Tran. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mechanics, **43** (7), 60-67 (2023);
https://doi.org/10.30546/2706-7734.43.7.2023.60

# Free vibrations of a nonhomogeneous rod-cylindrical shell-fluid system

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Received: 12.04.2023 / Revised: 05.05.2023 / Accepted: 02.06.2023

**Abstract.** *In the present paper, we consider free vibrations of an inhomogeneous anisotropic, fluid-contacting cylindrical shell stiffened with inhomogeneous rods. The Hamilton-Ostogradsky variational principle was used when solving the problem. It was accepted that the nonhomogeneity of rods used in the strengthening change by the exponential law. The nonhomogenity of the cylindrical shell change by the linear law in the direction of the thickness. The fluid was accepted as ideal. Rigid contact condition between the rods and the cylindrical shell was considered. Using the contact conditions, the frequency equation was structured, the roots were found implemented by the numerical method, characteristical curves were built.*

**Keywords.** nonhomogeneous, cylindrical shell · fluid · frequency · free vibrations

**Mathematics Subject Classification (2010):** 74H45, 74K25

## 1 Introduction

Studying constructions and structural elements under the dynamic action with taking into account their adequate features is of great importance. Depending on mechanical and thermal processing, the type of technology, composition of the material, the homogeneity and anisotropy features are created in the material of the structure. On the other hand, such constructions are in contact with different nature media. In many cases, there arises a need to stiffen such thin-walled structures for increasing their serviceability. In [7], free vibrations of an orthotropic homogeneous cylindrical shell contacting with viscous fluid and soil and stiffened with homogeneous rings, are studied. The papers [5, 6] study vibrations of a sharp nonhomogeneity feature cylindrical shell. In [2] the vibrations of a fluid-contacting inhomogeneous cylindrical shell stiffened with homogeneous rods, are studied. [1] studies vibrations of a nonhomogeneous cylindrical shell dynamically contacting with fluid and

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stiffened with homogeneous rods. The conducted analyses show that vibrations of a nonhomogeneous cylindrical shell dynamically contacting with moving fluid and stiffened with inhomogeneous rods have not been studied.

## 2 Problem statement

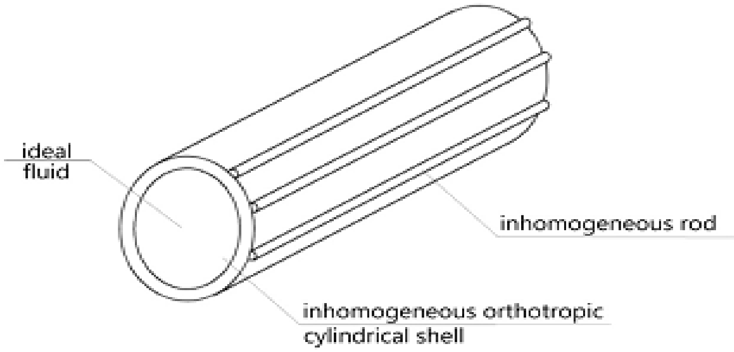


**Fig 2.1.** Inhomogeneous, fluid-contacting orthotropic cylindrical shell stiffened with inhomogeneous rods

Assume that ideal fluid-contacting cylindrical shell inhomogeneous along its generatrix has been stiffened with inhomogeneous rods (Fig. 2.1). According to the Hamilton-Ostogradsky variation principle:

$$\delta \int_{t_0}^{t_1} (K - W - A_m)\, dt = 0 \tag{2.1}$$

$$K = V_k + \sum_{i=1}^{k_1} K_i; W = V_p + \sum_{i=1}^{k_1} \Pi_i \tag{2.2}$$

Here $V_p$, $V_k$ are potential and kinetic energies of the cylindrical shell, $\Pi_i$ , $K_i$ are potential and kinetic energies of the *i*-th rod, $A_m$ is the work done by the forces acting on the cylindrical shell as viewed from fluid at the displacement points of the shell. For taking into account the homogeneity of the cylindrical shell and rods, we will assume that their elasticity module and densites is a coordinate function. In this case, the expressions for their enerjies will be as follows:

$$V_p = \frac{Rh}{2} \iint (\sigma_{11}\varepsilon_{11} + \sigma_{12}\varepsilon_{12} + \sigma_{22}\varepsilon_{22}) ds \tag{2.3}$$

$$\sigma_{11} = b_{11}(x)\,\varepsilon_{11} + b_{12}(x)\,\varepsilon_{22},\ \ \sigma_{22} = b_{12}(x)\,\varepsilon_{11} + b_{22}(x)\,\varepsilon_{22}, \sigma_{12} = b_{66}(x)\,\varepsilon_{12}$$

$$\varepsilon_{11} = \frac{\partial \varphi}{\partial x};;\ \varepsilon_{22} = \frac{\partial v}{\partial y};;\ \varepsilon_{12} = \frac{\partial U}{\partial y} + \frac{\partial v}{\partial x}$$

$$\tilde{b}_{11} = \int_0^l b_{11}(x)\, dx; \tilde{b}_{12} = \int_0^l b_{12}(x)\, dx; \tilde{b}_{22} = \int_0^l b_{22}(x)\, dx;\ ; \tilde{b}_{66} = \int_0^l b_{66}(x)\, dx;$$

$$b_{11}(x) = \frac{E_1(x)}{1-\nu_1\nu_2}; b_{22}(x) = \frac{E_2(x)}{1-\nu_1\nu_2}; b_{66}(x) = G_{12}(x) = G(x);$$

$$b_{12}(x) = \frac{\nu_2 E_1(x)}{1-\nu_1\nu_2} = \frac{\nu_1 E_2(x)}{1-\nu_1\nu_2}$$

$$V_k=\int_0^{2\pi}\int_0^l \rho\left(x\right)\left(\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right)dxd\varphi$$

$$\Pi_i=\frac{1}{2}\int_0^l\left[\tilde{E}_i\left(x\right)F_i\left(\frac{\partial u_i}{\partial x}\right)^2+\tilde{E}_i\left(x\right)J_{xi}\left(\frac{\partial^2 v_i}{\partial x^2}\right)^2+\right.$$

$$\left.+\tilde{E}_i\left(x\right)J_{zi}\left(\frac{\partial^2 w_i}{\partial x^2}\right)^2+G_i\left(x\right)J_{kpi}\left(\frac{\partial \varphi_{kpi}}{\partial x}\right)^2\right]dx \tag{2.4}$$

$$K_i=\int_0^l\tilde{\rho}_i\left(x\right)F_i\left[\left(\frac{\partial u_i}{\partial t}\right)^2+\left(\frac{\partial v_i}{\partial t}\right)^2+\left(\frac{\partial w_i}{\partial t}\right)^2+\frac{J_{kpi}}{F_i}\left(\frac{\partial \varphi_{kpi}}{\partial t}\right)^2\right]dx$$

$$A_m=-\int_0^l\int_0^{2\pi}q_z w dxd\varphi \tag{2.5}$$

In the expressions (2.3) - (2.5), $u, v, w$ are the displacements of the points of the cylindrical shell, $\tilde{E}_i(x)$ is the modules of elasticity of the *i*-th rod, $\tilde{\rho}_i(x)$ is the density of the material of the *i*-th rod, $F_i$ is the area of the cross-section of the *i*-th rod, $I_{xi,}I_{kpi}$ are the enertia moments of the cross section of the *i*-th rod,$G_i(x)$ is the elasticity modulus of the *i*-th rod in shear, $u_i, v_i,\ \ w_i$ are displeacements of the points of the *i*-th rod, $k_1$ is the amount of rods, $q_z$ is pressure force to the cylindrical shell as viewed from fluid.

Under the cylindrical shell strengthened with rods we understand a cylindrical shell and a system consisting of rods rigidly strengthened to it along the coordinate lines. It is considered that the coordinate axes coincide with the principal curvature lines of the cylindrical shell and are in rigid contact along these lines. So, the following conditions between the cylindrical shell and rods are satisfied [3]:

$$u_i\left(x\right)=u\left(x,y_i\right),\ v_i\left(x\right)=w\left(x,y_i\right)w_i\left(x\right)=w\left(x,y_i\right) \tag{2.6}$$

$$\varphi_i\left(x\right)=\varphi_1\left(x,y_i\right);\varphi_1\left(x,y_i\right)=-\left.\frac{\partial w}{\partial x}\right|_{y=y_i}$$

The pressure *p* created in fluid is in the form of the following expression [8]:

$$p=\Phi_{\alpha n}\rho_m\left(\omega_0^2\frac{\partial^2 w}{\partial t_1^2}+2U\omega_0\frac{\partial^2 w}{R\partial\xi\partial t_1}+U^2\frac{\partial^2 w}{R^2\partial\xi^2}\right) \tag{2.7}$$

Here

$$\Phi_{mk}=\begin{cases} I_k\left(\beta r\right)/I_k'\left(\beta R\right), & M_1<1\\ J_k\left(\beta_1 r\right)/J_k'\left(\beta_1 R\right), & M_1>1\\ \frac{r^k}{kR^{k-1}}, & M_1=1\end{cases},$$

$$M_1=\frac{U+R\omega/m}{a_0},\beta^2=R^{-2}\left(1-M_1^2\right)m^2,\beta_1^2=R^{-2}\left(M_1^2-1\right)m^2,$$

$I_k$ is the *k*-th order modified first kind Bessel function, $J_k$ is the *k*-th order first kind Bessel function, $a_0$ is sound propagation in fluid, $U$ is motion speed of fluid, $\xi=\frac{x}{l}$, $m$ is the wave number in the direction of the axis $x$.

The following contact conditions between the fluid and cylindrical shell are satisfied [8]:

$$v_r|_{r=R}=-\left(\omega\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}\right) \tag{2.8}$$

$$q_z=-p_{|r=R} \tag{2.9}$$

We will consider that the Navier conditions are satisfied at the edges of the cylindrical shell [4]:

$$v = w = M_{11} = N_{11} = 0;\ \text{for}\ (x = 0, x = l) \tag{2.10}$$

So, the solution of the problem of vibrations of a cylindrical shell dynamically contacting with fluid and strengthened with rods is reduced to joint integration of the of total energy (2.6), (2.8), (2.9) of the construction consisting of a cylindrical shell with flowing fluid in the inner area and strengthened with discretely distributed inhomogeneous rods under the boundary conditions (2.10).

## 3 Problem solution

We will look for the displacements of the shell in the following form:

$$\begin{aligned} u &= u_0 cos\frac{\pi m x}{l}\sin k\varphi \sin\omega t \\ v &= v_0 \sin\frac{\pi m x}{l}\cos k\varphi \sin\omega t \\ w &= w_0 \sin\frac{\pi m x}{l}\sin k\varphi \sin\omega t \end{aligned} \tag{3.1}$$

Here $u_0, v_0, w_0$ are unknown constants, $m$ , $n$ are wave numbers in the direction of the generatrix and in the circular direction.

Using solutions (3.1), formulas (2.3) and (2.4), contact conditions (2.6), we obtain:

$$\begin{gathered} \Pi_i = \frac{m^2\pi^2}{2l^2}\left[F_i I_1\ sin^2 k\varphi_i u_0^2 + (J_{xi}I_2 + J_{kpi}I_3)cos^2 k f_i v_0^2 + \right. \\ \left. +(J_{zi}I_2 + J_{kpi}I_3)sin^2 k f_i w_0^2 + kJ_{kpi}\ I_3 sin 2k\varphi_i v_0 w_0\right] sin^2\omega t \\ K_i = \omega^2 F_i\left[I_{10} sin^2 k\varphi_i u_0^2 + I_{11}\left(1 + \frac{J_{kpi}}{F_i R^2}\right)cos^2 k\varphi_i v_0^2 + \right. \\ \left. +I_{11}\left(1 + \frac{J_{kpi}k^2}{F_i R^2}\right)sin^2 k\varphi_i \omega_0^2 + I_{11}\ \frac{J_{kpi}}{F_i R^2} sin 2k\varphi_i v_0 w_0\right] sin^2\omega t \\ V_p = \frac{\pi R h}{2}\left[\left(\frac{\pi^2 m^2}{e^2}I_4 + \frac{k^2}{R^2}I_5\right)u_0^2 + \left(\frac{k^2}{R^2}I_6 + \frac{\pi^2 m^2}{e^2}I_5\right)v_0^2 + I_5 w_0^2 + \right. \\ \left. + \left(\frac{2\pi k m}{lR}I_7 + \frac{2\pi k m}{lR}I_5\right)u_0 v_0 - \frac{2\pi m}{l}I_7 u_0 w_0 - \frac{2k}{R}I_6 v_0 w_0\right] sin^2\omega t \\ V_k = \omega^2\pi\left(I_8 u_0^2 + I_9 v_0^2 + I_8 w_0^2\right) sin^2\omega t \\ A = -\frac{\pi l}{2R}\rho_m \Phi_{mn}\left(-\omega^2 + 2Um\omega - U^2 m^2\right) w_0^2 \end{gathered} \tag{3.2}$$

In the expressions (3.2)

$$I_1 = \int_o^l \widetilde{E_i}(x)\, cos^2\frac{m\pi x}{l}\, dx;$$

$$I_2 = \int_o^l \widetilde{E_i}(x)\, sin^2\frac{m\pi x}{l}\, dx \qquad I_3 = \int_o^l \widetilde{G_i}(x)\, sin^2\frac{m\pi x}{l}\, dx$$

$$I_4 = \int_0^l b_{11}(x)\, sin^2\frac{\pi x}{l} dx\ ; I_5 = \int_0^l b_{66}(x)\, cos^2\frac{px}{l} dx;\ I_6 = \int_0^l b_{22}(x)\, sin^2\frac{\pi x}{l} dx;$$

$$I_7 = \int_0^l b_{12} sin^2 \frac{\pi x}{l} dx;\ I_8 = \int_0^l \rho\left(x\right) cos^2 \frac{\pi x}{l} dx;\ I_9 = \int_0^l \rho\left(x\right) sin^2 \frac{\pi x}{l} dx;$$

$$I_{10} = \int_0^l \tilde{\rho}\left(x\right) cos^2 \frac{\pi x}{l} dx; I_{11} = \int_0^l \widetilde{\rho}\left(x\right) sin^2 \frac{px}{l} dx$$

By means of expressions (3.2) we obtain:

$$K - W - A = \left\{ \left\{ \omega^2 \left( \pi I_8 + \sum_{i=1}^{k_1} F_i I_{10} sin^2 k\varphi_i \right) - \left[ \frac{\pi Rh}{2} \left( \frac{\pi^2 m^2}{e^2} I_4 + \frac{k^2}{R^2} I_5 \right) + \right.\right.\right.$$

$$\left.\left. + \frac{m^2\pi^2}{2l^2} \sum_{i=1}^{k_1} F_i I_1 sin^2 k\varphi_i \right] \right\} u_0^2 + \left\{ \omega^2 \left( \pi I_9 + \sum_{i=1}^{k_1} F_i I_{11} \left( 1 + \frac{J_{kpi}}{F_i R^2} \right) cos^2 k\varphi_i \right) - \right.$$

$$\left. - \left[ \frac{\pi Rh}{2} \left( \frac{k^2}{R^2} I_6 + \frac{\pi^2 m^2}{e^2} I_5 \right) + \frac{m^2\pi^2}{2l^2} \sum_{i=1}^{k_1} (J_{xi} I_2 + J_{kpi} I_3) cos^2 k\varphi_i \right] \right\} v_0^2 +$$

$$+ \left\{ \omega^2 \left( \pi I_8 + \sum_{i=1}^{k_1} F_i I_{11} \left( 1 + \frac{J_{kpi} k^2}{F_i R^2} \right) sin^2 k\varphi_i \right) - \right.$$

$$- \left[ \frac{\pi Rh}{2} I_5 + \frac{m^2\pi^2}{2l^2} \sum_{i=1}^{k_1} \left( J_{zi} I_2 + J_{kpi} I_3 \right) sin^2 k\varphi_i \right] -$$

$$\left. - \frac{\pi l}{2R} \rho_m \Phi_{mn} \left( -\omega^2 + 2Um\omega - U^2 m^2 \right) \right\} w_0^2 -$$

$$- \frac{\pi Rh}{2} \left( \frac{2\pi km}{lR} I_7 + \frac{2\pi km}{lR} I_5 \right) u_0 v_0 + \frac{\pi^2 mRh}{l} I_7 u_0 w_0 +$$

$$\left. + \left\{ \sum_{i=1}^{k_1} F_i I_{11} \frac{J_{kpi}}{F_i R^2} sin 2k\varphi_i - \left[ -\pi khI_6 + \frac{m^2\pi^2}{2l^2} kJ_{kpi}\ I_3 sin 2k\varphi_i \right] \right\} v_0 w_0 \right\} sin^2 \omega t$$

Applying the Hamilton-Ostrogradsky variational principle, we obtain the following system of equations with respect to the constants $u_0, v_0, w_0$:

$$2 \left\{ \omega^2 \left( \pi I_8 + \sum_{i=1}^{k_1} F_i I_{10} sin^2 k\varphi_i \right) - \left[ \frac{\pi Rh}{2} \left( \frac{\pi^2 m^2}{e^2} I_4 + \frac{k^2}{R^2} I_5 \right) + \right.\right.$$

$$\left.\left. + \frac{m^2\pi^2}{2l^2} \sum_{i=1}^{k_1} F_i I_1\ sin^2 k\varphi_i \right] \right\} u_0 - \frac{\pi Rh}{2} \left( \frac{2\pi km}{lR} I_7 + \frac{2\pi km}{lR} I_5 \right) v_0 +$$

$$+ \frac{\pi^2 mRh}{l} I_7 w_0 = 0 \tag{3.3}$$

$$- \frac{\pi Rh}{2} \left( \frac{2\pi km}{lR} I_7 + \frac{2\pi km}{lR} I_5 \right) u_0 -$$

$$- \left\{ \omega^2 \left( \pi I_9 + \sum_{i=1}^{k_1} F_i I_{11} \left( 1 + \frac{J_{kpi}}{F_i R^2} \right) cos^2 k\varphi_i \right) - \right.$$

$$-\left[\frac{\pi Rh}{2}\left(\frac{k^2}{R^2}I_6+\frac{\pi^2 m^2}{e^2}I_5\right)+\frac{m^2\pi^2}{2l^2}\sum_{i=1}^{k_1}\left(J_{xi}I_2+J_{kpi}I_3\right)cos^2k\varphi_i]\right\}v_0+$$

$$+\left\{\sum_{i=1}^{k_1}F_iI_{11}\frac{J_{kpi}}{F_iR^2}sin2k\varphi_i-[-\pi khI_6+\frac{m^2\pi^2}{2l^2}kJ_{kpi}\ I_3sin2k\varphi_i]\right\}w_0=0$$

$$\frac{\pi^2 mRh}{l}I_7u_0+\left\{\sum_{i=1}^{k_1}F_iI_{11}\frac{J_{kpi}}{F_iR^2}sin2k\varphi_i-[-\pi khI_6+\frac{m^2\pi^2}{2l^2}kJ_{kpi}\ I_3sin2k\varphi_i]\right\}v_0-$$

$$-\left[\frac{\pi Rh}{2}I_5+\frac{m^2\pi^2}{2l^2}\sum_{i=1}^{k_1}\left(J_{zi}I_2+J_{kpi}I_3\right)sin^2k\varphi_i\right]-$$

$$\left.-\frac{\pi l}{2R}\rho_m\Phi_{mn}\left(-\omega^2+2Um\omega-U^2m^2\right)\right\}w_0=0$$

Since the obtained system (3.3) is the system of linear homogeneous algebraic equations, the necessary and sufficient condition for its non-trivial solution is the equality of its principal determinant to zero. As a result we obtain the following frequency equation:

$$det\,\|\,a_{pq}\|=0\ ,\quad p,q=1,2,3 \tag{3.4}$$

The constants $a_{pq}$ are the coeffients of the unknown constants $u_0, v_0, w_0$. Equation (3.1) is a transcendental equation with respect to the desired frequency ?. Its roots have been calculated by the numerical method.

## 4 Numerical results

It was accepted that

$$\rho\left(x\right)=\rho_0\left(1+\alpha\frac{x}{l}\right),E_1\left(x\right)=E_{10}\left(1+\beta\frac{x}{l}\right),$$

$$E_2\left(x\right)=E_{20}\left(1+\gamma\frac{x}{l}\right),\tilde{E}_i\left(x\right)=\tilde{E}_{i0}\left(1+\delta e^{\frac{x}{l}}\right),\tilde{\rho}_i\left(x\right)=\tilde{\rho}_{i0}\left(1+\varepsilon e^{\frac{x}{l}}\right),$$

$$G\left(x\right)=G_0(1+\beta\frac{x}{l}),G_i\left(x\right)=G_{i0}(1+\delta e^{\frac{x}{l}}),\omega_0=\sqrt{\frac{E_{10}}{(1-\nu^{2)}\rho_0R^2}},\omega_1=\omega/\omega_0,$$

Here $\alpha,\beta,\gamma,\delta,\varepsilon$ are inhomogeneity parameters. In the calculation, the following values were taken for the parameters characterizing the fluid, shell and rods:

$$E_{10}=18,3QPa,\ E_{20}=2,77\,QPa,\ G_0=3,5\,QPa,\rho_0=\rho_{i0}=1850kg/m^3,\tilde{E}_{i0}=6,67QPa$$

$$a_0=1800\frac{m}{sec},\ h=0,45cm,\rho_m/\rho_0=0,15,R=\ 160mm,L=800mm,I_{zi}=1,3mm^4,$$

$$I_{kpi}=0,23mm^4,I_{xi}=5,1mm^4,\ \nu_1=\nu_2=0,35,h_i=1,39\,cm,\ \ F_i=5,2\,mm^2,U/a_0=0,005$$

The results of the calculations were given in Fig. 4.1 in the form of dependence of the frequency parameter of the system on the amount of rods, in Fig. 4.2 in the form of dependence of the frequency parameter of the system on the inhomogeneity parameter for cylindrical shells made of different feature orthotropic materials.

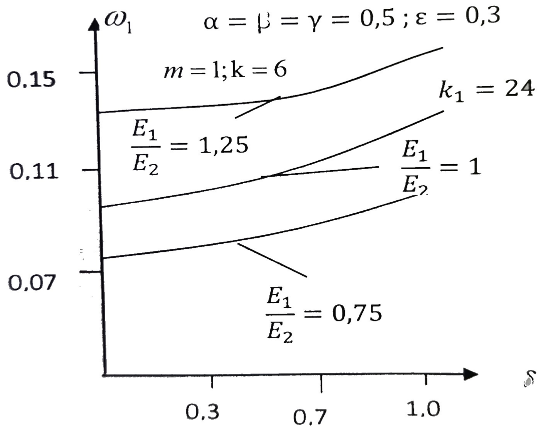


**Fig. 4.1.** Dependence of the frequency parameters on the number of rods

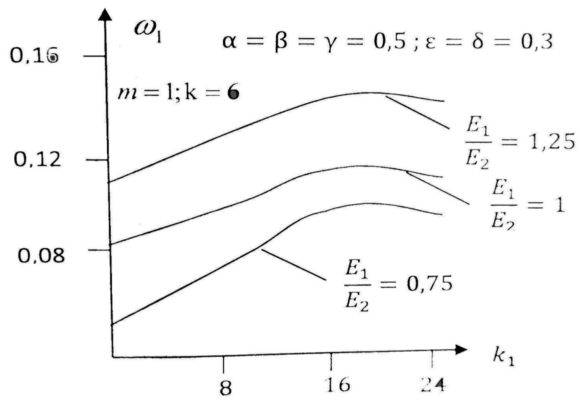


**Fig. 4.2.** Dependence of the frequency parameter on the inhomogenity parameter

## 5 Conclusions

Based on the conducted research, the following conclusions can be drawn:

1 Fig. 4.1 shows that increasing the number of rods, as first vibrations of the system increase and after certain kind decrease.
2 Fig. 4.2 shows that increasing the value of the inhomogenety parameter, natural vibrations of the system increase.

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