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Boundary estimates of solutions mixed boundary problems for elliptic equations with VMO coefficients

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Abstract. *In this paper we obtain generalized weighted Sobolev – Morrey estimates with weights from the Muckenhoupt class A_p by establishing boundedness of several important operators in harmonic analysis such as Hardy – Littlewood operators and Calderon –Zygmund singular integral operators in generalized weighted Morrey spaces. As a consequence, a priori estimates for the weak solutions mixed boundary problem uniformly elliptic equations of higher order in generalized weighted Sobolev – Morrey spaces in a smooth bounded domain $\Omega \subset R^n$ are obtained.*

Keywords. elliptic equation · mixed boundary problems · Morrey spaces · parabolic operators

Mathematics Subject Classification (2010): 35J40

1 Introduction

Later we define weighted Morrey spaces $L_{p,k}(\omega)$. Before Guliyev [14] give a concept of the generalized weighted Morrey spaces $M_{p,\varphi}(\omega)$ which could be viewed as extension of both $M_{p,\varphi}$ and $L_{p,k}(\omega)$, study the boundedness of the classical operators and their commutators in spaces $M_{p,\varphi}(\omega)$ was studied (see, also [17, 23]).

The reason to study continuity properties of these integrals in various functional spaces is that they permit to investigate the regularity of solutions to linear elliptic and parabolic partial differential equations and systems in terms of the data of the corresponding problems. The method, associated to the names of A. Calderon and A. Zygmund (see [3, 4]) uses explicit representation formula for the highest-order derivatives of the solution in terms of singular integrals acting on the known right-hand side plus another one acting on the very same derivatives. This last term appears in a commutator which norm can be made small enough if the coefficients have small oscillation over small balls. This way, suitable "integral continuity" of the principal coefficients ensure boundness of the commutator and therefore

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validity of the corresponding a priori estimate. The Sarason class of functions with vanishing mean oscillation verifies this requirement although they could be discontinuous. Their good behavior on small balls allows to extend the classical theory of elliptic and parabolic equations and systems with continuous coefficients to operators with discontinuous coefficients (see [5, 6]). A vast number of works are dedicated to boundary value problems for linear elliptic and parabolic operators with *VMO* coefficients in the framework of Sobolev and Sobolev-Morrey spaces (see [8,9, 15, 16,19 20, 21, 25]).

Later these results are extended on the generalized weighted Morrey spaces, which is obtained the boundness of the Calderon-Zygmund operators from one generalized weighted Morrey space $M_{p,\varphi_1}(\omega)$ to another $M_{p,\varphi_2}(\omega)$ (see [15, 19]), if the pair functions (φ_1,φ_2) satisfy the following condition

$$\int_r^\infty \frac{\underset{t<s<\infty}{ess\inf}\ \varphi_1(x,s)\omega(B(x,s))^{\frac{1}{p}}}{\omega(B(x,s))^{\frac{1}{p}}}\ \frac{dt}{t} \le C\varphi_2(x,r)\,, \tag{1.1}$$

where C does not depend on x and r.

Let $W_p^{2m}(\omega)$ be the standard notation for Sobolev spaces. In [1] for the solutions of uniformly elliptic equations in a smooth domain ω the following a priori estimate

$$\|u\|_{W_p^{2m}(\Omega)} \le C\|f\|_{L_p(\Omega)} \tag{1.2}$$

were obtained. In [24] on a bounded domain ω with smooth boundary $\partial\omega$ for the Laplace equation with weight be $\omega(x)$ belonging to the Muckenhoupt class A_p (see [5]) was proved the following a priori estimate

$$\|u\|_{W_p^2(\Omega,\omega)} \le C\,\|f\|_{L_p(\Omega,\omega)}$$

Weighted estimates for a wide class of singular integral operators has been obtained for weights in the class of Muckenhoupt A_p. Therefore, it is a natural question whether analogous weighted a priori estimates can be proved for the derivativies of solutions elliptic equations. In [10] the previous results of [5] (also [11-13]) for powers of the Laplacian operator with homogeneous Drichlet boundary conditions were extended to weighted Sobolev spaces, i.e.,it is proved that

$$\|u\|_{W_p^{2m}(\Omega,\omega)} \le C\,\|f\|_{L_p(\Omega,\omega).}$$

For $\omega \in A_p$, where the constant C depends on Ω, m, n, ω.

In [19, 22], Guliyev, Gadjiev and Galandarova study the boundedness of the sublinear operators generated by Calderon-Zygmund operators in local generalized Morrey spaces. By using these results they prove the solvability of the Dirichlet boundary value problem for a polyharmonic equation in modified local generalized Sobolev-Morrey spaces and obtain a priori estimates for the solutions of the Dirichlet boundary value problems for the uniformly elliptic equations in modified local generalized Sobolev-Morrey spaces defined on bounded smooth domains.

Let Ω be open set in $\mathbb{R}^n$ and $a(\cdot) \in L^1_{loc}(\Omega)$. We say

That $a(\cdot) \in BMO$ (bounded mean oscillation) if

$$\|a\|* = \underset{x\in\Omega,p>0}{\sup} \frac{1}{|\Omega(x,p)|} \int_{\Omega(x,p)} \left|a(y) - a_{\Omega(x,p)}\right| dy < \infty,$$

where $a_Q = \frac{1}{|Q|}\int_Q a(y)\,dy$ is the mean integral of $a(\cdot)$. The quantity $\|a\|*$ is a norm in BMO of function $a(\cdot)$ and BMO is a Banach space.

We say that $a(\cdot) \in VMO(\Omega)$ (vanishing mean oscillation) if $a \in BMO(\Omega)$ and $r > 0$ define

$$\eta(r) = \sup_{x \in \Omega, p \leq r} \frac{1}{|\Omega(x,p)|} \int_{\Omega(x,p)} \left|a(y) - a_{\Omega(x,p)}\right| dy < \infty,$$

and

$$\lim_{r \to 0} \eta(r) = \lim_{r \to 0} \sup_{x \in \Omega, p \leq r} \frac{1}{|\Omega(x,p)|} \int_{\Omega(x,p)} \left|a(y) - a_{\Omega(x,p)}\right| dy = 0.$$

The quantity $\eta(r)$ is called VMO - modulus of a.

We formulate the problem (4.1) again. We consider mixed problem for linear nondivergent equation order $2m$

$$Lu(x) = \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta}(x) D^\alpha D^\beta u(x) = f(x), \qquad x \in \Omega, \tag{1.3}$$

$$W^{2m}_{M_{p,f,o}}(\Omega, \omega) \cap W^m_0 M_{p,f,o}(\Omega, \varphi, o), \qquad p \in (1, \infty)$$

subject to the following conditions: there exists a constant $\lambda > 0$ such that

$$\lambda^{-1}|\xi|^{2m} \leq \sum_{|\alpha|,|\beta| \leq m} a_{\alpha,\beta}\xi_\alpha\xi_\beta \leq \lambda|\xi|^{2m} \tag{1.4}$$

$$a_{\alpha,\beta}(x) = a_{\beta,\alpha}(x), \quad |\alpha|, |\beta| \leq m,$$

i.d. the operator L uniform elliptic. The last assumption implies immediately essential boundedness of the coefficients $a_{\alpha,\beta} \in L_\infty(\Omega)$ and $a_{\alpha,\beta} \in VMO(\Omega)$, $f \in M_{p,f}(\Omega, \omega)$ with $1 < p < \infty$, $f : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is measurable.

To prove a local boundary estimate for the norm $D^\alpha D^\beta u$ we define the space $W^{2m,\gamma_0}_{p,\omega}(B_r^+)$ as a closure of $C_{\gamma_0} = \left\{u \in C_0^\infty\left(B\left(x^0, r\right)\right) : D^\alpha u(x) = 0 \text{ for } x_n \leq 0\right\}$ with respect to the norm of $W^{2m}_{p,\omega}$.

Theorem 1.1 *(Boundary Estimate). Suppose that* $u \in W^{2m,\gamma 0}_{p,\omega,o}(B_r^+)$ *and* $Lu \in M_{p,f,o}(B_r^+\omega)$, $|\alpha|, |\beta| \leq m$ *and for each e¿0 there exists* $r_0(e)$ *such that*

$$\left\|D^\alpha D^\beta u\right\| M_{p,f,o}\left(B_r^+\omega\right) \leq C \|Lu\| M_{p,f,o}\left(B_r^+\omega\right) \tag{1.5}$$

for any $r \in (0, r_0)$.

Proof. For $u \in W^{2m,\gamma 0}_{p,\omega,o}(B_r^+)$ the boundary representation formula holds (see [19])

$$D^\alpha D^\beta u(x) = P.V. \int_{B_r^+} D^\alpha D^\beta \Gamma(x, x-y) Lu(y) dy$$

$$+P.V. \int_{B_r^+} D^\alpha D^\beta \Gamma(x, x-y) \left[a_{\alpha\beta}(x) - a_{\alpha\beta}(y)\right] D^\alpha D^\beta u(y) dy \tag{1.6}$$

$$+Lu(x) \int_{S^{n-1}} D^a \Gamma(x,y) y_i ds_y + I_{\alpha\beta}(x),$$

$\forall i = \overline{1,n},\ |a|, |\beta| \leq m,$ where we have set

$$I_{a\beta}(x) = \int_{B_r^+} D^a D^\beta \Gamma(x, T(x) - y) Lu(y) dy$$

$$+\int_{B_r^+} D^a D^\beta \Gamma\left(x,T(x)-y\right)\left[a_{\alpha\beta}\left(x\right)-a_{a\beta}\left(y\right)\right]\ D^\alpha D^\beta u\left(y\right)dy,$$

$$\left|\alpha\right|,\left|\beta\right|\leq m-1,$$

$$I_{\alpha,m}\left(x\right)=I_{m,\alpha}\left(x\right)=\int_{B_r^+} D^\alpha D^\beta \Gamma\left(x,T(x)-y\right)\left(D^m T(x)\right)^l$$

$$\times\left\{\left[a_{\alpha\beta}\left(x\right)-a_{\alpha\beta}(y)\right]D^\alpha D^\beta u\left(y\right)+Lu(y)\right\}dy,$$

$$I_{mm}\left(x\right)=\int_{B_r^+} D^\alpha D^\beta \Gamma\left(x,T(x)-y\right)\left(D^m T(x)\right)^l\left(D^m T(x)\right)^s$$

$$\times\left\{\left[a_{\alpha\beta}\left(x\right)-a_{\alpha\beta}(y)\right]D^\alpha D^\beta u\left(y\right)+Lu(y)\right\}dy,$$

Where $D^m T\left(x\right)=\left(\left(D_m T\left(x\right)\right)^1,\ldots\ ,\ \left(D_m T\left(x\right)\right)^n\right)=T\left(l_n,x\right).$ Now we give some estimates singular and nonsingular integral operators.

The singular integral

$$Rf\left(x\right)=P.V.\int_{\mathbb{R}}^n K\left(x,x-y\right)f(y)dy$$

and its commutators

$$\left[a,R\right]f\left(x\right)P.V.\int_{\mathbb{R}}^n K\left(x,x-y\right)f\left(y\right)\left[a\left(x\right)-a\left(y\right)\right]dy=a\left(x\right)Rf\left(x\right)-R(af)(x)$$

are bounded in $L_p\left(\mathbb{R}^n\right)$ (see [7]). Moreover

$$|K(x,\xi)|\leq\Big|\xi|^{-n}\Big|K\left(x,\frac{\xi}{|\xi|}\right)\Big|\leq M\Big|\xi|^{-n}.$$

Then we have

$$|Rf(x)|\leq C\int_{\mathbb{R}}^n\frac{|f\left(y\right)|}{|x-y|^n}\,dy,$$

$$|\left[a,R\right]f(x)|\leq C\int_{\mathbb{R}}^n\frac{|a\left(x\right)-a(y)|\,|f(y)|}{|x-y|^n}\,dy$$

where the constants C are independent of f.

Lemma 1.1 *Let the function $\varphi:\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}_+$ satisfy the condition and $1<p<\infty$. Then for any $f\in M_{p,f}\left(\omega\right)$ and $a\in BMO$ there exist constants depending on $n,\ p,\varphi$ and the kernel such that*

$$\|Rf\|_{M_{p,\varphi}(\omega)}\leq C\,\|f\|_{M_{p,f}(\omega)},$$

$$\|\left[a,R\right]f\|_{M_{p,f}(\omega)}\leq C\,\|a\|*\|f\|_{M_{p,f}(\omega)},$$

where constants are independent of f.

For studying regularity properties of the solution of the Dirichlet problem (1.3) we need also of some additional local results.

Lemma 1.2 *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain and $a\in BMO\left(\Omega\right)$. Suppose the function $f:\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}_+$ satisfy the condition and $f\in M_{p,f}\left(\Omega,\omega\right)$ with $1<p<\infty$. Then*

$$\|Rf\|_{M_{p,f}(\Omega,\omega)}\leq C\,\|f\|_{M_{p,f}(\Omega,\omega)},$$

$$\|\left[a,R\right]f\|_{M_{p,f}(\Omega,\omega)}\leq C\,\|a\|*\|f\|_{M_{p,f}(\Omega,\omega)},\tag{1.7}$$

Where $C=C\left(n,p,f,\ \Omega,\ K\right)$ is independent of f.

Lemma 1.3 *Let the conditions of Lemma 1.1 satisfy and* $a \in VMO\left(\mathbb{R}_{+}^{n}\right)$ *with VMO-modulus* a. *Then for any* $\varepsilon > 0$ *there exists a positive number* $\rho_0 = \rho_0(\varepsilon, \gamma_\alpha)$ *such that for any ball* B_r *with a radius* $r \in (0, \rho_0)$ *and all* $f \in M_{p,f}(B_r, \omega)$ *the following inequality holds*

$$\|[a, R] f\| M_{p,f\left(B_r^{+},\omega\right)} \leq Ce \|f\| M_{p,f\left(B_r^{+},\omega\right)} \tag{1.8}$$

with $C = C(n, p, f, \Omega, K)$ *independent of* f.

To obtain above estimates it is sufficient to extend $K(x, \cdot)$ and $f(\cdot)$ as zero outside Ω. This extension keeps its BMO norm or VMO modulus according to [15].

For any $x, y \in \mathbb{R}_{+}^{n}$, $\tilde{x} = \left(x', -x_n\right)$ define the generalized reflection $T(x, y)$ as

$$T(x, y) = x - 2x_n \frac{a_{\alpha\beta}^{n}(y)}{a_{\alpha\beta}^{nn}(y)} \ ,$$

$$T(x) = T(x, x) : \mathbb{R}_{+}^{n} \to \mathbb{R}_{-}^{n} \ ,$$

where $a_{\alpha\beta}^{n}$ is the last row of the coefficients matrix $(a_{\alpha\beta})_{\alpha\beta}$. Then there exist positive a constant C depending on n and A, such that

$$C^{-1} |\tilde{x} - y| \leq |T(x)| \leq C |\tilde{x} - y| , \qquad \forall x, y \in \mathbb{R}_{+}^{n} .$$

For any $f \in M_{p,f}\left(\mathbb{R}_{+}^{n}, \omega\right)$ and $a \in BMO\left(\mathbb{R}_{+}^{n}\right)$ consider the nonsingular integral operators

$$\tilde{R}f(x) = \int_{\mathbb{R}_{+}^{n}} K(x, T(x) - y) f(y) \, dy,$$

$$\left[a, \ \tilde{R}\right] f(x) = a(x) \widetilde{R} f(x) - \tilde{R}(af)(x) .$$

The kernel $K(x, T(x) - y) : \mathbb{R}^{n} \times \mathbb{R}_{+}^{n} \to \mathbb{R}$ is not singular and verifies the conditions $1 - b$ and 2 from Calderon-Zygmund kernel. Moreover

$$K(x, T(x) - y) \leq M \left|T(x) - y\right|^{-n} \leq C \left| \tilde{x} - y\right|^{-n}$$

Implies

$$\left|\tilde{R}f(x)\right| \leq C \int_{\mathbb{R}_{+}^{n}} \frac{|f(y)|}{|\tilde{x} - y|^{n}} \, dy,$$

$$\left|\left[a, \ \tilde{R}\right] f(x)\right| \leq C \int_{\mathbb{R}_{+}^{n}} \frac{|a(x) - a(y)| \, |f(y)|}{|\tilde{x} - y|^{n}} \, dy$$

where constants C are independent of f.

The following estimates are simple consequence of the previous results.

Lemma 1.4 *Let* φ *be measurable function satisfying condition* (1.7) *and* $a \in BMO(\Omega)$, $p \in (1, \infty)$. *Then the operator* $\widetilde{R} f$ *and* $\left[a, \ \tilde{R}\right] f$ *are continuous in* $M_{p,f}\left(\mathbb{R}_{+}^{n}, \omega\right)$ *and for all* $f \in M_{p,f}\left(\mathbb{R}_{+}^{n}, \omega\right)$ *the following holds*

$$\left\|\widetilde{R} f \ \right\| M_{p,f}\left(\mathbb{R}_{+}^{n}, \omega\right) \leq C \|f\| M_{p,f}\left(\mathbb{R}_{+}^{n} , \omega\right) ,$$

$$\left\|\left[a, \tilde{R}\right] f\right\| M_{p,f}\left(\mathbb{R}_{+}^{n} , \omega\right) \leq C \|a\| * \|f\| \ M_{p,f}\left(\mathbb{R}_{+}^{n} , \omega\right) . \tag{1.9}$$

where constants C *are dependent on known quantities only.*

Lemma 1.5 *Let f be measurable function satisfying condition* (1.5), $\alpha \in VMO\left(\mathbb{R}_+^n\right)$ *with* VMO*-modulus* γ_α *and* $p \in (1,\infty)$. *Then for any* $\varepsilon > 0$ *there exists a positive number* $\rho_0 = \rho_0(\varepsilon, \gamma_\alpha)$ *such that for any ball* B_r^+ *with a radius* $r \in (0,\rho_0)$ *and all* $f \in M_{p,f}(B_r^+,\omega)$ *the following holds*

$$\left\|\left[a,\ \tilde{R}\right]f\right\|_{M_{p,f}\left(B_r^+,\omega\right)} \le Ce\|f\|_{M_{p,f}\left(B_r^+,\omega\right)} \tag{1.10}$$

where C is independent of $\varepsilon,\ f$ *and* r.

The proof is as [15].

Taking into account the VMO properties of the coefficients $a_{\alpha\beta}$, it is possible to choose r_0 so small that

$$\left\|D^\alpha D^\beta u\right\|_{M_{p,f,o}\left(B_r^+,\omega\right)} \le C\|Lu\|_{M_{p,f,o}\left(B_r^+,\omega\right)}$$

for each $r < r_0$. For arbitrary matrix function $w = \{w_{ij}\}_{i,j=1}^n \in M_{p,f}[(B_r^+,\omega)]^{n^2}$ define

$$S_{ij\alpha\beta}(w_{\alpha\beta})(x) = [a_{\alpha\beta}, B_{ij}]\, w_{\alpha\beta}(x), \qquad i,j = \overline{1,n}, |\alpha| \le m, |\beta| \le m,$$

$$\tilde{S}_{ij\alpha\beta}(w_{\alpha\beta})(x) = \left[a_{\alpha\beta}, \tilde{B}_{ij}\right] w_{\alpha\beta}(x), \qquad i,j = \overline{1,n-1}, |\alpha| \le m, |\beta| \le m,$$

$$\tilde{S}_{in\alpha\beta}(w_{\alpha\beta})(x) = \left[a_{\alpha\beta}, \tilde{B}_{ij}\right] w_{\alpha\beta}(D_nT(x))^l, \qquad i,j = \overline{1,n}, |\alpha| \le m, |\beta| \le m,$$

$$\tilde{S}_{nn\alpha\beta}(w_{\alpha\beta})(x) = \left[a_{\alpha\beta}, \tilde{B}_{ls}\right] w_{\alpha\beta}(D_nT(x))^l(D_nT(x))^s, \qquad |a| \le m, |\beta| \le m,$$

From (1.8) and (1.10) we can take r so small that

$$\sum_{i,j=1}^{n} \sum_{|\alpha|,|\beta|\le m} \left\|S_{ij\alpha\beta} + \tilde{S}_{ij\alpha\beta}\right\| < 1. \tag{1.11}$$

Now given $u \in W_{p,\omega}^{2m,\omega_0}(B_r^+)$ with $Lu \in M_{p,f}(B_r^+,\omega)$ we set

$$\tilde{H}(x) = RLu(x) + \tilde{R}Lu(x) + \tilde{R}Lu(x)(D_nT(x))^l +$$

$$+\tilde{R}Lu(x)(D_nT(x))^l(D_nT(x))^s + Lu(x)\int_{S^{n-1}} D^\alpha\Gamma(x,y)\, y_i, d\sigma_y.$$

Then estimates (1.7) and (1.9) imply $\tilde{H} \in M_{p,f}(B_r^+,\omega)$. Define the operator

$$Uw = \left\{\sum_{|\alpha|,|\beta\le m|} \left(S_{ij\alpha\beta}(W_{\alpha\beta}) + \tilde{S}_{ij\alpha\beta}(W_{\alpha\beta}) + \tilde{H}_{ij}(x)\right)\right\}_{i,j=1}^n$$

By virtue of (1.11) it is a contraction mapping in $[M_{p,f}(B_r^+,\omega)]^{n^2}$ and there is a unique fixed point $\tilde{w} = \{\tilde{w}_{\alpha\beta}\}_{|\alpha|,|\beta|\le m}^n$ such that $U\tilde{w} = \tilde{w}$. On the other hand, it follows from the representation formula (1.6) that also $D^\alpha\, D^\beta u = \{D^\alpha D^\beta u\}_{|\alpha|,|\beta|\le m}$ is a fixed point of U. Hence $D^\alpha\, D^\beta u = \tilde{w},\ \ D^\alpha\, D^\beta u \in M_{p,\omega}(B_r^+)$ and estimate (1.5) holds. Thus, theorem is proved.

Theorem 1.2 *Let operator L in problem* (1.1) *be uniformly elliptic and $a_{\alpha\beta} \in VMO\,(\Omega)$. Then for any function $f \in M_{p,f}\,(\Omega, \omega)$ the unique solution of the problem* (1.3) *has $2m$ derivatives in $M_{p,f}\,(\Omega, \omega)$. Moreover*

$$\left\| \sum_{|\alpha|,|\beta| \le m} D^\alpha D^\beta u \right\|_{M_{p,f,o}(\Omega,\omega)} \le C \left(\|u\|_{M_{p,f,o}(\Omega,\omega)} + \|f\|_{M_{p,f,o}(\Omega,\omega)} \right) \tag{1.12}$$

with the constant C depends on known quantities.

Proof. Since $M_{p,f}\,(\Omega, \omega) \subset L_{p,\omega}\,(\Omega)$ the problem (1.1) is uniquely solvable in the Sobolev space $W^{2m}_{p,\omega}\,(\Omega) \cap W^{m}_{p,\omega,o}\,(\Omega)$ according to [2] and [7]. By local attering of the boundary, covering with semi-balls, taking a partition of unity subordinated to that covering and applying of estimate (1.5) we get a boundary a priori estimate validity of (1.12)

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