sqrt{2g_0}}$

Note that the results of subsequent numerical calculations are consistent with the above statements.

Now, assuming that corresponds to the value $h/R=5\cdot10^{-2},\ \upsilon=0,5$, in accordance with formulas (1.23) we present the numerical dependences of $\frac{c}{c_0}$, $\delta_1\frac{c_0}{\omega}$, on for$g_0\in[0,1]$, which are given in

Figures 1 and 2.

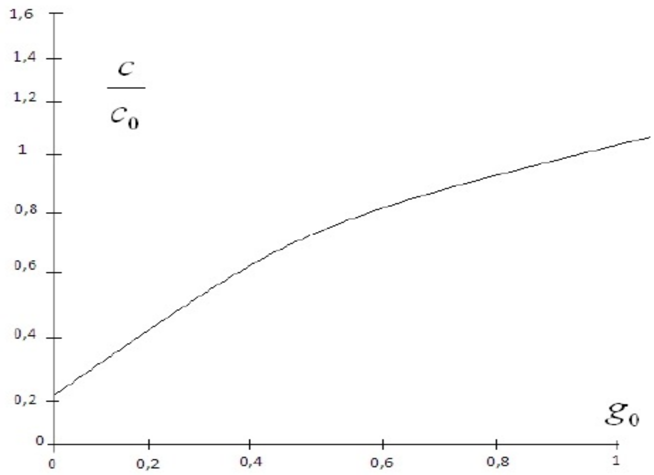

Fig. 1. Dependence of the wave propagation velocity on the stiffness parameter.

It follows that with an increase in the stiffness parameter, the rate of water propagation in the system increases and for the selected range g_0, approaches the value of 1.37. According to the data of the second figure, we can conclude that for the received η there is a value $g_0 \approx 0,25$, when passing through which the attenuation decreases.

Thus, we have obtained a rigorous analytical solution to the problem of a wave flow of an ideal incompressible fluid in an elastic tube, considering the inhomogeneity of the medium. In a particular case of homogeneity, the influence of the stiffness parameter on the wave characteristics of the system is analysed. It is shown that, depending on g_0, the attenuation may decrease.

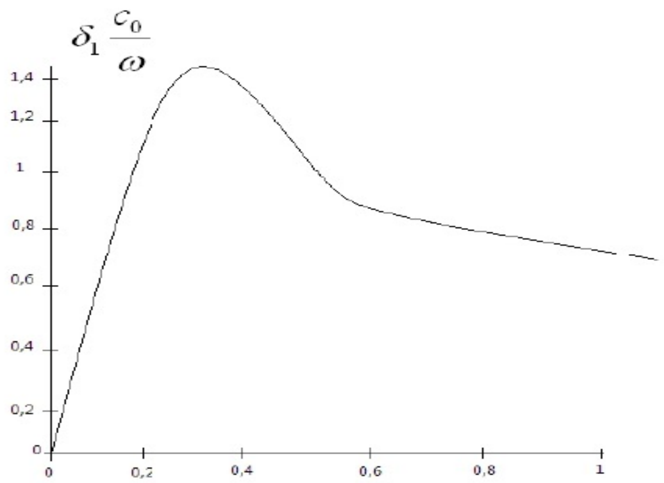

Fig. 2. Dependence of damping on the stiffness parameter.

References

1. Volmir. A.S., Shells in fluid and gas flows. A.S. Volmir.- M. Nauka, 1979, - 320 p.
2. Hydrodynamics of blood circulation: ed. By S.A. Regirer.-M.: Mir, 1971. 269 p.
3. Pedley, T.J. *The fluid mechanics of large blood vessels.* T.J. Pedley M.: Mir, 1983.- 400 p.

4. Laightfoot, E. *Transport phenomena in living systems.* E. Laightfoot. M. Mir, 1977. 520 p.
5. Kamke, E. The handbook of the ordinary differential equations. E. Kamke.-M. Nauka, 1971-576 p.
6. Naimark, M.A. *The linear differential operators* M.A. Naimark. M.: Nauka, 1969.526 p.
7. Amenzadeh, R.Yu. Waves in an inhomogeneous tube with a flowing two-phase fluid/ R.Yu. Amenzadeh. G.A. Gurbanov, F.B. Nagiyev, *Mechanics of composite materials.* 39(4) (2003), 555-566.