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On the dispersion of axisymmetric waves propagating in the initially strained highly elastic plate under two side contact with various fluids

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Abstract. *The paper investigates the influence of the initial axisymmetric homogeneous finite strains of a plate of highly elastic material, in contact on one side with the water layer and on the other side with the Glycerin layer, on the dispersion of the axisymmetric waves propagating in it. It is assumed that the flow of fluids in these layers is restricted by the upper and lower rigid walls. Also, it is assumed that the material of the plate is Lucite and the motion of that is described by the three-dimensional linearized equations and relations of the theory of elastic waves in prestressed bodies. However, the fluid flow is described by the linearized Euler equations for compressible, non-viscous fluids. The dispersion curves for quasi-Scholte modes are presented and discussed. In the context of this discussion, the main focus is on the influence of the difference between the fluids contacting the plate at the upper and lower levels on the dispersion of the studied waves. In particular, it is found that the limits of the propagation velocities of the asymmetric and symmetric quasi-Scholte waves at high wavenumbers differ from each other due to the mentioned differences between the fluids.*

Keywords. Axisymmetric wave dispersion · plate + fluid system · bi-lateral contact · compressible inviscid fluid · highly elastic material · finite initial strains · Lucite · glycerin · water.

Mathematics Subject Classification (2010): 35J40

1 Introduction

The treatises [1-8] give an overview of the relevant research. From these papers, it is clear that before the work in papers [9] and [10] published this year, all the investigations on the wave dispersion problems related to the plate+fluid system consider the case of plane strain. The paper [9] is the first to study the dispersion of axisymmetric waves in a hydroelastic system consisting of a finite, axisymmetrically pre-strained plate of highly elastic compressible

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material, a compressible inviscid fluid layer, and a rigid wall confining the flow of this fluid. However, the work [10] is concerned with the study of axisymmetric waves in an initially axisymmetric finitely pre-strained plate of highly compressible elastic material immersed in a compressible inviscid fluid whose flow is confined by rigid walls. Therefore, in the work [10], the plate is assumed to be in contact with the fluid on both sides. At the same time, in the work [10] it is assumed that the fluid of the "upper" and "lower" layers is the same. In the present work, we try to continue the investigation started in the work [10] for the case when the fluids of the mentioned "upper" and "lower" layers are different from each other. To obtain numerical results, glycerin is used as the fluid of the upper layer but water is used as the fluid of the "lower" layer, and Lucite is used as the plate material. Numerical results for the dispersion of the quasi-Scholte waves are presented and the effect of the difference of the fluids of the "upper" and "lower" fluid layers on these dispersion curves is studied for different values of the problem parameters characterizing the amount of initial strains in the plate and of the fluid layers' deeps.

2 Mathematical formulation of the problem

Consider the "rigid wall +fluid layer + plate + fluid layer +rigid wall" hydro-elastic system, the sketch of which is shown in Fig. 1. Assume that in the initial state, the thickness of the plate is$h$, and the thickness of the upper and lower fluid layers is equal to each other, and is h_d. At the same time, assume that the upper and lower layer fluids differ from each other.

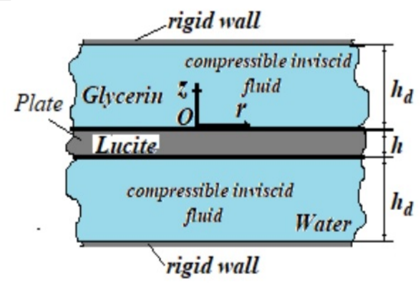


Fig. 1. Sketch of the hydro-elastic system under consideration

We determine with the Lagrange coordinates the position of the points of this plate in the natural state in the cylindrical system of coordinates $Or\theta z$associated with the upper face plane of the plate. According to the selected coordinate system, in the natural state, the plate occupies the region$\{0 \le r < \infty \ , 0 \le \theta \le 2\pi, -h \le z \le 0\}$. We assume that in the initial state (before any wave propagation), the plate is loaded at infinity (i.e. at $r \to \infty$) by the axisymmetric homogeneously distributed radial forces and as a result of this loading, the uniformly distributed strain-stress state appears in that. As we assume that the plate material is a highly elastic one, therefore the initial strain–stress state in the plate can be presented as follows:

$$u_r^0 = (\lambda_1 - 1)r, u_\theta^0 = 0, u_z^0 = (\lambda_3 - 1)z \qquad (2.1)$$

where λ_1 and λ_3 are constants and are called elongation parameters. In (2.1), the upper index 0 denotes that the related quantity belongs to the initial strain-stress state. This notation will also be used below.

As in the paper [10], we assume that the material of the plate is highly elastic and compressible, therefore, for describing the elasticity relations of the plate material we use the harmonic potential proposed by John [11] which has the following expression:

$$\Phi = \frac{1}{2}\lambda e_1^2 + \mu e_2, \tag{2.2}$$

where

$$e_1 = \sqrt{1+2\varepsilon_1} + \sqrt{1+2\varepsilon_2} + \sqrt{1+2\varepsilon_3} - 3,$$
$$e_2 = \left(\sqrt{1+2\varepsilon_1} - 1\right)^2 + \left(\sqrt{1+2\varepsilon_2} - 1\right)^2 + \left(\sqrt{1+2\varepsilon_3} - 1\right)^2. \tag{2.3}$$

Here λ and μ are material constants and ε_i $(i = 1, 2, 3)$ are the principal values of Green's strain tensor.

After selection of the elastic potential, the physical components of the symmetric stress tensor $s_{(ij)}$ are determined by the following expression:

$$s_{(ij)} = \frac{1}{2}\left(\frac{\partial}{\partial\varepsilon_{(ij)}} + \frac{\partial}{\partial\varepsilon_{(ji)}}\right)\Phi, \tag{2.4}$$

where under (ij), the indices rr, $\theta\theta$, zz, $r\theta$, rz, $z\theta$ are understood.

Substituting the expressions in (2.1) into the non-linear strain-displacement relations in the cylindrical coordinate system $Or\theta z$ (the explicit expressions of these relations can be found in many textbooks related to the theory of elasticity, as well as in the monograph [12]), we obtain the following initial strains:

$$\varepsilon_{rr}^0 = \varepsilon_{\theta\theta}^0 = \frac{1}{2}\left((\lambda_1)^2 - 1\right), \varepsilon_{zz}^0 = \frac{1}{2}\left((\lambda_3)^2 - 1\right), \varepsilon_{rz}^0 = \varepsilon_{r\theta}^0 = \varepsilon_{\theta z}^0 = 0. \tag{2.5}$$

It is clear from the expressions in (2.5) that in the initial state, the principal values ε_1^0, ε_2^0, and ε_3^0 of the strain tensor coincide with ε_{rr}^0, $\varepsilon_{\theta\theta}^0$, and ε_{zz}^0, respectively. Therefore, according to the expressions (2.3), (2.5), and (2.2), in the initial state the following expression for the selected elastic potential Φcan be written:

$$\Phi^0 = \frac{1}{2}\lambda(2\lambda_1 + \lambda_3 - 3)^2 + \mu(2(\lambda_1 - 1)^2 + (\lambda_3 - 1)^2). \tag{2.6}$$

Using this expression, we obtain the following expressions for the initial stresses from the relation (2.4):

$$s_{zz}^0 = [\lambda(2\lambda_1 + \lambda_3 - 3) + 2\mu(\lambda_3 - 1)](\lambda_3)^{-1},$$
$$s_{rr}^0 = s_{\theta\theta}^0 = [\lambda(2\lambda_1 + \lambda_3 - 3) + 2\mu(\lambda_1 - 1)](\lambda_1)^{-1}. \tag{2.7}$$

According to the problem statement, we assume that

$$s_{zz}^0 = [\lambda(2\lambda_1 + \lambda_3 - 3) + 2\mu(\lambda_3 - 1)](\lambda_3)^{-1} = 0 \Rightarrow \lambda(2\lambda_1 + \lambda_3 - 3) = 2\mu(1 - \lambda_3) \Rightarrow$$
$$\lambda_3 = \frac{\lambda(3 - 2\lambda_1) + 2\mu}{\lambda + 2\mu} = \frac{(3 - 2\lambda_1) + 2\mu/\lambda}{1 + 2\mu/\lambda}. \tag{2.8}$$

Using the relations in (2.8), we obtain the following expressions for the stresses s_{rr}^0 and $s_{\theta\theta}^0$:

$$s_{rr}^0 = s_{\theta\theta}^0 = \frac{2\mu}{\lambda_1}(\lambda_1 - \lambda_3). \tag{2.9}$$

Thus, through the expressions (2.1), (2.5), (2.8) and (2.9), we determine completely the initial axisymmetric stress-strain state in the plate for the selected material constants λ and this state is determined only by the elongation parameter λ_1. Consequently, for the fixed ratio λ/μ the magnitude of the initial stress-strain state in the plate can be estimated only through the parameter λ_1.

Thus, we assume that after appearing of the foregoing initial stress-strain state in the plate that is in bilateral contact with the fluid, the axisymmetric waves in the composite hydro-elastic system propagate in the radial direction. For writing the equations and relations of this propagation, we use the Lagrange coordinates r' and z' in the coordinate system $O'r'\theta'z'$ related to the foregoing initial state, and these coordinates are determined through the coordinates r and z in the coordinate system $Or\theta z$ by the following expressions:

$$r' = \lambda_1 r, z' = \lambda_3 z, \tag{2.10}$$

according to which, in the initial state the thickness of the plate becomes h' where $h' = \lambda_3 h.$

Taking the foregoing assumptions and notation into consideration, and based on the monographs [12 – 16], and others listed therein, we write the equations and relations of the three-dimensional linearized theory of elastic waves for finite pre-strained elastic bodies with the Lagrange coordinates r' and z' for the axisymmetric case.

The linearized equations of motion:

$$\frac{\partial Q_{r'r'}}{\partial r'} + \frac{\partial Q_{z'r'}}{\partial z'} + \frac{1}{r'}\left(Q_{r'r'} - Q_{\theta'\theta'}\right) = \rho' \frac{\partial^2 u_{r'}}{\partial t^2},$$

$$\frac{\partial Q_{r'z'}}{\partial r'} + \frac{\partial Q_{z'z'}}{\partial z'} + \frac{1}{r'} Q_{r'z'} = \rho' \frac{\partial^2 u_{z'}}{\partial t^2}. \tag{2.11}$$

The linearized elasticity relations:

$$Q_{r'r'} = \omega'_{1111}\frac{\partial u_{r'}}{\partial r'} + \omega'_{1122}\frac{u_{r'}}{r'} + \omega'_{1133}\frac{\partial u_{z'}}{\partial z'},$$

$$Q_{z'z'} = \omega'_{3311}\frac{\partial u_{r'}}{\partial r'} + \omega'_{3322}\frac{u_{r'}}{r'} + \omega'_{3333}\frac{\partial u_{z'}}{\partial z'}, Q_{r'z'} = \omega'_{1313}\frac{\partial u_{r'}}{\partial z'} + \omega'_{1331}\frac{\partial u_{z'}}{\partial r'},$$

$$Q_{z'r'} = \omega'_{3113}\frac{\partial u_{r'}}{\partial z'} + \omega'_{3131}\frac{\partial u_{z'}}{\partial r'}. \tag{2.12}$$

The following notation is used in (2.11) and (2.12): $Q_{r'r'}$, $Q_{\theta'\theta'}$, $\ldots$, $Q_{z'r'}$ are the perturbations of the components of the non-symmetric Kirchhoff stress tensor, $u_{r'}$ and $u_{z'}$ are the perturbations of the components of the displacement vector in the system of coordinates $O'r'\theta'z'$, and ρ' is the plate material density related to the unit volume in the initially strained state; $\rho' = \left(\lambda_1^2\lambda_3\right)^{-1}\rho$, where ρ is the plate material density in its natural state.

Moreover, according to Guz [13], in the case under consideration, the components ω'_{1111}, ω'_{1122},..., ω'_{3131} are determined by the following expressions:

$$\omega'_{1111} = (\lambda + 2\mu)\frac{1}{\lambda_3}, \omega'_{2222} = \omega'_{1111}, \omega'_{3333} = \frac{\lambda_3}{\lambda_1^2}(\lambda + 2\mu), \omega'_{1122} = \frac{1}{\lambda_3}\lambda,$$

$$\omega'_{1133} = \frac{1}{\lambda_1}\lambda, \omega'_{1313} = 2\mu\frac{\lambda_3}{\lambda_1(\lambda_1 + \lambda_3)}, \omega'_{3113} = \omega'_{1331} = 2\mu\frac{\lambda_1}{\lambda_3(\lambda_1 + \lambda_3)}. \tag{2.13}$$

This completes the field equations and relations within the framework of which the motion of the plate is described.

We now consider the equations and relations describing the axisymmetric flow of compressible inviscid fluid in the cylindrical coordinate system

According to the monograph of Guz [17], when setting up these equations and relations, the coordinates r' and z' are assumed to be Euler coordinates. In this case, the difference between the Euler and Lagrangian coordinates is ignored because the perturbations are very small.

The linearized Euler and state equations of the fluid flow are:

$$\frac{\partial V^{\pm}_{r'\,r'}}{\partial t} = -\frac{1}{\rho^{\pm}_{10}}\frac{\partial p'^{\pm}_1}{\partial r'}, \frac{\partial V^{\pm}_{z'}}{\partial t} = -\frac{1}{\rho^{\pm}_{10}}\frac{\partial p'^{\pm}_1}{\partial z'},$$

$$p'^{\pm}_1 = (a^{\pm}_0)^2 \rho'^{\pm}_1, (a^{\pm}_0)^2 = \left(\frac{\partial p'^{\pm}_1}{\partial \rho'^{\pm}_1}\right)_0, \tag{2.14}$$

The continuity equation is:

$$\frac{\partial \rho'^{\pm}_1}{\partial t} + \rho^{\pm}_{10}\left(\frac{\partial V^{\pm}_{r'}}{\partial r'} + \frac{V^{\pm}_{r'}}{r'} + \frac{\partial V^{\pm}_{z'}}{\partial z'}\right) = 0. \tag{2.15}$$

In (2.14) and (2.15) the following notation is used: $a^{\pm}_0$ is the sound speed in the fluid, $\rho^{\pm}_1$ and $p'^{\pm}_1$ are the perturbations of the fluid density and the fluid pressure, respectively, $\rho^{\pm}_{10}$ is the fluid density in the initial state, and $V^{\pm}_{r'}$ and $V^{\pm}_{z'}$ are the components of the velocity vector. The signs "+" (the sign "-") in this notation means that the corresponding quantity relates to the upper fluid layer which occupies the region $\{0 \le r' \le \infty;\ 0 \le \theta' \le 2\pi;\ \ 0 \le z' \le h_d\}$ (the lower fluid layer which occupies the region $\{0 \le r' \le \infty;\ 0 \le \theta' \le 2\pi;$ $(-h' - h_d) \le z' \le -h'\}$ where $h' = \lambda_3 h$).

We now formulate the compatibility and impermeability conditions for the plate + fluid system under consideration (Fig. 1). According to the foregoing notation, these conditions can be written as follows.

Compatibility conditions on the upper interface plane are:

$$Q_{z'z'}|_{z'=0} = -\left. p'^{+}_1 \right|_{z'=0}, Q_{z'r'}|_{z'=0} = 0, \left.\frac{\partial u_{z'}}{\partial t}\right|_{z'=0} = \left. V^{+}_{z'}\right|_{z'=0}. \tag{2.16}$$

Compatibility conditions on the lower interface plane are:

$$Q_{z'z'}|_{z'=-h'} = -\left. p'^{-}_1 \right|_{z'=-h'}, Q_{z'r'}|_{z'=-h'} = 0, \left.\frac{\partial u_{z'}}{\partial t}\right|_{z'=-h'} = \left. V^{-}_{z'}\right|_{z'=-h'}. \tag{2.17}$$

Impermeability condition on the upper rigid wall is:

$$\left. V^{+}_{z'}\right|_{z'=h_d} = 0. \tag{2.18}$$

Impermeability condition on the lower rigid wall is:

$$\left. V^{-}_{z'}\right|_{z'=-h'-h_d} = 0. \tag{2.19}$$

This completes the formulation of the problem under consideration.

3 Method of solution

According to the monograph [13], for solution to the system of equations (2.11), (2.12), and (2.13) we use the presentations:

$$u_{r'} = -\frac{\partial^2}{\partial r'\partial z'}\mathrm{X}, u_{z'} = \frac{1}{\omega'_{1111} + \omega'_{1313}}\left(\omega'_{1111}\Delta'_1 + \omega'_{3113}\frac{\partial^2}{\partial z'^2} - \rho'\frac{\partial^2}{\partial t^2}\right)\mathrm{X}, \tag{3.1}$$

where the function X is determined from the equation

$$\left[\left(\Delta' + (\xi'_2)^2\frac{\partial^2}{\partial z'^2}\right)\left(\Delta' + (\xi'_3)^2\frac{\partial^2}{\partial z'^2}\right) - \rho'\left(\frac{\omega'_{1111} + \omega'_{1331}}{\omega'_{1111}\omega'_{1331}}\Delta' + \frac{\omega'_{3333} + \omega'_{3113}}{\omega'_{1111}\omega'_{1331}}\right)\frac{\partial^2}{\partial t^2} + \frac{\rho'}{\omega'_{1111}\omega'_{1331}}\frac{\partial^4}{\partial t^4}\right]\mathrm{X} = 0, \Delta' = \frac{d^2}{dr'^2} + \frac{1}{r'}\frac{d}{dr'}, \tag{3.2}$$

where

$$(\xi'_{2,3})^2 = d \pm \left[d^2 - \omega'_{3333}\omega'_{3113}\left(\omega'_{1111}\omega'_{1331}\right)^{-1}\right]^{\frac{1}{2}},$$

$$d = \left[\omega'_{1111}\omega'_{3333} + \omega'_{1331}\omega'_{3113} - (\omega'_{1133} + \omega'_{1313})^2\right]\left(2\omega'_{1111}\omega'_{1331}\right)^{-1}. \tag{3.3}$$

Following the usual procedure, for the axisymmetric waves, we represent the function X in the form

$$\mathrm{X} = F e^{\chi k z'} J_0(kr')\cos(\omega t). \tag{3.4}$$

Substituting the expression in (3.4) into the equation (3.2) and doing some mathematical calculations, we obtain the following characteristic equation for determination of the constant χ:

$$A_1\chi^4 + B_1\chi^2 + C_1 = 0, \tag{3.5}$$

where

$$A_1 = (\xi'_2)^2(\xi'_3)^2, B_1 = \left(B\frac{c^2}{\lambda_1^2\lambda_3 c_2^2} - ((\xi'_2)^2 + (\xi'_3)^2)\right),$$

$$C_1 = -A\frac{c^2}{\lambda_1^2\lambda_3 c_2^2} + \frac{c^4}{\lambda_1^4\lambda_3^2 c_2^4} + 1$$

$$A = \frac{\omega'_{1111} + \omega'_{1331}}{\omega'_{1111}\omega'_{1331}}, B = \frac{\omega'_{3333} + \omega'_{3113}}{\omega'_{1111}\omega'_{1331}}, c = \frac{\omega}{k}, c_2 = \sqrt{\mu/\rho}. \tag{3.6}$$

In this way, we determine the following expressions for the roots of the equation (3.5):

$$\chi_1 = \sqrt{D_1}, \chi_2 = \sqrt{D_2}, \chi_3 = -\chi_1, \chi_4 = -\chi_2, \tag{3.7}$$

where

$$D_1 = -\frac{B_1}{2A_1} + \sqrt{\left(\frac{B_1}{2A_1}\right)^2 - \frac{C_1}{A_1}}, D_2 = -\frac{B_1}{2A_1} - \sqrt{\left(\frac{B_1}{2A_1}\right)^2 - \frac{C_1}{A_1}}. \tag{3.8}$$

After determination of the foregoing roots, the function X is presented as follows:

$$\mathrm{X} = \left[F_1\varphi_1(\chi_1 kz') + F_2\varphi_2(\chi_1 kz') + F_3\varphi_3(\chi_2 kz') + F_4\varphi_4(\chi_2 kz')\right] J_0(kr')\cos(\omega t). \tag{3.9}$$

where

$$\varphi_1(\chi_1 kz') = \begin{cases} e^{\chi_1 kz'}\ if\ D_1 > 0 \\ \cos(\alpha kz')\ if\ D_1 < 0 \end{cases}, \varphi_2(\chi_1 kz') = \begin{cases} e^{-\chi_1 kz'}\ if\ D_1 > 0 \\ \sin(\alpha kz')\ if\ D_1 < 0 \end{cases},$$

$$\varphi_3(\chi_2 kz') = \begin{cases} e^{\chi_2 kz'} \; if \; D_2 > 0 \\ \cos(\beta kz') \; if \, D_2 < 0 \end{cases}, \varphi_4(\chi_2 kz') = \begin{cases} e^{-\chi_2 kz'} \; if \; D_2 > 0 \\ \sin(\beta kz') \; if \, D_2 < 0 \end{cases}'$$

$$\alpha = Im\sqrt{D_1}, \beta = Im\sqrt{D_2}. \tag{3.10}$$

Substituting the expressions in (3.9) and (3.10) into the relations in (3.1) we determine the expressions for the displacements, and substituting the latter ones into the relations in (2.12) we determine the perturbation of the stresses. For reducing the volume of the paper we do not write these expressions here.

Note that in finding the solutions in (3.10), it is assumed that the values of D_1 and D_2 in (3.3) are real numbers, because in concrete numerical investigations for the chosen problem parameters this very case is observed. It is obvious that in the other cases, i.e. in the cases where the values of D_1 and D_2 are complex numbers, the solutions presented in (3.10) will have corresponding changes.

This completes the determination of the quantities related to the plate.

Now we consider the solution procedure to equations (2.14) and (2.15), and for this purpose, according to the monograph by Guz [17], we use the following presentations:

$$V_{r'}^{\pm} = \frac{\partial}{\partial r}\Phi^{\pm}, V_{z'}^{\pm} = \frac{\partial}{\partial z}\Phi^{\pm}, p_1^{'\pm} = -\rho_{10}^{\pm}\frac{\partial}{\partial t}\Phi^{\pm}, \rho_1^{'\pm} = -\frac{\rho_{10}^{\pm}}{(a_0^{\pm})^2}\frac{\partial}{\partial t}\Phi^{\pm}, \tag{3.11}$$

where the function $\Phi^{\pm}$ is found from the following equation:

$$\left(\Delta - \frac{1}{(a_0^{\pm})^2}\frac{\partial^2}{\partial t^2}\right)\Phi^{\pm} = 0, \Delta = \frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'} + \frac{\partial^2}{\partial z'^2}. \tag{3.12}$$

According to the expression in (3.4), we present the function Φ in the form of:

$$\Phi^{\pm} = G^{\pm} e^{\delta^{\pm} kz'} J_0(kr')\sin(\omega t) \tag{3.13}$$

where $G^{\pm}$ and $\delta^{\pm}$ are the unknown constants. Substituting the expression (3.13) into the equation (3.12) and doing some mathematical manipulations, we determine the constant $\delta^{\pm}$ as follows:

$$(\delta^{\pm})^2 = 1 - \frac{c^2}{(a_0^{\pm})^2} \quad \Rightarrow \quad \delta_1^{\pm} = \sqrt{1 - \frac{c^2}{(a_0^{\pm})^2}}, \delta_2^{\pm} = -\delta_1^{\pm}. \tag{3.14}$$

Here, c is the propagation velocity of the axisymmetric wave and $a_0^{\pm}$ is the sound speed in the selected fluid.

Thus, taking into consideration the foregoing preparations, we can write the following expression for the function $\Phi^{\pm}$for upper and lower fluid layers:

$$\Phi^{\pm} = \left[G_1^{\pm}\psi_1^{\pm}(\delta_1^{\pm} kz') + G_2^{\pm}\psi_2^{\pm}(\delta_1^{\pm} kz')\right] J_0(kr')\sin(\omega t). \tag{3.15}$$

Also, using the expressions in (3.11) we obtain the following expressions for the velocities and pressure in the lower and upper fluid layers:

$$V_{r'}^{\pm} = \left[G_1^{\pm}\psi_1^{\pm}(\delta_1^{\pm} kz') + G_2^{\pm}\psi_2^{\pm}(\delta_1^{\pm} kz')\right] J_0(kr')\sin(\omega t),$$

$$V_{z'}^{\pm} = \left[G_1^{\pm}\frac{d\psi_1^{\pm}}{dz'} + G_2^{\pm}\frac{d\psi_2^{\pm}}{dz'}\right] J_0(kr')\sin(\omega t),$$

$$p_1^{'\pm} = -\rho_{10}^{\pm}\omega\left[G_1^{\pm}\psi_1^{\pm}(\delta_1^{\pm} kz') + G_2^{\pm}\psi_2^{\pm}(\delta_1^{\pm} kz')\right] J_0(kr')\cos(\omega t) \tag{3.16}$$

where

$$\psi_1^{\pm} = \begin{cases} e^{\delta_1^{\pm} kz'} \; if \; (1 - c^2/(a_0^{\pm})^2) > 0 \\ \cos(\alpha_1^{\pm} kz') \; if \; (1 - c^2/(a_0^{\pm})^2) < 0 \end{cases},$$

$$\psi_2^{\pm} = \begin{cases} e^{-\delta_1^{\pm} kz'} \; if \; (1 - c^2/(a_0^{\pm})^2) > 0 \\ \sin(\alpha_1^{\pm} kz') \; if \; (1 - c^2/(a_0^{\pm})^2) < 0 \end{cases},$$

$$\alpha_1^{\pm} = \sqrt{(c^2/(a_0^{\pm})^2 - 1)}. \tag{3.17}$$

Finally, using the conditions in (2.16) – (2.19), we obtain the system of homogeneous linear algebraic equations with respect to the unknown constants F_1, F_2, F_3, F_4, G_1^+, G_2^+, G_1^-, and G_2^-. Equating to zero the determinant of the coefficient matrix of these equations, we obtain the following dispersion equation with respect to ω and k:

$$\det \left\| a_{ij}(\omega, k, a_0^+/c_2, a_0^-/c_2, \rho_{10}^{\prime +} \big/ \rho, \rho_{10}^{\prime -}/\rho, \lambda/\mu, kh, h_d/h) \right\| = 0, i; j = 1, 2, ..., 8. \tag{3.18}$$

To reduce the size of the paper, the explicit expressions of the components a_{ij} of the coefficient matrix (a_{ij}) are not given here. These expressions can be obtained from the preceding relations and equations after some obvious mathematical manipulations.

Solving the dispersion equation (3.18), we construct the dispersion curves, i.e. the graphs of the dependence between c/c_2 and kh for the fixed problem parameters a_0^+/c_2, a_0^-/c_2, ρ/ρ_{10}^+, ρ/ρ_{10}^-, λ/μ,h_d/h, and λ_1. We recall that throuth the parameter λ_1 we will investigate the influence of the initial strains in the plate on the mentioned dispersion curves. Note that the dispersion equation is solved numerically by employing the well-known "bi-section" method.

4 Numerical results and discussions

Under obtaining numerical results we assume that the plate material is Lucite with the mechanical constants $\mu = 1.86 * 10^9 Pa$, $\lambda = 3.96 * 10^9 Pa$, $\rho = 1160 kg/m^3$, and $c_2 = \sqrt{\mu/\rho} = 1265\, m/s$, and the fluid of the upper layer is Glycerin with density $\rho_{10}^+ = 1260\, kg/m^3$and with the sound speed $a_0^+ = 1927\, m/s$, and the fluid of the lower layer is water with density $\rho_{10}^- = 1000\, kg/m^3$and with the sound speed $a_0^- = 1459.5\, m/s$.

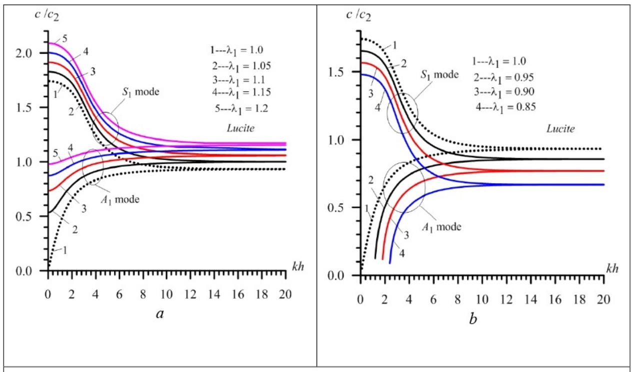


Fig. 2 The influence of the initial radial stretching (a) and compression (b) of the plate made of Lucite on the dispersion of the S_1 and A_1 modes of the axisymmetric waves for the case where the plate is not in contact with the fluid

First, we consider the dispersion curves of the S_1 and A_1modes of the axisymmetric waves propagating in the plate of Lucite, which is not in contact with the fluid, and analyze the influence of the initial strains in this plate, i.e., the influence of the parameter λ_1 on these curves, which are shown in Fig.2. Note that this plot is done separately for the cases$\lambda_1 \geq 1$ (Fig. 2a) and $\lambda_1 \leq 1$ (Fig. 2b). From the analysis of the results presented in Fig. 2, it is evident that the initial radial elongation (compression) of the plate leads to a monotonic increase (decrease) of the wave propagation velocity in all the modes considered. Moreover, it is evident from the results in Fig. 2a that in the case $\lambda_1 \geq 1$, the limits the wave propagation velocity in the low and high wavenumbers in modes A_1 and S_1 increase monotonically with λ_1, and the limits of the wave propagation velocity in the high wavenumbers coincide with the axisymmetric Rayleigh wave velocity in the relevant initially stretched Lucite half-space.

However, the results in Fig. 2b indicate that the initial radial compression of the plate leads to a decrease in the limiting value of the wave propagation velocity at low wavenumbers in the S_1 mode and the cut-off wavelength occurs in the A_1 mode, in addition to the fact that the wave propagation velocity decreases monotonically with the decrease of the parameter λ_1 in the modes considered. Note that these results are in a qualitative sense consistent with the corresponding results presented and analyzed in papers [18, 19], where the corresponding case with plane strain was considered. Note also that the dispersion curves presented in Fig. 2, obtained under $\lambda_1 = 1.0$ coincide with the corresponding dispersion curves obtained for the case with plane strain [18, 19]. This agreement and matching provide some guarantee of the reliability of the calculation algorithm used and of the PC programs used to obtain these results.

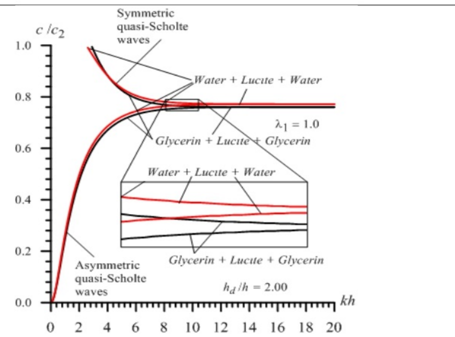


Fig. 3 Dispersion curves for the asymmetric and symmetric modes of the quasi-Scholte waves for the hydroelastic systems "Glycerin + Lucite + Glycerin" and "Water + Lucite + Water"

Note that the dispersion curves shown in Fig. 2 are constructed from the first and subsequent second-lowest roots obtained from the numerical solution of the dispersion equation for the case where the plate is not in contact with the fluids. Let us now consider the dispersion curves also obtained from the first and second lowest roots of the numerical solution of the dispersion equation (3.18). First, we consider the case in which the fluids of the upper and lower fluid layers are the same and examine only the case in which $h_d/h = 2$and assume that $\lambda_1 = 1.0$. The dispersion curves for this case are shown in Fig. 3 and these curves are given simultaneously for the systems ”water+Lucite+water” and ”glycerin + Lucite+glycerin”. From these curves, it can be seen that in the case where the plate is in contact with the liquid on both sides, the asymmetric and symmetric quasi-Scholte modes appear instead of the A_1 and S_1 modes in Fig. 2, and the limits of the wave propagation velocity in these modes at high wavenumbers are the corresponding propagation velocity of the Scholte waves. Moreover, the asymmetric (symmetric) quasi-Scholte wave propagation velocity approaches the corresponding Scholte wave propagation velocity from below (from above).

Fig. 3 also shows that the Scholte wave propagation velocity for the "Lucite + glycerin" pair is lower than for the "Lucite + water" pair.

Although the results presented are given for the case when $h_d/h = 2$and the above conclusions are drawn on the basis of these results, the numerical results obtained for other possible values of the ratio h_d/h (these results are not given here) show that these conclusions are valid in a qualitative sense also for the other values of this ratio.

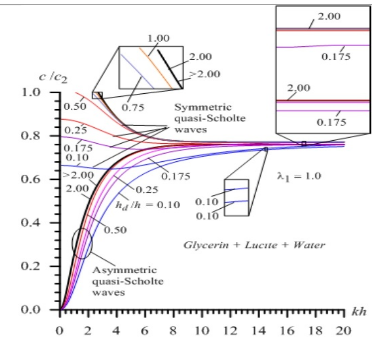
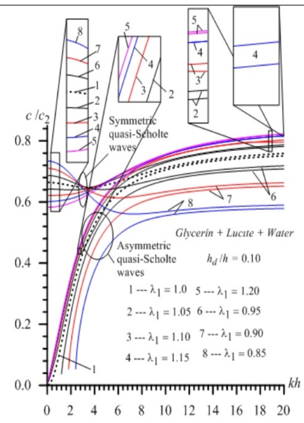


Fig. 4 Dispersion curves of asymmetric and symmetric modes of the quasi-Scholte waves obtained for the hydroelastic system "Glycerin + lucite + Water" under different values of the ratio h_d / h in the case $\lambda_1 = 1.0$

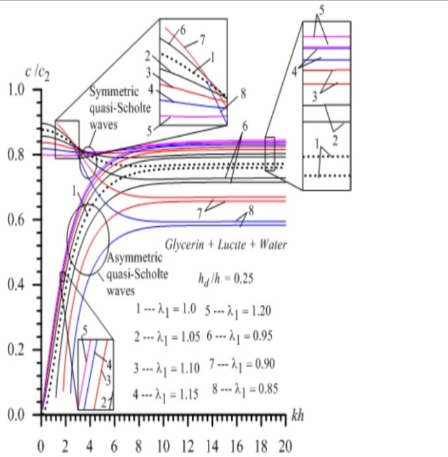
Now we consider the dispersion curves constructed for the hydro-elastic system "Glycerin + Lucite + Water". First, we consider dispersion curves related to the asymmetric and symmetric modes of the quasi-Scholte waves obtained for different values of the ratio h_d/h under $\lambda_1 = 1.0$ and shown in Fig. 4. From them it can be seen that a decrease in the values of the ratio h_d/h causes a decrease in the values of the wave propagation velocity in both asymmetric and symmetric modes. Moreover, it is clear from these results that the cut-off wavenumbers that appear in the dispersion curves for the quasi-Scholte waves of the symmetric modes at relatively large values of the ratio h_d/h disappear when this ratio is decreased.

It can be seen from Fig. 4 that the limit values under the low wavenumbers of the asymmetric waves do not depend on the ratio h_d/h, while these limit values of the symmetric waves clearly depend on the ratio h_d/h and decrease with decreasing h_d/h. Moreover, based on the results in Fig. 4, it can be said that in the cases where $h_d/h \geq 2$, the results for the system "rigid wall + fluid + plate + fluid + rigid wall" agree with very high accuracy with the corresponding results for the system "fluid half-space + plate + fluid half-space".

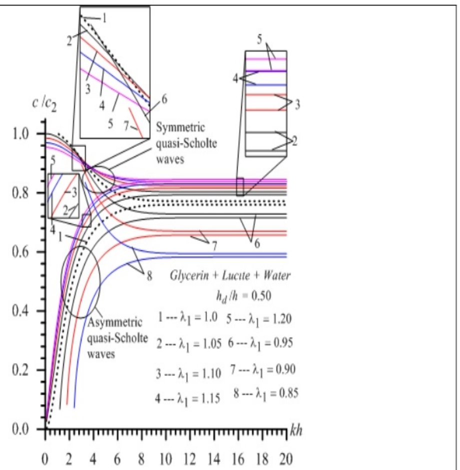
The main conclusion, which also follows from the results in Fig. 4, is the following: The limiting values of the wave propagation velocity of the asymmetric and symmetric quasi-Scholte waves at high wavenumbers, unlike to the case considered in Fig. 3, are different from each other and do not depend on the h_d/h ratio. Moreover, for the asymmetric quasi-Scholte waves, this limit is the propagation velocity of the Scholte waves for the "Lucite + Glycerin" pair, but for the symmetric quasi-Scholte waves, it is the propagation velocity of the Scholte waves for the "Lucite + Glicerin" pair. In other words, the wave propagation velocity of the asymmetric waves approaches from below the velocity of the Scholte waves for the "Lucite + Glycerin" pair, but the wave propagation velocity of the symmetric waves approaches from above the velocity of the Scholte waves for the "Lucite + Water" pair.


c /c2
Symmetric quasi-Scholte waves
Asymmetric quasi-Scholte waves
Glycerin + Lucite + Water
hd /h = 0.10
1 --- λ1 = 1.0 5 --- λ1 = 1.20
2 --- λ1 = 1.05 6 --- λ1 = 0.95
3 --- λ1 = 1.10 7 --- λ1 = 0.90
4 --- λ1 = 1.15 8 --- λ1 = 0.85
kh

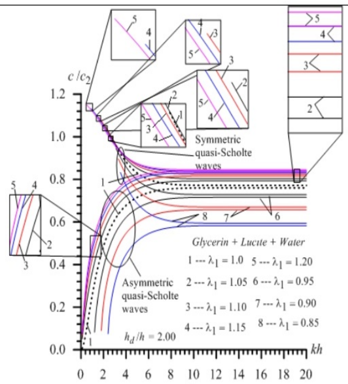
a


c /c2
Symmetric quasi-Scholte waves
Asymmetric quasi-Scholte waves
Glycerin + Lucite + Water
hd /h = 0.25
1 --- λ1 = 1.0 5 --- λ1 = 1.20
2 --- λ1 = 1.05 6 --- λ1 = 0.95
3 --- λ1 = 1.10 7 --- λ1 = 0.90
4 --- λ1 = 1.15 8 --- λ1 = 0.85
kh

b


c /c2
Symmetric quasi-Scholte waves
Asymmetric quasi-Scholte waves
Glycerin + Lucite + Water
hd /h = 0.50
1 --- λ1 = 1.0 5 --- λ1 = 1.20
2 --- λ1 = 1.05 6 --- λ1 = 0.95
3 --- λ1 = 1.10 7 --- λ1 = 0.90
4 --- λ1 = 1.15 8 --- λ1 = 0.85
kh

c



d

Fig. 5. The influence of the initial strains of the plate on the dispersion of the acymmetric and symmetric quasi-Scholte waves in the cases where $h_d / h = 0.1$ (a), 0.25 (b), 0.5 (c), and 2.0 (d)

Now we consider the influence of the initial strains of the plate on the dispersion curves of the asymmetric and symmetric quasi-Scholte waves. These results are presented in Fig. 5 for different values of the parameter λ_1 under $h_d/h = 0.1$(Fig. 5a), 0.25 (Fig. 5b), 0.5 (Fig. 5c), and 2.0 (Fig. 5d). From these results, it appears that for all selected values of the ratio h_d/h, the initial stretching (compression) of the plate leads to an increase (a decrease) in the propagation velocity of asymmetric waves. At the same time, under the initial compression of the plate, the cut-off dimensionless wavenumbers appear on the dispersion curves related to the asymmetric mode of quasi-Scholte waves, and the values of these wavenumbers increase with the decrease of the parameter λ_1. Moreover, the analysis of the results in Fig. 5 shows that the character of the influence of the parameter λ_1 on the dispersion curves of the asymmetric mode in the qualitative sense does not depend on the h_d/h ratio and on the dimensionless wavenumber kh.

At the same time, in all considered cases the values of the wave propagation velocity in the asymmetric mode of the quasi-Scholte wave under $kh \to \infty$ approach from below the velocity of the Scholte waves for the pair "pre-strained plate of Lucite + Glycerin" in all the cases considered. The limits of the propagation velocity of the asymmetric waves under $kh \to 0$ when the plate is prestressed approach the finite values, which are different from zero, but not so much. When the plate is pre-compressed, the propagation velocity of the above waves under $kh \to (kh)_{cf}$, approaches zero, where $(kh)_{cf}$ is the cut-off value of the dimensionless wavenumber. As mentioned above, the $(kh)_{cf}$ occurs in the pre-compressed cases of the plate and increases with decreasing parameter λ_1. The results show that the influence of the ratio h_d/h on the values of $(kh)_{cf}$ is insignificant.

Above we analyzed the dispersion curves in the context of the asymmetric mode of quasi-Scholte waves. Now we analyze the dispersion curves in relation to the symmetric mode of the quasi-Scholte waves and, first of all, based on the results in Fig. 5, we can state that the character of the influence of the initial strains of the plate on these dispersion curves depends on the dimensionless wavenumber and the ratio h_d/h. That is, there is such a value of the dimensionless wavenumber kh (denote it by $(kh)*$), before which the initial stretching of the plate leads to a decrease, but the initial compression of the plate leads to an increase of the wave propagation velocity. However, in the cases where $kh > (kh)*$, the initial stretching (compression) leads to an increase (decrease) in the propagation velocity in the symmetric mode of the quasi-Scholte waves. From the results, it can be seen that the values of $(kh)*$ depend on the parameter λ_1 and the ratio h_d/h. Moreover, from the analysis of the graphs in Fig. 5, it can be seen that for relatively small values of the ratio h_d/h (e.g., at $h_d/h = 0.1$), there is no any cut-off wavenumber on the dispersion curves related to the symmetric mode. However, for relatively large values of the ratio h_d/h (e.g., at $h_d/h = 0.25$), the initial compression of the plate causes the cut-off wavenumber to appear, and the values of the cut-off wavenumbers increase as the parameter λ_1 is decreased. In the cases where $h_d/h = 0.25$ and 2.0, the cut-off wavenumber exists on the analyzed dispersion curves even without initial compression of the plate. From the corresponding numerical results, it is clear that in these cases the initial compression (stretching) of the plate leads to a decrease (increase) of the cut-off wavenumbers.

Note that the propagation velocity of waves in symmetric mode has the finite linit as $kh \to 0$ or as $kh \to (kh)_{cf}$ and these limits depend on the parameter λ_1. This dependence can be illustrated with the help of Fig. 5. For example, from the analysis of the results shown in Fig. 5a, it can be seen that in the case $h_d/h = 0.1$, an increase in the values of the parameter λ_1 leads to a decrease in the values of the limiting velocities mentioned above. Similarly, concrete conclusions can be drawn about the influence of the parameter λ_1 on the aforementioned limiting velocities in the cases $h_d/h = 0.25$ (Fig. 5b), 0.5 (Fig. 5c) and 2.0 (Fig. 5d).

The results also show that the limit values of the propagation velocity of the symmetric waves under $kh \to \infty$ approach from above the corresponding propagation velocity of the Scholte waves for the pair "pre-strained Lucity + Water".

This completes the analyses of the numerical results.

5 Conclusions

Thus, in the present work, the influence of the initial axisymmetric homogeneous finite strains of a plate of highly elastic material, in contact on one side with the water layer and on the other side with the Glycerin layer, on the dispersion of the axisymmetric waves propagating therein is studied. It is assumed that the flow of fluids in these layers is restricted by the upper and lower rigid walls. It is also assumed that the material of the plate is Lucite and the motion of the plate is described by the three-dimensional linearized equations and relations of the theory of elastic waves in prestressed bodies. However, the fluid flow is described by the linearized Euler equations for compressible, non-viscous fluids. The dispersion curves for quasi-Scholte waves are presented and discussed. In the context of this discussion, the main focus is on the influence of the difference between the fluids contacting the plate on the upper and lower planes on the dispersion of the studied waves. In particular, it is found that the limits of the propagation velocities of the asymmetric and symmetric quasi-Scholte waves at high wavenumbers differ from each other due to the mentioned differences between the fluids.

Moreover, the influence of the initial strains in the plate on the dispersion curves of the quasi-Scholte waves is analyzed. In this regard, it is found that the initial stretching (compression) of the plate leads to an increase (decrease) in the propagation velocity of the asymmetric quasi-Scholte waves. At the same time, it is found that the influence of the initial strains of the plate on the propagation velocity of the symmetric quasi-Scholte waves has a more complicated character and depends on the dimensionless wavenumber and the fluid layer thickness (i.e., h_d/hratio).

The numerical results show that the limits of the propagation velocity of the assymmetric (symmetric) waves at high wavenumbers from below (above) approach the propagation velocity of the Scholte waves for the pair "pre-strained plate of Lucite +Glycerin" ("pre-strained plate of Lucite +Water").

The other and more concrete conclusions about the influence of the problem parameters on the dispersion of the studied waves can be found in the text of the paper.

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