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The mixed boundary value problems for uniformly elliptic equations in modified local generalized Sobolev-Morrey spaces

Shahla M. Galandarova

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Abstract. *We consider problems which arise as mathematical models of various applied problems, mechanics, physics and so on. By using these results we prove the solvability of the mixed boundary value problem for a polyharmonic equation in modified local generalized Sobolev-Morrey spaces. We obtain a priori estimates for the solutions of the mixed boundary value problems for the uniformly elliptic equations in modified local generalized Sobolev-Morrey spaces defined on bounded smooth domains.*

Keywords. solvability of the mixed problem, modified local generalized Sobolev- Morrey spaces, a priori estimates, uniformly elliptic equations.

Mathematics Subject Classification (2010): 35J35

1 Introduction

We consider problems which arise as mathematical models of various applied problems, mechanics, physics and so on. For instance reaction-drift-diffusion processes of electrically charged species phase transition processes in porous media.

The estimates for the solutions mixed boundary value problem for the biharmonic equations in generalized Morrey spaces are obtained. The better inclusion between the Morrey and Holder spaces permits to obtain regularity of the solution to boundary problems.

Let $\Omega \subset R^n, n \geq 2,\ \partial\Omega = \Gamma_1 \cup \Gamma_2$.

Definition 1.1 *The generalized Sobolev- Morrey space $W_{m,\rho,\varphi}(\Omega)$consists of all Sobolev functions $u \in W_{m,\rho}(\Omega)$ with distributional derivatives $D^s u \in M_{p,\varphi}(\Omega)$, endowed with the norm*

$$\|u\|_{W_{m,\rho,\varphi}(\Omega)} = \sum_{0\leq s\leq m} \|D^s u\|_{M_{p,\varphi}(\Omega)}.$$

Shahla M. Galandarova
Azerbaijan State University of Economics (UNEC) Baku, Azerbaijan
E-mail: shahlaqalandarova@gmail.com

The space $W_{m,p,\varphi}(\Omega)\bigcap W_{1,p,0}(\Omega)$ consist of all functions $u\in W_{m,p}(\Omega)\bigcap W_{1,p,0}\ (\Omega)$ with $D^s u\in M_{p,\varphi}\Omega$ and endowed with the same norm. $W_{1,p,0}(\Omega)$ in the closure functions of $C^\infty(\Omega)$ vanishing on Γ_1, with respect to the norm in $W_p^1\ (\Omega)$.

We consider mixed boundary value problem for biharmonic equation.

$$\Delta^2 u=f \text{ in } \Omega \tag{1.1}$$

$$u\,|_{\Gamma_1}=\frac{\partial u}{\partial n}\,|_{\Gamma_1}=0 \text{ and } \frac{\partial^2 u}{\partial n^2}\,|_{\Gamma_2}=g \tag{1.2}$$

The classical Morrey spaces $L_{p,\lambda}$ are originally introduced to study the local behavior of solutions to elliptic partial differential equations. In fact, the better inclusion between the Morrey and Holder spaces permits to obtain regularity of the solution to elliptic boundary value problems. For the properties and applications of the classical Morrey spaces we refer the readers to [30,34].

In [8] Chiarenza and Frasca showed boundness of the Hardy-Littlewood maximal operator in $L_{p,\lambda}(R^n)$ that allows them to prove continuityin these spaces of some classical integral operators. The results in [8] allow us to study the regularity of the solutions of of elliptic parabolic equations and systems in $L_{p,\lambda}$ (see [9,11,12,33,35-37] and the references therein). In [31] Mizuhara extended the Morrey's consept of integral average over a ball with a certain growth, taking a weight function $\varphi(x,r): R^n\times R_+\to R_+$ instead of r^λ. So he put the beginning of the study of the generalized Morrey spaces $M_{p,\phi,}p>1$ with φbelonging to various classes of weight functions. In [32] Nakai proved boundedness of the maximal and

Calderón- Zygmund operators in $M_{p,\varphi}$ imposing suitable integral and doubling conditions on φ. Taking a weight $w(x,t)=\phi(x,t)^p t^n$ the conditions of Mizuhara- Nakai become

$$\int_r^\infty \varphi(x,\tau)^{p\frac{d\tau}{\tau}}\le C\varphi(x,r)^p\,, C^{-1}\le\frac{\varphi(x,\,t)}{\varphi\,(x,\,r)}\le C,\forall r\le t\le 2r\,,$$

where the constants do not depend on t,r and $x\in R$.

In series of works, the first author studies the continuity in generalized Morrey spaces of sublinear operators generated by various integral operators as Calderon-Zygmund, Riesz and others (see [4,21]). The following theorem obtained in [21] extends the results of Nakai to the generalized Morrey spaces with weight $\omega\ (x,t)=\varphi\ (x,t)\ t^n$ (for the definition of the spaces see Section 2).

Theorem A ([21, Theorem 6.2]) *Let $1\le p<\infty$and (φ_1,φ_2) satisfy the condition*

$$\int_r^\infty \varphi_1\ (x,\tau)\ \frac{d\tau}{\tau}\le C\varphi_2\ (x,\ r)\,, \tag{1.3}$$

where C does not depend on x and r. Then the Calderon- Zygmund operators are bound from $M_{p,\phi_1}(R^n)$ to $M_{p,\phi_2}(R^n)$ for $p>1$ and from $M_{1,\phi_1}(R^n)$ to the weak space $WM_{p,\varphi_2}(R^n)$.

This result is extended on spaces with weaker condition on the weight pair $(\varphi_1,\,\varphi_2)$ (see [4]). A further development of the generalized Morrey spaces can be found in the works [4] and the references therein. In [4], Guliyev et al. obtained a weaker than (1.1) condition on the pair (φ_1,φ_2) which is optimal and ensure the boundedness of the classical integral operators from $M_{p,\phi_1}(R^n)$ to $M_{p,\phi_2}(R^n)$. Precisely, if

$$\int_r^\infty \frac{ess\ \sup\limits_{t<s<\infty}\ \phi_1\ (x,s)\,s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt=C\phi_2\,(x,r)\,, \tag{1.4}$$

then the Calderon- Zygmund operators are bound from $M_{p,\phi_1}(R^n)$ to $M_{p,\phi_2}(R^n)$ for $p>1$ and *from* $M_{1,\phi_1}(R^n)$ *to the weak space* $WM_{p,\varphi_2}(R^n)$.

We use this integral inequality to obtain the Calderon- Zygmund type estimate for the $M_{p,\varphi}$- regularity of the solution. These results allow us to study the regularity of the solutions of various linear elliptic and parabolic boundary value problems in $M_{p,\varphi}$ (see [27,28,38]).

Later these results are extended on the local generalized Morrey spaces, which is obtained the boundedness of the Calderon- Zygmund operators from one local generalized Morrey space $LM^{\{x_0\}}_{p,\phi_1}(R^n)$ to another $LM^{\{x_0\}}_{p,\phi_2}(R^n), x_0 \in R^n$, if the pair functions (φ_1,φ_2) satisfy the following condition

$$\int_r^\infty \frac{ess \sup\limits_{t<s<\infty} \phi_1(x_{0,s})\, s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt = C\phi_2(x_0,r) \tag{1.5}$$

where C does not depend on r.

In this paper we study the boundedness of the sublinear operators, generated by Calderon-Zygmund operators in local generalized Morrey spaces. By using these results, we obtain the regularity of the solutions of higher order uniformly elliptic mixed boundary value problem in modified local generalized Sobolev- Morrey spaces defined on bounded smooth domains.

The paper is organized as follows. In Section 3 we prove the boundedness of the sublinear operators, generated by Calderon- Zygmund operators in the local generalized Morrey spaces. Further, we obtain the regularity estimates for the solvability of the the mixed boundary value problem for polyharmonic equation in modified local generalized Sobolev-Morrey spaces. In Section 4 we provide priori estimates for the solutions of the the mixed boundary value problems for the uniformly elliptic equations in modified local generalized Sobolev- Morrey spaces defined on bounded smooth domains.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Definitions and statement of the problem

Definition 2.1 *Let $\varphi : \Omega \times R_+ \to R_+$ be a measurable function and $1 \le p < \infty$. For any domain Ω the generalized Morrey space $M_{p,\varphi}(\Omega)$ (the weak generalized Morrey space $WM_{p,\varphi}(\Omega)$ consists of all $f \in L^{loc}_p(\Omega)$ such that*

$$\|f\|_{M_{p,\varphi}(\Omega)} = \sup_{x\in\Omega, 0<r<d} \frac{1}{\varphi(x,r)} \frac{1}{|B(x,r)|^{\frac{1}{p}}} \|f\|\, WL_p(\Omega(x,r)) < \infty,$$

$$\left(\|f\|_{WM_{p,\varphi}(\Omega)} = \sup_{x\in\Omega, 0<r<d} \frac{1}{\varphi(x,r)} \frac{1}{|B(x,r)|^{\frac{1}{p}}} \|f\|_{WL_p(\Omega(x,r))} < \infty \right)$$

where $d = \sup\limits_{x,y\in\Omega} |x-y|, B(x,r) = \{y \in R^n : |x-y| < r\}$ and $\Omega(x,r) = \Omega \bigcap B(x,r)$.

In the case $\varphi(x,r) = r^{\frac{\lambda-n}{p}}, M_{p,\varphi} = L_{p,\lambda}$ where $0<\lambda<n$. If $\lambda = 0$, then $L_{p,0}(R^n) = L_p(R^n)$, if $\lambda = n$ then $L_{p,n}(R^n) = L_\infty(R^n)$. In the case $\lambda < 0$ or $\lambda > n, L_{p,\lambda}(R^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on R^n.

Definition 2.2 *Let $\varphi(x,r)$ be a positive measurable function on $\Omega\times(0,d)$ and $1\le p<\infty$. Fixed $x_0\in\Omega$, we denote by $LM_{p,\varphi}^{\{x_0\}}\,\Omega\left(WLM_{p,\varphi}^{\{x_0\}}(\Omega)\right)$ the local generalized Morrey space (the weak local generalized Morrey space), the space of all functions $f\in L_p^{loc}(\Omega)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}(\Omega)}=\sup_{0<r<d}\frac{1}{\varphi(x_0,r)}\frac{1}{|B\,(x_0,\,r)|^{\frac{1}{p}}}\,\|f\|_{L_p(\Omega\,(x_0,r))}$$

$$\left(\|f\|_{WLM_{p,\varphi}^{\{x_0\}}(\Omega)}=\sup_{0<r<d}\frac{1}{\varphi(x_0,r)}\frac{1}{|B\,(x_0,r)|^{\frac{1}{p}}}\,\|f\|_{WL_p(\Omega\,(x_0,r))}\right).$$

Definition 2.3 *Let $\varphi(x,r)$ be a positive measurable function on $\Omega\times(0,\,d)$ and $1\le p<\infty$. We denote by $\tilde{M}_{p,\varphi}(\Omega)$ $(M_{p,\varphi}(\Omega))$ the modified generalized Morrey space (the modified weak generalized Morrey space), the space of all functions $f\in L_p(\Omega)$ with finite norm*

$$\|f\|_{\tilde{M}_{p,\varphi}(\Omega)}=\|f\|_{M_{p,\varphi}(\Omega)}+\|f\|_{L_p(\Omega)}$$

$$\left(\|f\|_{W\tilde{M}_{p,\varphi}(\Omega)}=\|f\|_{WM_{p,\varphi}(\Omega)}+\|f\|_{WL_p(\Omega)}\right).$$

Definition 2.4 *Let $\varphi(x,r)$ be a positive measurable function on $\Omega\times(0,d)$ and $(1\le p<\infty)$. Fixed $x_0\in\Omega$, we denote by $\widetilde{LM}_{p,\phi}^{\{x_0\}}(\Omega)$ $(\widetilde{LM}_{p,\phi}^{\{x_0\}}(\Omega))$ the modified local generalized Morrey space (the modified weak local generalized Morrey space), the space of all functions $f\in L_p(\Omega)$ with finite norm*

$$\|f\|_{\widetilde{LM}_{p,\phi}^{\{x_0\}}(\Omega)}=\|f\|_{LM_{p,\phi}^{\{x_0\}}(\Omega)}+\|f\|_{L_p(\Omega)}$$

$$\left(\|f\|_{W\widetilde{LM}_{p,\phi}^{\{x_0\}}(\Omega)}=\|f\|_{WLM_{p,\phi}^{\{x_0\}}(\Omega)}+\|f\|_{WL_p(\Omega)}\right).$$

Definition 2.5 *The modified generalized Sobolev- Morrey space $W_{p,\varphi}^{2m}(\Omega)$ consist of all functions $u\in W_p^{2m}(\Omega)$ with distributional derivatives*
$D_u^s\in\tilde{M}_{p,\phi,0}(\Omega)\,\phi,0\le|s|\le 2m$, endowed with the norm

$$\|u\|_{W_{p,\phi,0}^{2m}(\Omega)}=\sum_{0\,\le|s|\le 2m}\|D^s u\|_{\tilde{M}_{p,\phi}(\Omega)}.$$

The modified local generalized Sobolev-Morrey space $W_{p,\varphi}^{2m,\{x_0\}}(\Omega)$ consist of all functions $u\in W_p^{2m}(\Omega)$ with distributional derivatives $D_u^s\in\widetilde{LM}_{p,\phi}^{\{x_0\}}(\Omega)\,,0\le|s|\le 2m$, endowed with the norm

$$\|u\|_{W_{p,\phi,0}^{2m,\{x_0\}}(\Omega)}=\sum_{0\le|s|\le 2m}\|D^s u\|_{\widetilde{LM}_{p,\phi}(\Omega)}.$$

The space $W_{p,\phi,0}^{2m,\{x_0\}}(\Omega)\cap W_{p,0}^1(\Omega)$ consists of all functions $u\in W_{p,0}^1(\Omega)$ with $D_u^s\in LM_{p,\phi}^{\{x_0\}}(\Omega)\,,0\le|s|\le 2m$ in the closure functions of $C^\infty(\Omega)$ vanishing on Γ_1 and is endowed by the same norm. Recall that $W_{p,0}^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm in $W_p^1(\Omega)$, where functions vanishing on Γ_1.

At first we consider the mixed boundary value problem for polyharmonic equation

$$\begin{cases} (-\Delta)^m u = f & in \quad \Omega \\ u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{m-1} u}{\partial n^{m-1}} = g & \\ & on\ \partial\Omega = \Gamma_1 \cup \Gamma_2 \\ \frac{\partial^m u}{\partial n^m}|_{\Gamma_2} = g, & \end{cases} \tag{2.1}$$

where $\Omega \subset R^n$, $n \geq 2$ is a bounded domain with sufficiently smooth boundary.

For the solutions of the problem (2.1) we give some estimates for the Green function and the Poisson kernels. Later we obtain a priori estimates for solvability of problem (2.1) in the local generalized Morrey spaces.

Let $G_m, (x,\ y)$ be the Green function and $K_j\,(x,y)\,, j = \overline{0, m-1}$ be the Poisson kernels of problem (2.1). Then the solution of problem (2.1) can be written as

$$u\ (x) = \int_{\Omega} G_m\,(x,y)\,f\,(y)\,dy + \sum_{j=0}^{m-1} \int_{\partial\Omega} K_j\,(x,y)\,g\,(y)\,d\sigma_y$$

for correspondingly f and g. For example, when $m = 2$ and $n = 2$ we will that there is a constant $C\ (\Omega)$ such that

$$|G_2\,(x,y)|\ \leq\ C\,(\Omega)\ d\,(x)\ d\,(y)\ \min\left\{1, \frac{d\,(x)\,d\,(y)}{|x-y|^2}\right\}, \tag{2.2}$$

which was proved in [29], where d is the distance of x to the boundary $\partial\Omega$

$$d\,(x) = \underset{\widetilde{x \in \partial\Omega}}{inf}\ |x - \tilde{x}|\,. \tag{2.3}$$

However, we would like to mention that for G_m and K_j estimates are the optimal tools for deriving regularity results in spaces that involve to behavior at the boundary. Coming back to the $m = n = 2$ it follows from (2.2) that the solution of problem (2.1) satisfies the following estimates for appropriate f at $g = 0$

$$\left\|ud^{-2}\right\|_{L_\infty(\Omega)} \leq C\,(\Omega)\,\|f\|_{L_1(\Omega)}\,,$$

$$\|u\|_{L_\infty(\Omega)}\ \leq\ C\,(\Omega)\ \left\|fd^2\right\|_{L_1(\Omega)}\,.$$

We also derive estimates for derivative of kernels. We will focus on estimate that contain growth rates near the boundary. These estimates are optimal. Indeed, when we consider $G_m\ (x,y)$ for $\Omega\ =\ B\ (x,y)$ a ball in R^n the growth rates near the boundary are sharp (see [18]). For $m\ =\ 1$ or $m\ \geq\ 2$ and $\Omega\ =\ B\,(x,r)$ it is known that the Green function is positive and can even be estimated from below by a positive function with the same singular behavior (see [19]). Let us remind that for $m \geq 2$ the Green function in general is not positive. For general domains the optimal behavior in absolute values is captured in our estimates. Sharp estimates for K_{m-1} and K_{m-2}in the case of a ball can be found in [20]. In [5] Barbatis considered the pointwise estimates for the Green functions of higher order parabolic problems on domains and derived pointwise estimates for the kernel. For higher order parabolic systems the classical estimates obtained by Eidelman [17] were not considered in domains with boundary. For a survey on spectral theory of higher order elliptic operators, including some estimates for the corresponding kernels, we refer to [14]. We also consider analogously problems [21,22,23,24,25].

Let G a function on $\Omega\ \times\ \Omega$and $\alpha,\ \beta\ \in\ \mathrm{N}^n$. Derivatives of G are denoted by

$$D_x^{\alpha} D_y^{\beta} f\,(x,y)\ =\ \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2.}\ \cdots\ \partial x_n^{\alpha_n}} \frac{\partial^{|\beta|}}{\partial y_1^{\beta_1} \partial y_2^{\beta_2\cdot}\ \cdots \partial y_n^{\beta_n}} G\,(x,y)\,,$$

where $|\alpha| = \sum\limits_{k=1}^{n} \alpha_k,\ |\beta| = \sum\limits_{k=1}^{n} \beta_k.$

For completeness we will give some estimates for $G_m(x,y)$ and $K_j(x,y)$ depending on the distance to the boundary and auxiliary results with proof. We will do by estimating the $j-th$ derivative through an integration of the $(j+1)-th$ derivative along a path to the boundary. The dependence on the distance to the boundary $d(x)$ will appear closing a path which length is proportional to $d(x)$. The path will be constructed in Lemma 2.10.

Theorem 2.1 *([15,29]) Let $G_m(x,y)$ be the Green function of problem* (2.1).*Then for every $x,y \in \Omega$ the following estimates hold:*

1. if $2m-n>0$, then

$$|G_m(x,y)| \leq d^{m-\frac{n}{2}}(y) \min\left\{1, \frac{d(x)\,d(y)}{|x-y|^2}\right\}^{\frac{n}{2}};$$

2. if $2m-n=0$, then

$$|G_m(x,y)| \leq \log\left(1+\min\left\{1, \frac{d(x)\,d(y)}{|x-y|^2}\right\}^{m}\right);$$

3. if $2m-n<0$, then

$$|G_m(x,y)| = |x-y|^{2m-n} min\left\{1, \frac{d(x)\,d(y)}{|x-y|^2}\right\}^{m};$$

Theorem 2.2 *([15,29]). Let $K_j(x,y),\ j=\overline{0,m-1}$ be the Poisson kernels of problem* (2.1). *Then for every $x \in \Omega y \in \partial\Omega$ the following estimates hold:*

$$|K_j(x,y)| \leq \frac{d^m(x)}{|x-y|^{n-j+m-1}} \tag{2.4}$$

Remark 2.1 If $n-1<j\leq m-1$, then from (2.4) we get the inequality

$$|K_j(x,y)| \leq d^{1+j-n}(x) \tag{2.5}$$

on $\Omega \times \partial\Omega$.

Remark 2.2 The estimates in Theorem 2.2 hold for any uniformly elliptic operator of order $2m$.

In [19] the estimates in Theorem 2.1 are given for the case that $\Omega = B(x,r)$in R^n. In there the authors use an explicit formula for the Green´s function, given in [6].

For general domains one cannot expect an explicit formula for the Greens functions and the Poisson kernels. We will use the estimates for $G_m(x,y)$ and $K_j(x,y)$ given in [29]. In [29] for sufficiently regular domains Ω some estimates for the Greens function and Poisson kernels was proved.

3 Sublinear operators, generated by Calderon - Zygmund operators in local generalized Morrey spaces

Let Ω be an open bounded subset of R^n. Suppose that Trepresents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\Omega)$

$$|Tf(x)| \le c_0 \int_\Omega \frac{|f(y)|\,dy}{|x-y|^n}, \qquad x \notin supp(f)\,, \tag{3.1}$$

where c_0 is independent of f and x.

The following local estimates for the sublinear operator satisfying condition (3.1) are valid.

Lemma 3.1 *Let* $1 \le p < \infty$, Ω*be an open bounded subset of* R^n, $x_0 \in \Omega$, $0 < r \le d$, $d = \sup_{x,y\in\Omega}|x-y| < \infty$. *Let also* T *be a sublinear operator satisfying condition* (3.1)*, and bounded from* $L_p(\Omega)$ *to* $WL_p(\Omega)$ *,and bounded on* $L_p(\Omega)$ *for* $p > 1$.

(i). Then the inequality

$$\|Tf\|_{WL_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,t))}\,dt + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)} \tag{3.2}$$

holds for any $\Omega(x_0,r)$ *and for any* $f \in L_p(\Omega)$.

(ii) Moreover, for $p > 1$ *the inequality*

$$\|Tf\|_{L_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,t))}\,dt + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)} \tag{3.3}$$

holds for any $\Omega(x_0,r)$ *and for any* $f \in L_p(\Omega)$.

Proof. Let $1 \le p < \infty$. Since

$$r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,t))}\,dt \ge r^{\frac{n}{p}} \|f\|_{L_p(\Omega(x_0,r))} \int_r^d t^{-\frac{n}{p}-1} dt$$
$$\approx \|f\|_{L_p(\Omega(x_0,r))} \left(d^{\frac{n}{p}} - r^{\frac{n}{p}}\right), \; r \in (0,d)$$

we get that

$$\|f\|_{L_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,t))}\,dt + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)}\,, \; r \in (0,d)\,. \tag{3.4}$$

(i). Assume that $1 \le p < \infty$. Let $r \in (0, d/2)$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{\Omega(x_0,2r)}$ and $f_2 = f\chi_{\Omega/\Omega(x_0,2r)}$. Taking into account the linearity of T, we have

$$\|Tf\|_{WL_p(\Omega(x_0,r))} \le \|Tf_1\|_{WL_p(\Omega(x_0,r))} + \|Tf_2\|_{WL_p(\Omega(x_0,r))}\,. \tag{3.5}$$

Since $f_1 \in L_p(\Omega)$, in view of (3.4), the boundedness of T from $L_p(\Omega)$ to $WL_p(\Omega)$ implies that

$$\|Tf_1\|_{WL_p(\Omega(x_0,r))} \le \|Tf_1\|_{WL_p(\Omega)} \lesssim \|f_1\|_{L_p(\Omega)} \approx \|f\|_{L_p(\Omega(x_0,r))}$$
$$\lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,t))}\,dt + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)}\,, \tag{3.6}$$

where the constant is dependent of f, x_0 and r.

We have

$$|Tf_2(x)| \lesssim \int_{\Omega/\Omega(x_0,2r)} \frac{|f(y)|\ dy}{|x-y|^{n-1}},\ x\in\ \Omega(x_0,r).$$

It´s clear that $x \in \Omega(x_0,r)$, $y \in \Omega \quad \Omega(x_0,2r)$ implies $(1/2)\ |x_0-y| \le |x-y| < (3/2)\ |x_0-y|$.

Therefore we obtain that

$$\|Tf_2\|_{L_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{\Omega/\Omega(x_0,2r)} \frac{|f(y)|\, dy}{|x_0-y|^{n-1}}$$

By Fubin's theorem, we get that

$$\int_{\Omega/\Omega(x_0,2r)} \frac{|f(y)|}{|x_0-y|^{n-1}}\, dy \approx \int_{\Omega/\Omega(x_0,2r)} |f(y)| \left(1+\int_{|x_0-y|}^{d} \frac{ds}{s^n}\right) dy$$

$$= \int_{\Omega/\Omega(x_0,2r)} |f(y)|\ dy + \int_{\Omega/\Omega(x_0,\,2r)} |f(y)| \left(\int_{|x_0-y|}^{d} \frac{ds}{s^n}\right) dy$$

$$= \int_{\Omega/\Omega(x_0,2r)} |f(y)|\ dy + \int_{2r}^{d} \left(\int_{2r\le|x_0-y|\le s} |f(y)|\ dy\right) \frac{ds}{s^n}.$$

$$\le \int_{\Omega} |f(y)|\, dy + \int_{2r}^{d} \left(\int_{2r(x_0,s)} |f(y)|\, dy\right) \frac{ds}{s^n}.$$

Applying Hölders inequality, we arrive at

$$\int_{\Omega/\Omega(x_0,2r)} \frac{|f(y)|\, dy}{|x_0-y|^{n}} \lesssim \|f\|_{L_p(\Omega)} + \int_{2r}^{d} s^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,s))}\ ds.$$

Thus the inequality

$$\|Tf_2\|_{L_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{r}^{d} s^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,s))}\, ds + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)} \tag{3.7}$$

holds for all $r\in\ (0,d/2)$.

On the other hand, since

$$\|Tf_2\|_{WL_p(\Omega(x_0,r))} \le \|Tf_2\|_{L_p(\Omega(x_0,r))}$$

using (3.7),we get that

$$\|Tf_2\|_{WL_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{r}^{d} s^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,s))}\, ds + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)} \tag{3.8}$$

holds true for all $r\in(0,d/2)$.

Finally, combining (3.6) and (3.8), we obtain that

$$\|Tf\|_{WL_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{r}^{d} s^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,s))}\, ds + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)}$$

holds for all $r\in[0,d/2)$ with a constant independent of $f,\ x_0$ and r.

Let now $r \in [d/2, d)$. Then, using $(L_p(\Omega), WL_p(\Omega))$ - boundedness of T, we obtain

$$\|Tf\|_{WL_p(\Omega(x_0,r))} \le \|Tf\|_{WL_p(\Omega)} \le \|f\|_{L_p(\Omega)} \approx r^{\frac{n}{p}} \|f\|_{L_p(\Omega)},$$

and, inequality (3.2) holds.

(ii). Assume that $1 < p < \infty$. Let again $r \in (0, d/2)$. We write $f = f_1 + f_2$with $f_1 = f\chi_{\Omega(x_0,2r)}$and $f_2 = f\chi_{\Omega/\Omega(x_0,2r)}$. Taking into account the linearity of T, we have

$$\|Tf\|_{L_p(\Omega(x_0,r))} \le \|Tf_1\|_{L_p(\Omega(x_0,r))} + \|Tf_2\|_{L_p(\Omega(x_0,r))}. \tag{3.9}$$

Since $f_1 \in L_p(\Omega)$, in view of (3.4), the boundedness of Ton $L_p(\Omega)$ implies that

$$\|Tf_1\|_{L_p(\Omega(x_0,r))} \le \|Tf_1\|_{L_p(\Omega)} \approx \|f\|_{L_p(\Omega(x_0,2r))}$$

$$\lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,t))} + dt + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)}, \tag{3.10}$$

where the constant is independent of f, x_0 and r .

Combining (3.9), (3.10) and (3.7), we get inequality (3.3) holds for all $r \in (0, d/2)$ with a constant independent of $f,\ x_0$and r.

If $r \in [d/2, d)$,then, using the boundedness of Ton $L_p(\Omega)$,we obtain that

$$\|Tf\|_{L_p(\Omega(x_0,r))} \le \|Tf\|_{L_p(\Omega)} \lesssim \|f\|_{L_p(\Omega)} \approx r^{\frac{n}{p}} \|f\|_{L_p(\Omega)},$$

and, inequality (3.3) holds.

Now we are going to use the following statement on the boundedness of the weighted Hardy operator.

$$\mathrm{H}^*_\omega g(t) := \int_t^d g(s)\ \omega(s)\ ds, 0 < t \le d < \infty,$$

where ω is a fixed function non-negative and measurable on $(0, d)$.

The following theorem was proved in [25].

Theorem 3.1 *Let $\upsilon_1,\ \upsilon_2$ and ω be positive almost everywhere and measurable functions on $(0, d)$. The inequality*

$$ess\sup \upsilon_2(t)\ H^*_\omega g(t) \le C\, ess \sup_{0<t<d} \upsilon_1(t)\, g(t) \tag{3.11}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, d)$ if and only if

$$B := ess\sup \upsilon_2(t) \int_t^d \frac{\omega(s)ds}{\underset{0<t<d}{ess} \sup_{s<\tau<d} \upsilon_1(\tau)} < \infty \tag{3.12}$$

Moreover, if C^* is the minimal value of Cin (3.11)*, then $C^* = B$.*

Remark 3.1 In (3.11) and (3.12) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

Theorem 3.2 *Let $1 \le p < \infty$, Ωbe an open bounded subset of R^n, $x_0 \in \Omega$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^d \frac{ess \inf\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\,\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\varphi_2(x_0,r)\ , \tag{3.13}$$

where C does not depend on r. Let also T be a sublinear operator satisfying condition (3.1)*, and bounded from $L_p(\Omega)$ to $WL_p(\Omega)$, and bounded on $L_p(\Omega)$ for $p > 1$. Then there exists $c = c(p, \varphi_1, \varphi_2, n) > 0$ such that*

$$\|Tf\|_{W\widetilde{LM}^{\{x_0\}}_{p,\varphi_2}(\Omega)} \le c\,\|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)}\,.$$

Moreover, for $p > 1$ there exists $c = c(p, \varphi_1, \varphi_2, n) > 0$ such that

$$\|Tf\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_2}(\Omega)} \le c\,\|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)}\,.$$

Moreover, for $p > 1$ there exists $c = c(p, \varphi_1, \varphi_2, n) > 0$ such that

$$\|Tf\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_2}(\Omega)} \le c\,\|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)}\,.$$

Proof. By Theorem 3.2 and Lemma 3.1 with $\upsilon_2(r) = \varphi_2(x_0,r)^{-1}$, $\upsilon_1(r) = \varphi_1(x_0,r)^{-1} r^{\frac{n}{p}}$ and $\omega(r) = r^{-\frac{n}{p}}$ we have

$$\begin{aligned}\|Tf\|_{W\widetilde{LM}^{\{x_0\}}_{p,\varphi_2}(\Omega)} &\lesssim \sup_{0,<r<d} \varphi_1(x_0,r)^{-1} \int_r^d, \|f\|_{WL_p(\Omega(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} + \|Tf\|_{WL_p(\Omega)} \\ &\lesssim \sup_{0<r<d} \varphi_1(x_0,r)^{-1} r^{\frac{n}{p}} \|f\|_{L_p(\Omega(x_0,r))} + \|f\|_{L_p(\Omega)} \\ &= \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}(\Omega)} + \|f\|_{L_p(\Omega)} = \|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)}\end{aligned}$$

and for $1 < p < \infty$

$$\begin{aligned}\|Tf\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_2}(\Omega)} &\lesssim \sup_{0<r<d} \varphi_1(x_0,r)^{-1} \int_r^d \|f\|_{L_p(\Omega(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} + \|Tf\|_{L_p(\Omega)} \lesssim \\ &\lesssim \sup_{0<r<d} \varphi_1(x_0,r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(\Omega(x_0,r))} + \|f\|_{L_p(\Omega)} \\ &= \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}(\Omega)} + \|f\|_{L_p(\Omega)} = \|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)}\,.\end{aligned}$$

From Theorem 3.4 we get the following corollary.

Corollary 3.1. Let $1 \le p \le \infty, \Omega$ be an open bounded subset of R^n, $x_0 \in \Omega$, and (φ_1, φ_2)
satisfy the condition

$$\int_r^d \frac{ess \inf\limits_{t<\tau<\infty} \varphi_1(x,\tau)\,\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\varphi_2(x,r)\,, \tag{3.14}$$

where C does not depend on x and r. Let also T be a sublinear operator satisfying condition (3.1) and bounded from $L_p(\Omega)$ to $WL_p(\Omega)$, and bounded on $L_p(\Omega)$ for $p > 1$. Then there exists $c = c(p, \varphi_1, \varphi_2, n) > 0$ such that

$$\|Tf\|_{W\widetilde{M}_{p,\varphi_2}(\Omega)} \le c\,\|f\|_{\widetilde{M}_{p,\varphi_1}(\Omega)}\,.$$

Moreover, for $p > 1$ there exists $c = c(p, \varphi_1, \varphi_2, n) > 0$ such that

$$\|Tf\|_{\widetilde{M}^{p,\varphi_2}(\Omega)} \le c\,\|f\|_{\widetilde{M}_{p,\varphi_1}(\Omega)}\,.$$

4 The mixed boundary value problem for polyharmonic equation in modified local generalized Sobolev- Morrey spaces.

Now we will derive regularity estimates for solution of problem (2.1) when $g=0$

$$\begin{cases} (-\Delta)^m u = f \quad in \quad \Omega \\ \frac{\partial^k u}{\partial n^k} = 0 \quad on \quad \partial\Omega = \Gamma_1 \cup \Gamma_2, \\ \frac{\partial^m u}{\partial n^m}|_{\Gamma_2} = g \end{cases} \tag{4.1}$$

where $0 \le k \le m-1$, $\Omega \subset R^n$ is bounded.

We get the estimates of solution problem (4.1) in modified local generalized Sobolev-Morrey spaces.

$$\|u\|_{W^{2m,\{x_0\}}_{p,\varphi_2}(\Omega)} \le \|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}}.$$

Note that

$$Kf(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \sum_{|\alpha|=2m} D^\alpha_{x_i} G_m(x-y) f(y)\, dy$$

is the Calderon-Zygmund operator. Here and later, we take, that function f define in R^n, also this function is continuity extended to exterior of domain Ω with zero. The function $D^m_{x_i} G_m(x,y) \in C^\infty(R^n\ \{0\})$ and this function is homogeneous of order $m-n$. Hence consequence, that $D^{2m}_{x_i} G_m(x,y)$ homogeneous of order $2m-n$ and tends to zero on unit sphere (see [15]). Then from general theory giving in [7] consequence that K bounded operator on $L_p(R^n)$ for $1<p<\infty$. Moreover, maximal singularity operator.

$$\tilde{K}f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \sum_{|\alpha|=2m} D^\alpha G_m(x,y) f(y)\, dy \right|$$

also a bounded on $L_p(R^n)$ for $1<p<\infty$.

From Theorem 3.4 we get the following corollary.

Corollary 4.1. *Let $1 < p < \infty, \Omega$ be an open bounded subset of R^n, $x_0 \in \Omega$, and (φ_1, φ_2) satisfy the condition (3.13.). Then operators M and K are bounded from $\widetilde{LM}^{\{x_0\}}_{p,\phi_1}(O)$to $\widetilde{LM}^{\{x_0\}}_{p,\phi_2}(O)$.*

From Corollary 3.5 we get the following.

Corollary 4.2. *Let $1<p<\infty$, Ω be an open bounded subset of R^n, and (φ_1,φ_2) satisfy the condition* (3.14).*Then operators M and K are bounded from $\tilde{M}_{p,\varphi_1}(\Omega)$ to $\tilde{M}_{p,\varphi_2}(\Omega)$.*

Dirichlet boundary value problems for uniformly elliptic equations

Theorem 4.1 *Let $1<p<\infty, x_0 \in \Omega, \Omega \subset R^n$ be a bounded domain with $\partial\Omega \subset C^2$, and (φ_1,φ_2) satisfy condition* (3.13). *Let also $f \in \widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)$ and function u is a solution of problem* (4.1). *Then there is exist constant C which dependent only at n, φ and Ω such that*

$$\|u\|^{2m,\{x_0\}}_{W_{p,\varphi_2,0}}(\Omega) \le C\|f\|^{\{x_0\}}_{\widetilde{LM}_{p,\varphi_1}}(\Omega). \tag{4.2}$$

Proof. The proved consequence from the above estimates of the Green´s function from [27]: the following inequalities

$$|u(x) + |D_{x_i}u(x)|| \le Mf(x), \tag{4.3}$$

$$\left|D_{x_i x_j} u(x)\right| \le Kf(x) + Mf(x) + |f(x)| \tag{4.4}$$

hold uniformly for any $x \in \Omega$.

With similarly ideas can be proved estimated

$$|u(x)| + \left| \sum_{|\alpha| \le m} D^{\alpha}_{x_i} u(x) \right| \le Mf(x), \tag{4.5}$$

$$\left| \sum_{|\alpha| \le 2m} D^{\alpha} u(x) \right| \le \widetilde{K} f(x) + Mf(x) + |f(x)|. \tag{4.6}$$

Now we passing to prove of Theorem 4.1. From Corollary 4.1 imply that the operators $M$and are bounded in $LM^{\{x_0\}}_{p,\varphi}(\Omega)$. Therefore statement 4.3 and estimate (4.2) the immediately consequence from inequalities (4.5), (4.6) and Corollary 4.1.

Theorem 4.1 is proved.

From inequalities (4.5), (4.6) and Corollary 4.2 we get the following corollary.

Corollary 4.4. *Let* $1 < p < \infty, \Omega \subset R^n$ *be a bounded domain with and* (φ_1, φ_2) *satisfy the condition* (3.14). *Let also* $f \in \widetilde{M}_{p,\varphi_1}(\Omega)$ *and function u is a solution of problem* (4.1). *Then there is exist constant C which dependent only at* n, φ *and* Ω *such that*

$$\|u\|_{W^{2m}_{p,\phi_{2,0}}(\Omega)} = C \|f\|_{\widetilde{M}_{p,\phi_1}(\Omega)}.$$

5 Estimates of solutions any higher order uniformly elliptic equation with smooth coefficients in modified local generalized Sobolev- Morrey spaces.

Consider the boundary value problem

$$\begin{cases} Lu = f & in\ \Omega \\ B_j u = \psi_j & on\ \partial\Omega \end{cases} \tag{5.1}$$

for $j = 0, ..., m-1$. The following assumptions hold.

1. The operator

$$Lu = \sum_{|\alpha| \le 2m} \alpha_{\alpha,j}(x)\ D^{\alpha} u$$

is uniformly elliptic: there exists a constant $\gamma > 0$, such that

$$\gamma^{-1} |\xi|^2 \le \sum_{\alpha,j} \alpha_{\alpha,j}(x)\ \xi_\alpha \xi_j \le \gamma |\xi|^2, a.e.\ x \in \Omega,\ \forall \xi \in R^n$$

$$\alpha_{\alpha,j}(x) = \alpha_{j,\alpha}(x)$$

2. The boundary operators

$$B_j = \sum_{|\beta| \le m_j} b_{j\beta} D^{\beta},\ \text{for}\ \ j = 0,\ m-1$$

satisfy the complementing condition relative to L (see the complementing condition on page 663 of [17] .)

3. Let $l_1 > \max_j (2m - m_j)$ and $l_0 = \max_j (2m - m_j)$. The coefficients $\alpha_{\alpha j}$ belong to $C^{l_1+1}(\bar{\Omega})$ and $b_{j\beta}$ belong to $C^{l_1+1}(\partial\Omega)$

4. The boundary $\partial\,\Omega$is C^{l_1+2m+1}.

5. $f \in LM^{\{x_0\}}_{p,\varphi}(\Omega)$ with $1 < p < \infty$ and $\varphi : \Omega \times R_+ \to R_+$ measurable.

Theorem 5.1 *Let us consider the boundary value problem* (5.1) *and satisfy conditions 1-5 and also condition of Theorem 4.3. Then there is exist constant C which dependent only at* n, φ *and* Ω *such that*

$$\|u\|_{W^{2m,\{x_0\}}_{p,\phi_2,0}} = C\|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\phi_1}(\Omega)}. \tag{5.2}$$

Theorem 5.1 similarly ideas of Theorem 4.1 is proved.

For this it will be enough to consider the Krasovsky work [29]. We will recall the theorem in [29] which gives the estimates of the Green´s function and the Poisson kernels. The proved consequence from these estimates. As proof of Theorem 4.3 we use estimates (4.5), (4.6) and Corollary 4.1. Therefore, statement of theorem and estimate (5.2) the immediate consequence from inequalities (4.5), (4.6). Theorem 5.1 is proved.

From inequalities (4.5), (4.6) and Corollary 4.2 we get the following corollary.

Corollary 5.2. *Let us consider the boundary value problem* (5.1) *and satisfy conditions 1-5 and also condition of Corollary 4.4. Then there is exist constant C which dependent only at* n, φ *and* Ω *such that*

$$\|u\|_{W^{2m}_{p,\phi_2,0}} = C\|f\|_{\widetilde{M}_{p,\phi_2}(\Omega)}.$$

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