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Nonlinear feedback control of motion and power of moving sources during heating of the rod

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Abstract. *The article proposes an approach to solving the problem of synthesizing the control of the motion and power of lumped point-wise heat sources. For concreteness, the problem of control with nonlinear feedback by moving heat sources during the heating of the rod is considered. The power and motion of point sources involved in the right side of the parabolic type differential equation are determined depending on the measured values of the state of the process at the measurement points. As a result, the right side of the differential equation depends nonlinearly on the values of the state of the process at given points of the rod. Formulas for the components of the gradient of the functional with respect to feedback parameters are obtained, which make it possible to use first-order optimization methods for the numerical solution of synthesis problems.*

Keywords. loaded equation, rod heating, feedback control, moving source, measurement point, feedback parameters

Mathematics Subject Classification (2010): 49M05, 49K20

1 Introduction

The article studies the problem of synthesis of control of the heating process of a rod by lumped point-wise heat sources moving along the rod. The current values of power and control of the movement of sources are determined by the results of measurements of temperature at given points of the rod. The paper proposes to use a nonlinear dependence of the control actions of the power and movement of sources on the measured temperature values. After substituting these dependencies into the differential equation, a loaded differential equation is obtained, in which the loading points are the state measurement points. The constant coefficients involved in these dependencies are the desired feedback parameters that need to be optimized. Thus, the problem of control synthesis for moving heat sources

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with nonlinear feedback is reduced to the problem of parametric optimal control described by a loaded equation.

It is known that the problems of control of objects with feedback, described by both ordinary and partial differential equations, are the most difficult both in the theory of optimal control and for the practice of their application [6,7,8,11,12,13,14,15].

If there are sufficiently general approaches to study control synthesis problems for objects with lumped parameters [8,11,12,15], then for objects with distributed parameters [6,7,12,14] there are no such approaches yet. This is due to the complexity, diversity and mathematical models and options for the corresponding formulations of control problems for such objects [6,12]. The implementation of currently known methods for controlling objects with feedback in real time is also very difficult; it requires the use of expensive telemechanics, measuring and computer technology [7,12,14].

Nevertheless, in practice, as is known, a fairly large number of automatic control systems, automatic control of both objects with lumped and distributed parameters operate [3,4,6,7,8,9,12,14,16].

In this paper, the problem of optimizing the feedback parameters is reduced to the problem of parametric optimal control. To solve it, it is proposed to apply first-order numerical optimization methods using the obtained formulas for the components of the gradient of the objective functional with respect to the synthesized feedback parameters being optimized.

The described approach to the synthesis of the control of moving sources can be used to control other evolutionary processes described by other types of differential equations and types of initial–boundary conditions.

2 Formulation of the problem

Consider the following problem, which describes the process of heating a rod by moving point-wise heat sources [7]:

$$u_t(x,t) = a^2 u_{xx}(x,t) - \lambda_0[u(x,t)-\theta] + \sum_{i=1}^{N_c} q_i\,(t)\,\delta\,(x - z_i\,(t)), \tag{2.1}$$

$$x \in (0,l)\,, \quad t \in (t_0, t_f],$$

$$u_x(0,t) = \lambda_1(u(0,t)-\theta), \quad u_x(l,t) = -\lambda_2(u(l,t)-\theta), \quad t \in (t_0,t_f], \tag{2.2}$$

Here $u(x,t)$ is the temperature of the rod at the point $x \in [0,l]$ at time $t \in [t_0,t_f]$; l is the rod length; t_0 is start and t_f is the end time of the heating process; $a > 0$, λ_0, λ_1, λ_2 are given parameters of the heating process; $\delta(\cdot)$ is the Dirac delta function, $q_i\,(t)$ and $z_i\,(t)$ are piece-wise continuous functions with respect to t, which determine the power and location of the i^{th} heat source moving along the rod and satisfies constraints:

$$\underline{q_i} \le q_i\,(t) \le \overline{q_i}, \quad t \in [t_0,t_f], \quad i = 1,2,\ldots,N_c, \tag{2.3}$$

$$0 \le z_i\,(t) \le l, \quad t \in [t_0,t_f], \quad i = 1,2,\ldots,N_c, \tag{2.4}$$

where $\underline{q_i}$, $\overline{q_i}$, $i = 1,2,\ldots,N_c$ are given; N_c is the number of point-wise heat sources.

θ is the time-constant ambient temperature, the exact value of which is not specified. But is known the set Θ of possible values of θ and the distribution density function $\rho_\Theta(\theta)$ is such that:

$$\rho_\Theta(\theta) \ge 0, \quad \theta \in \Theta, \quad \int\limits_\Theta \rho_\Theta(\theta)d\theta = 1.$$

The temperature of the rod at the initial moment of time is not set, but the set of its possible values is given, determined by parametric functions depending on the s-dimensional vector of parameters b:

$$u(x,t_0) = \varphi(x;b), \quad x \in [0,l], \quad b \in \mathrm{B} \subset \mathbb{R}^s. \tag{2.5}$$

Here B is a given set of values of the parameters of the initial function $\varphi(x;b)$, while the distribution density function $\rho_{\mathrm{B}}(b)$ is known such that:

$$\rho_{\mathrm{B}}(b) \geq 0, \quad b \in \mathrm{B}, \quad \int\limits_{\mathrm{B}} \rho_{\mathrm{B}}(b)db = 1.$$

Motions of point-wise heat sources $z_i(t)$ are controlled and are determined by the initial-value problems with second-order ordinary differential equations and initial conditions

$$\ddot{z}_i(t) = a_i \dot{z}_i(t) + b_i z_i(t) + \vartheta_i(t), \quad t \in (t_0, t_f], \tag{2.6}$$

$$z_i(t_0) = z_i^0, \quad \dot{z}_i(t_0) = z_i^1, \quad i = 1,2,\ldots,N_c. \tag{2.7}$$

Here a_i, b_i are the given parameters of the movement of sources; z_i^0 and z_i^1 are given initial values of heat sources; $\vartheta_i(t)$ is a piece-wise continuous function that determines the motion control of the i^{th} heat source and satisfies the following constraints:

$$\underline{\vartheta_i} \leq \vartheta_i(t) \leq \overline{\vartheta_i}, \quad t \in [t_0, t_f], \quad i = 1,2,\ldots,N_c. \tag{2.8}$$

$\underline{\vartheta_i}$, $\overline{\vartheta_i}$, $i = 1,2,\ldots,N_c$ are given.

The problem is to determine the functions that control the process under consideration: $q(t) = (q_1(t), q_2(t), \ldots, q_{N_c}(t))$, $\vartheta(t) = (\vartheta_1(t), \vartheta_2(t), \ldots, \vartheta_{N_c}(t))$, $w = w(t) = (q(t), \vartheta(t))$, minimizing the given functional:

$$J(w) = \int\limits_{\mathrm{B}} \int\limits_{\Theta} I(w;b,\theta) \rho_{\Theta}(\theta) \rho_{\mathrm{B}}(b) d\theta db, \tag{2.9}$$

$$I(w;b,\theta) = \int\limits_0^l \mu(x)[u(x,t_f) - U(x)]^2 dx + \tag{2.10}$$

$$+\varepsilon_1 \|q(t) - \hat{q}\|^2_{L_2^{N_c}[t_0,t_f]} + \varepsilon_2 \|\vartheta(t) - \hat{\vartheta}\|^2_{L_2^{N_c}[t_0,t_f]}.$$

Here $U(x)$, $x \in [0,l]$ is a given piece-wise continuous function that determines the desired final temperature distribution on the rod at the moment $t = t_f$; $\mu(x) \geq 0$, $x \in [0,l]$ is the weight function; $u(x,t) = u(x,t;w,b,\theta)$ is the solution of the initial–boundary-value problem (2.1), (2.2), (2.5) with admissible given control $w(t)$, parameters of the initial condition $\varphi(x;b)$ and ambient temperature θ.

Let at the given N_o points of the rod $\xi_j \in [0,l]$, $j = 1,2,\ldots,N_o$, temperature measurements are continuously over time is taken:

$$\breve{u}_j(t) = u(\xi_j, t), \quad t \in [t_0, t_f], \quad \xi_j \in [0,l], \quad j = 1,2,\ldots,N_o.$$

The results of these measurements are used to form the current values of the controls in the form of the following dependencies that are nonlinear with respect to the distance between point-wise sources and measurement points and linear with respect to the measured $u(\xi_j, t)$ and nominal γ_{1i}^j and γ_{2i}^j values of the j^{th} measurement point.

Here the constants α_{1i}^{j}, β_{1i}^{j}, γ_{1i}^{j}, α_{2i}^{j}, β_{2i}^{j}, γ_{2i}^{j}, ξ_j, $i = 1,2,\ldots,N_c$, $j = 1,2,\ldots,N_o$, are synthesized feedback parameters. The parameter γ_{1i}^{j} and γ_{2i}^{j} characterizes the required value of the nominal temperature at the point $x = \xi_j$, which must be achieved due to the i^{th} point-wise source. It is clear that this value should be close to the given desired value $U(\xi_j)$, $i = 1,2,\ldots,N_c$, $j = 1,2,\ldots,N_o$. Parameters α_{1i}^{j}, α_{2i}^{j} and β_{1i}^{j}, β_{2i}^{j} by analogy with synthesis problems for objects with lumped parameters will be called gain factors.

$$q_i(t) = \sum_{j=1}^{N_o} \left(\alpha_{1i}^{j}(z_i(t) - \xi_j)^2 + \beta_{1i}^{j}\right)\left[u(\xi_j,t) - \gamma_{1i}^{j}\right], \tag{2.11}$$

$$t \in [t_0, t_f], \quad i = 1,2,\ldots,N_c,$$

$$\vartheta_i(t) = \sum_{j=1}^{N_o} \left(\alpha_{2i}^{j}(z_i(t) - \xi_j)^2 + \beta_{2i}^{j}\right)\left[u(\xi_j,t) - \gamma_{2i}^{j}\right], \tag{2.12}$$

$$t \in [t_0, t_f], \quad i = 1,2,\ldots,N_c.$$

The are natural constraints on the locations of measurement points

$$0 \le \xi_j \le l, \quad j = 1,2,\ldots,N_o. \tag{2.13}$$

Substituting dependencies (2.11), (2.12) into equations (2.1), (2.6), we obtain:

$$u_t(x,t) = a^2 u_{xx}(x,t) - \lambda_0[u(x,t) - \theta] + \sum_{i=1}^{N_c} \delta(x - z_i(t)) \times \tag{2.14}$$

$$\times \left\{ \sum_{j=1}^{N_o} \left(\alpha_{1i}^{j}(z_i(t) - \xi_j)^2 + \beta_{1i}^{j}\right)\left[u(\xi_j,t) - \gamma_{1i}^{j}\right] \right\}, \quad x \in (0,l), \quad t \in (t_0, t_f],$$

$$\ddot{z}_i(t) = a_i \dot{z}_i(t) + b_i z_i(t) + \sum_{j=1}^{N_o} \left(\alpha_{2i}^{j}(z_i(t) - \xi_j)^2 + \beta_{2i}^{j}\right)\left[u(\xi_j,t) - \gamma_{2i}^{j}\right], \tag{2.15}$$

$$t \in (t_0, t_f], \quad i = 1,2,\ldots,N_c.$$

The specificity of equations (2.14), (2.15) is, firstly, that they are point loaded with respect to the spatial variable. Second, equations (2.14), (2.15) with respect to the time variable must be solved simultaneously. Note that linearly loaded equations have been studied in such works as [1,2,5,10].

Combine parameters $\alpha_1 = ((\alpha_{1i}^{j}))$, $\beta_1 = ((\beta_{1i}^{j}))$, $\gamma_1 = ((\gamma_{1i}^{j}))$, $\alpha_2 = ((\alpha_{2i}^{j}))$, $\beta_2 = ((\beta_{2i}^{j}))$, $\gamma_2 = ((\gamma_{2i}^{j}))$, $\xi = (\xi_j)$ into one $\mathcal{N} = 6N_o(N_c + 1)$ dimensional synthesized vector of feedback parameters $y = (\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \xi)$, $i = 1,2,\ldots,N_c$, $j = 1,2,\ldots,N_o$.

The objective functional in this case can be written as follows:

$$J(y) = \int_{\mathrm{B}} \int_{\Theta} I(y; b, \theta) \rho_\Theta(\theta) \rho_{\mathrm{B}}(b) d\theta db, \tag{2.16}$$

$$I(y; b, \theta) = \int_0^l \mu(x)[u(x,t_f) - U(x)]^2 dx + \varepsilon \left\| y - \hat{y} \right\|_{\mathbb{R}^{\mathcal{N}}}^2. \tag{2.17}$$

Thus, the original considered control problem for moving point-wise sources (2.1)–(2.10) with feedback (2.11), (2.12) is reduced to a parametric optimal control problem (2.16), (2.17), (2.14), (2.2), (2.5), (2.15), (2.7) [13,16].

Let us note the following features of the obtained parametric optimal control problem.

First, the process under study is described by a system of loaded differential equations with partial and ordinary derivatives.

Secondly, the problem is specific because of the objective functional (2.9)–(2.10), which estimates the behavior of not a single trajectory, but a bunch of phase trajectories with values of initial conditions and ambient temperature from given sets.

In general, the resulting problem can be described to the class of finite-dimensional optimization problems with respect to the vector $y \in \mathbb{R}^{\mathcal{N}}$. In this problem, in order to calculate the objective functional for admissible values of the feedback parameters, it is required to solve initial–boundary-value problems with respect to differential equations with partial and ordinary derivatives.

3 Approach to determining feedback parameters

For the numerical solution of problem (2.1)–(2.10), namely, to finding the local minimum of the objective functional (2.16), (2.17), it is proposed to use the external penalty method to take into account constraints on power (2.3) and constraints on the motion controls (2.8) of the moving point-wise heat sources [16].

The constraints (2.3) for the power and (2.8) for the motion controls of each i^{th} point-wise heat sources with continuous feedback (2.11) and (2.12) will become the following constraints for the optimized parameters y and the temperature at the measurement points $u(\xi_j, t)$, $j = 1, 2, \ldots, N_o$:

$$\underline{q_i} \leq \sum_{j=1}^{N_o} \left(\alpha_{1i}^j (z_i(t) - \xi_j)^2 + \beta_{1i}^j\right) \left[u(\xi_j, t) - \gamma_{1i}^j\right] \leq \overline{q_i}, \quad t \in [t_0, t_f], i = 1, 2, \ldots, N_c,$$

$$\underline{\vartheta_i} \leq \sum_{j=1}^{N_o} \left(\alpha_{2i}^j (z_i(t) - \xi_j)^2 + \beta_{2i}^j\right) \left[u(\xi_j, t) - \gamma_{2i}^j\right] \leq \overline{\vartheta_i}, \quad t \in [t_0, t_f], i = 1, 2, \ldots, N_c,$$

which we denote and present in the following equivalent form:

$$g_1^i(t; y) = |\breve{g}_1^i(t; y)| - \frac{\overline{q_i} - \underline{q_i}}{2} \leq 0, \quad t \in [t_0, t_f], \quad i = 1, 2, \ldots, N_c, \tag{3.1}$$

$$\breve{g}_1^i(t; y) = \sum_{j=1}^{N_o} \left(\alpha_{1i}^j (z_i(t) - \xi_j)^2 + \beta_{1i}^j\right) \left[u(\xi_j, t) - \gamma_{1i}^j\right] - \frac{\overline{q_i} + \underline{q_i}}{2},$$

$$g_2^i(t; y) = |\breve{g}_2^i(t; y)| - \frac{\overline{\vartheta_i} - \underline{\vartheta_i}}{2} \leq 0, \quad t \in [t_0, t_f], \quad i = 1, 2, \ldots, N_c, \tag{3.2}$$

$$\breve{g}_2^i(t; y) = \sum_{j=1}^{N_o} \left(\alpha_{2i}^j (z_i(t) - \xi_j)^2 + \beta_{2i}^j\right) \left[u(\xi_j, t) - \gamma_{2i}^j\right] - \frac{\overline{\vartheta_i} + \underline{\vartheta_i}}{2}.$$

Taking into account the above constraints (3.1) and (3.2) we will choose the penalty functional with respect to functional (2.16), (2.17) in the following form:

$$J_{\mathcal{R}}(y) = \int_{\mathrm{B}} \int_{\Theta} I(y; b, \theta) \rho_{\Theta}(\theta) \rho_{\mathrm{B}}(b) d\theta db, \tag{3.3}$$

$$I_{\mathcal{R}}(y;b,\theta)=\int\limits_0^l \mu(x)[u(x,t_f)-U(x)]^2dx+\varepsilon\left\|y-\hat{y}\right\|^2_{\mathbb{R}^{\mathcal{N}}}+\mathcal{R}_1G_1\left(y\right)+\mathcal{R}_2G_2\left(y\right), \quad (3.4)$$

$$G_1\left(y\right)=\sum_{i=1}^{N_c}\int\limits_{t_0}^{t_f}\left[g_1^{i,+}(t;y)\right]^2dt,\quad G_2\left(y\right)=\sum_{i=1}^{N_c}\int\limits_{t_0}^{t_f}\left[g_2^{i,+}(t;y)\right]^2dt,$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the penalty coefficient tending to $+\infty$. The functions $g_j^{i,+}(\cdot)$ means that $g_j^{i,+}(\cdot)=g_j^i(\cdot)$ if $g_j^{i,+}(\cdot)>0$, $g_j^{i,+}(\cdot)>0$ and $g_j^{i,+}(\cdot)=0$ if $g_j^i(\cdot)\le 0$, $j=1,2$, $i=1,2,\ldots,N_c$.

To minimize the functional (3.3), (3.4), it is proposed to use the iterative procedure of the gradient projection method [16]:

$$y^{k+1}=\mathcal{P}_{(2.13)}\left[y^k-\alpha^k\,\mathbf{grad}_yJ_{\mathcal{R}}(y^k)\right], \quad (3.5)$$

$$\alpha^k=\arg\min_{\alpha\ge 0}J_{\mathcal{R}}\left(\mathcal{P}_{(2.13)}\left[y^k-\alpha\,\mathbf{grad}_yJ_{\mathcal{R}}(y^k)\right]\right),\quad k=0,1,\ldots.$$

Here α^k is one-dimensional minimization step, $y^0\in\mathbb{R}^{\mathcal{N}}$ is arbitrary starting vector from the set of feedback parameters; $\mathcal{P}_{(2.13)}[\cdot]$ is the projection operator onto the constraints defined by (2.13).

In order to implement the procedure (3.5), it is assumed that the formula for the gradient of the functional (3.3), (3.4) with respect to feedback parameters.

Theorem 3.1 *Under the conditions imposed above on the functions and parameters involved in problem* (2.14), (2.2), (2.5), (2.15), (2.7), *the functional* (3.3), (3.4) *is differentiable with respect to the feedback parameters, and the gradient components are determined by formulas:*

$$\frac{\partial J_{\mathcal{R}}(y)}{\partial\alpha_{1i}^j}=\int\limits_{\mathrm{B}}\int\limits_{\Theta}\left\{-\int\limits_{t_0}^{t_f}\left(\psi\left(z_i\left(t\right),t\right)-2\mathcal{R}_1 sgn\left(\check{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\right)\left(z_i\left(t\right)-\xi_j\right)^2\times\right.$$
$$\left.\times\left[u\left(\xi_j,t\right)-\gamma_{1i}^j\right]dt+2\varepsilon\left(\alpha_{1i}^j-\hat{\alpha}_{1i}^j\right)\right\}\rho_{\Theta}(\theta)\rho_{\mathrm{B}}(b)d\theta db, \quad (3.6)$$

$$\frac{\partial J_{\mathcal{R}}(y)}{\partial\beta_{1i}^j}=\int\limits_{\mathrm{B}}\int\limits_{\Theta}\left\{-\int\limits_{t_0}^{t_f}\left(\psi\left(z_i\left(t\right),t\right)-2\mathcal{R}_1 sgn\left(\check{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\right)\times\right.$$
$$\left.\times\left[u\left(\xi_j,t\right)-\gamma_{1i}^j\right]dt+2\varepsilon\left(\beta_{1i}^j-\hat{\beta}_{1i}^j\right)\right\}\rho_{\Theta}(\theta)\rho_{\mathrm{B}}(b)d\theta db, \quad (3.7)$$

$$\frac{\partial J_{\mathcal{R}}(y)}{\partial\gamma_{1i}^j}=\int\limits_{\mathrm{B}}\int\limits_{\Theta}\left\{\int\limits_{t_0}^{t_f}\left(\psi\left(z_i\left(t\right),t\right)-2\mathcal{R}_1 sgn\left(\check{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\right)\times\right.$$
$$\left.\times\left(\alpha_{1i}^j(z_i\left(t\right)-\xi_j)^2+\beta_{1i}^j\right)dt+2\varepsilon\left(\gamma_{1i}^j-\hat{\gamma}_{1i}^j\right)\right\}\rho_{\Theta}(\theta)\rho_{\mathrm{B}}(b)d\theta db, \quad (3.8)$$

$$\frac{\partial J_{\mathcal{R}}(y)}{\partial \alpha_{2i}^{j}} = \int\limits_{\mathrm{B}}\int\limits_{\Theta}\Bigg\{-\int\limits_{t_0}^{t_f}\left(\varphi_i(t) - 2\mathcal{R}_2 sgn\left(\breve{g}_2^i(t;y)\right) g_2^{i,+}(t;y)\right)(z_i(t)-\xi_j)^2\times$$

$$\times\left[u(\xi_j,t)-\gamma_{2i}^j\right]dt + 2\varepsilon\left(\alpha_{2i}^j - \hat{\alpha}_{2i}^j\right))\Bigg\}\rho_\Theta(\theta)\rho_{\mathrm{B}}(b)d\theta db, \tag{3.9}$$

$$\frac{\partial J_{\mathcal{R}}(y)}{\partial \beta_{2i}^{j}} = \int\limits_{\mathrm{B}}\int\limits_{\Theta}\Bigg\{-\int\limits_{t_0}^{t_f}\left(\varphi_i(t) - 2\mathcal{R}_2 sgn\left(\breve{g}_2^i(t;y)\right) g_2^{i,+}(t;y)\right)\times$$

$$\times\left[u(\xi_j,t)-\gamma_{2i}^j\right]dt + 2\varepsilon\left(\beta_{2i}^j - \hat{\beta}_{2i}^j\right)\Bigg\}\rho_\Theta(\theta)\rho_{\mathrm{B}}(b)d\theta db, \tag{3.10}$$

$$\frac{\partial J_{\mathcal{R}}(y)}{\partial \gamma_{2i}^{j}} = \int\limits_{\mathrm{B}}\int\limits_{\Theta}\Bigg\{\int\limits_{t_0}^{t_f}\left(\varphi_i(t) - 2\mathcal{R}_2 sgn\left(\breve{g}_2^i(t;y)\right) g_2^{i,+}(t;y)\right)\times$$

$$\times\left(\alpha_{2i}^j(z_i(t)-\xi_j)^2+\beta_{2i}^j\right)dt + 2\varepsilon\left(\gamma_{2i}^j - \hat{\gamma}_{2i}^j\right)\Bigg\}\rho_\Theta(\theta)\rho_{\mathrm{B}}(b)d\theta db, \tag{3.11}$$

$$\frac{\partial J_{\mathcal{R}}(y)}{\partial \xi_j} = \int\limits_{\mathrm{B}}\int\limits_{\Theta}\Bigg\{-\sum_{i=1}^{N_c}\int\limits_{t_0}^{t_f}\left(\psi(z_i(t),t) - 2\mathcal{R}_1 sgn\left(\breve{g}_1^i(t;y)\right) g_1^{i,+}(t;y)\right)\times \tag{3.12}$$

$$\times\left(\alpha_{1i}^j(z_i(t)-\xi_j)^2+\beta_{1i}^j\right)u_x(\xi_j,t)\,dt-$$

$$-\sum_{i=1}^{N_c}\int\limits_{t_0}^{t_f}\left(\varphi_i(t) - 2\mathcal{R}_2 sgn\left(\breve{g}_2^i(t;y)\right) g_2^{i,+}(t;y)\right)\left(\alpha_{2i}^j(z_i(t)-\xi_j)^2+\beta_{2i}^j\right)u_x(\xi_j,t)\,dt+$$

$$+\sum_{i=1}^{N_c}\int\limits_{t_0}^{t_f}\left(\psi(z_i(t),t) - 2\mathcal{R}_1 sgn\left(\breve{g}_1^i(t;y)\right) g_1^{i,+}(t;y)\right)2\alpha_{1i}^j(z_i(t)-\xi_j)\left[u(\xi_j,t)-\gamma_{1i}^j\right]dt+$$

$$+\sum_{i=1}^{N_c}\int\limits_{t_0}^{t_f}\left(\varphi_i(t) - 2\mathcal{R}_2 sgn\left(\breve{g}_2^i(t;y)\right) g_2^{i,+}(t;y)\right)2\alpha_{2i}^j(z_i(t)-\xi_j)\left[u(\xi_j,t)-\gamma_{2i}^j\right]dt+$$

$$+2\varepsilon\left(\xi_j-\hat{\xi}_j\right)\Bigg\}\rho_\Theta(\theta)\rho_{\mathrm{B}}(b)d\theta db,$$

$i = 1,2,\ldots,N_c$, $j = 1,2,\ldots,N_o$. *The functions* $\psi(x,t)$ *and* $\varphi_i(t)$, $i = 1,2,\ldots,N_c$, *are solutions to the following conjugate problems:*

$$\psi_t(x,t) = -a^2\psi_{xx}(x,t) + \lambda_0\psi(x,t) - \sum_{j=1}^{N_o}\delta(x-\xi_j)\times \tag{3.13}$$

$$\times\left\{\sum_{i=1}^{N_c}\left(\psi\left(z_i\left(t\right),t\right)-2\mathcal{R}_1 sgn\left(\breve{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\right)\left(\alpha_{1i}^j(z_i\left(t\right)-\xi_j)^2+\beta_{1i}^j\right)\right\}-$$

$$-\sum_{j=1}^{N_o}\delta\left(x-\xi_j\right)\left\{\sum_{i=1}^{N_c}\left(\varphi_i(t)-2\mathcal{R}_2 sgn\left(\breve{g}_2^i(t;y)\right)g_2^{i,+}(t;y)\right)\left(\alpha_{2i}^j(z_i\left(t\right)-\xi_j)^2+\beta_{2i}^j\right)\right\},$$

$$x\in(0,l)\,,\quad t\in[t_0,t_f),$$

$$\psi(x,t_f)=-2\mu(x)\left(u(x,t_f)-U(x)\right),\quad x\in[0,l]\,, \tag{3.14}$$

$$\psi_x(0,t)=\lambda_1\psi(0,t),\quad \psi_x(l,t)=-\lambda_2\psi(l,t),\quad t\in[t_0,t_f), \tag{3.15}$$

$$\ddot{\varphi}_i(t)=-a_i\dot{\varphi}_i(t)+b_i\varphi_i(t)+ \tag{3.16}$$

$$+\left(\psi\left(z_i\left(t\right),t\right)-2\mathcal{R}_1 sgn\left(\breve{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\right)\left\{\sum_{j=1}^{N_o}2\alpha_{1i}^j(z_i\left(t\right)-\xi_j)\left[u\left(\xi_j,t\right)-\gamma_{1i}^j\right]\right\}+$$

$$+\left(\varphi_i(t)-2\mathcal{R}_2 sgn\left(\breve{g}_2^i(t;y)\right)g_2^{i,+}(t;y)\right)\left\{\sum_{j=1}^{N_o}2\alpha_{2i}^j(z_i\left(t\right)-\xi_j)\left[u\left(\xi_j,t\right)-\gamma_{2i}^j\right]\right\}+$$

$$+\psi_x\left(z_i\left(t\right),t\right)\left\{\sum_{j=1}^{N_o}\left(\alpha_{1i}^j(z_i\left(t\right)-\xi_j)^2+\beta_{1i}^j\right)\left[u\left(\xi_j,t\right)-\gamma_{1i}^j\right]\right\},$$

$$t\in[t_0,t_f),\quad i=1,2,\ldots,N_c,$$

$$\dot{\varphi}_i(t_f)=-a_i\varphi_i(t_f),\quad \varphi_i(t_f)=0,\quad i=1,2,\ldots,N_c. \tag{3.17}$$

Proof. To prove the differentiable of the functional $J_{\mathcal{R}}(y)$ with respect to y, we use the increment method.

Under conditions imposed on the parameters involved in the problem, the formula takes place:

$$\mathbf{grad}_y J_{\mathcal{R}}(y)=\mathbf{grad}_y\int\limits_{\mathrm{B}}\int\limits_{\Theta}I_{\mathcal{R}}(y;b,\theta)\rho_\Theta(\theta)\rho_{\mathrm{B}}(b)d\theta db= \tag{3.18}$$

$$=\int\limits_{\mathrm{B}}\int\limits_{\Theta}\mathbf{grad}_y I_{\mathcal{R}}(y;b,\theta)\rho_\Theta(\theta)\rho_{\mathrm{B}}(b)d\theta db.$$

Therefore, we define formulas for $\mathbf{grad}_y I_{\mathcal{R}}(y;b,\theta)$ for arbitrary admissible values $b\in\mathrm{B}$ and $\theta\in\Theta$.

Denote the third term on the right side (2.14).

$$V(t;y)=\sum_{i=1}^{N_c}\delta\left(x-z_i\left(t\right)\right)\left\{\sum_{j=1}^{N_o}\left(\alpha_{1i}^j(z_i\left(t\right)-\xi_j)^2+\beta_{1i}^j\right)\left[u\left(\xi_j,t\right)-\gamma_{1i}^j\right]\right\},$$

$$x\in(0,l)\,,\quad t\in(t_0,t_f],$$

Denote by $u(x,t)=u(x,t;y,b,\theta)$, $z(t)=z(t;y,b,\theta)$ the solutions of the initial–boundary-value problem (2.14), (2.2), (2.5) and initial-value problems (2.15), (2.7) for the given values of the parameters b and θ. Suppose that the parameters y have been incremented Δy: $\tilde{y}=y+\Delta y$, then the solutions of the problems (2.14), (2.2), (2.5) and (2.15), (2.7):

$$\tilde{u}\left(x,t;\tilde{y},b,\theta\right)=u(x,t;y,b,\theta)+\Delta u(x,t;y,b,\theta),$$

$$\tilde{z}\,(t;\tilde{y},b,\theta) = z(t;y,b,\theta) + \Delta z(t;y,b,\theta).$$

It is clear that $\Delta u(x,t;y,b,\theta)$ and $\Delta z(t;y,b,\theta)$ satisfies the conditions of initial–boundary-value and the initial-value problems:

$$\Delta u_t(x,t) = a^2 \Delta u_{xx}(x,t) - \lambda_0 \Delta u(x,t) + \Delta V(t;y), \quad x \in (0,l)\,, t \in (t_0,t_f], \tag{3.19}$$

$$\Delta u(x,0) = 0, \quad x \in [0,l]\,, \tag{3.20}$$

$$\Delta u_x(0,t) = \lambda_1 \Delta u(0,t), \quad t \in (t_0,t_f], \tag{3.21}$$

$$\Delta u_x(l,t) = -\lambda_2 \Delta u(l,t), \quad t \in (t_0,t_f].$$

$$\Delta \ddot{z}_i(t) = a_i \Delta \dot{z}_i(t) + b_i \Delta z_i\,(t) + \Delta \vartheta_i(t), \quad t \in (t_0,t_f], \tag{3.22}$$

$$\Delta z_i(t_0) = 0, \quad \Delta \dot{z}_i(t_0) = 0, \quad i = 1,2,\ldots,N_c. \tag{3.23}$$

Will receive an increment of the functional (2.17)

$$\Delta I(y;b,\theta) = I(y+\Delta y;b,\theta) - I(y;b,\theta) = \tag{3.24}$$

$$= 2\int\limits_0^l \mu(x)\left(u(x,t_f) - U(x)\right)\Delta u(x,t_f)dx + 2\varepsilon\left\langle y-\hat{y},\Delta y\right\rangle +$$

$$+o\left(\|\Delta u(x,t)\|_{L_2[\Omega]}, \|\Delta y\|_{\mathbb{R}^{\mathcal{N}}}\right), \quad \Omega = [0,l]\times[t_0,t_f].$$

Let us shift the right-hand sides of the differential equations (3.19) and (3.22) to the left, multiply both parts of the obtained equalities by the so far arbitrary functions $\psi(x,t)$ and $\varphi_i(t)$, $i = 1,2,\ldots,N_c$, respectively, integrate over $x \in [0,l]$ and $t \in [t_0,t_f]$ and adding with (3.24):

$$\Delta I(y;b,\theta) = 2\int\limits_0^l \mu(x)\left(u(x,t_f) - U(x)\right)\Delta u(x,t_f)dx + 2\varepsilon\left\langle y-\hat{y},\Delta y\right\rangle +$$

$$+\int\limits_{t_0}^{t_f}\int\limits_0^l \psi(x,t)\left(\Delta u_t(x,t) - a^2\Delta u_{xx}(x,t) + \lambda_0\Delta u(x,t) - \Delta V(t;y)\right)dxdt +$$

$$+\sum_{i=1}^{N_c}\int\limits_{t_0}^{t_f}\varphi_i(t)\left(\Delta\ddot{z}_i(t) - a_i\Delta\dot{z}_i(t) - b_i\Delta z_i\,(t) - \Delta\vartheta_i(t)\right)dt +$$

$$+o\left(\|\Delta u(x,t)\|_{L_2[\Omega]}, \|\Delta z\,(t)\,\|_{L_2^{N_c}[t_0,t_f]}, \|\Delta y\|_{\mathbb{R}^{\mathcal{N}}}\right).$$

Consider an increment of the penalty terms of functional:

$$\Delta G_1(y) = G_1(y+\Delta y) - G_1(y) = 2\mathcal{R}_1\sum_{i=1}^{N_c}\sum_{j=1}^{N_o}\int\limits_{t_0}^{t_f}\operatorname{sgn}\left(\breve{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\times \tag{3.25}$$

$$\times\Big\{\left(\Delta\alpha_{1i}^j(z_i\,(t)-\xi_j)^2 + \Delta\beta_{1i}^j\right)\left[u\,(\xi_j,t) - \gamma_{1i}^j\right] - \Delta\gamma_{1i}^j\left(\alpha_{1i}^j(z_i\,(t)-\xi_j)^2 + \beta_{1i}^j\right) +$$

$$+\Delta\xi_j\left(\left(\alpha_{1i}^j(z_i\,(t)-\xi_j)^2 + \beta_{1i}^j\right)u_x\,(\xi_j,t) - 2\alpha_{1i}^j(z_i\,(t)-\xi_j)\right)\left[u\,(\xi_j,t) - \gamma_{1i}^j\right] +$$

$$+\left(\alpha_{1i}^j(z_i\,(t)-\xi_j)^2 + \beta_{1i}^j\right)\Delta u\,(\xi_j,t) + 2\alpha_{1i}^j(z_i\,(t)-\xi_j)\left[u\,(\xi_j,t) - \gamma_{1i}^j\right]\Delta z_i\,(t)\Big\} +$$

$$+o\left(\|\Delta u\,(\xi_j,t)\,\|_{L_2[\Omega]}, \|\Delta z\,(t)\,\|_{L_2^{N_c}[t_0,t_f]}, \|\Delta y\|_{\mathbb{R}^{\mathcal{N}}}\right),$$

$$\Delta G_2(y) = G_2(y+\Delta y) - G_2(y) = 2\mathcal{R}_2 \sum_{i=1}^{N_c} \sum_{j=1}^{N_o} \int_{t_0}^{t_f} \operatorname{sgn}\left(\breve{g}_2^i(t;y)\right) g_2^{i,+}(t;y) \times \tag{3.26}$$

$$\times \Big\{ \left(\Delta\alpha_{2i}^j (z_i(t)-\xi_j)^2 + \Delta\beta_{2i}^j\right) \left[u(\xi_j,t) - \gamma_{2i}^j\right] - \Delta\gamma_{2i}^j \left(\alpha_{2i}^j (z_i(t)-\xi_j)^2 + \beta_{2i}^j\right) +$$

$$+\Delta\xi_j \left(\left(\alpha_{2i}^j (z_i(t)-\xi_j)^2 + \beta_{2i}^j\right) u_x(\xi_j,t) - 2\alpha_{2i}^j (z_i(t)-\xi_j)\right) \left[u(\xi_j,t) - \gamma_{2i}^j\right] +$$

$$+ \left(\alpha_{2i}^j (z_i(t)-\xi_j)^2 + \beta_{2i}^j\right) \Delta u(\xi_j,t) + 2\alpha_{2i}^j (z_i(t)-\xi_j) \left[u(\xi_j,t) - \gamma_{2i}^j\right] \Delta z_i(t) \Big\} +$$

$$+o\left(\|\Delta u(\xi_j,t)\|_{L_2[\Omega]}, \|\Delta z(t)\|_{L_2^{N_c}[t_0,t_f]}, \|\Delta y\|_{\mathbb{R}^{\mathcal{N}}}\right).$$

Having carried out the appropriate transformations and grouping, taking into account (3.19)–(3.23), and adding increment of penalty functions (3.25), (3.26), we will have

$$\Delta I_{\mathcal{R}}(y;b,\theta) = 2\int_0^l \mu(x)\left(u(x,t_f) - U(x)\right)\Delta u(x,t_f)dx + \int_0^l \psi(x,t_f)\Delta u(x,t_f)dx +$$

$$+\int_{t_0}^{t_f}\int_0^l \left(-\psi_t(x,t) - a^2\psi_{xx}(x,t) + \lambda_0\psi(x,t)\right)\Delta u(x,t)dxdt -$$

$$-a^2\int_{t_0}^{t_f} \left(\psi_x(l,t) + \lambda_2\psi(l,t)\right)\Delta u(l,t)dt - a^2\int_{t_0}^{t_f} \left(\psi_x(0,t) - \lambda_1\psi(0,t)\right)\Delta u(0,t)dt -$$

$$-\sum_{j=1}^{N_o}\sum_{i=1}^{N_c} \Bigg\{ \int_{t_0}^{t_f} \Bigg(\left(\psi(z_i(t),t) - 2\mathcal{R}_1 \operatorname{sgn}\left(\breve{g}_1^i(t;y)\right) g_1^{i,+}(t;y)\right) \left(\alpha_{1i}^j (z_i(t)-\xi_j)^2 + \beta_{1i}^j\right) +$$

$$+ \left(\varphi_i(t) - 2\mathcal{R}_2 \operatorname{sgn}\left(\breve{g}_2^i(t;y)\right) g_2^{i,+}(t;y)\right) \left(\alpha_{2i}^j (z_i(t)-\xi_j)^2 + \beta_{2i}^j\right) \Bigg) \Delta u(\xi_j,t)\,dt \Bigg\} +$$

$$+\sum_{i=1}^{N_c}\sum_{j=1}^{N_o} \Delta\alpha_{1i}^j \Big\{ -\int_{t_0}^{t_f} \left(\psi(z_i(t),t) - 2\mathcal{R}_1 \operatorname{sgn}\left(\breve{g}_1^i(t;y)\right) g_1^{i,+}(t;y)\right) (z_i(t)-\xi_j)^2 \times$$

$$\times \left[u(\xi_j,t) - \gamma_{1i}^j\right] dt + 2\varepsilon\left(\alpha_{1i}^j - \hat{\alpha}_{1i}^j\right) \Big\} +$$

$$+\sum_{i=1}^{N_c}\sum_{j=1}^{N_o} \Delta\beta_{1i}^j \Big\{ -\int_{t_0}^{t_f} \left(\psi(z_i(t),t) - 2\mathcal{R}_1 \operatorname{sgn}\left(\breve{g}_1^i(t;y)\right) g_1^{i,+}(t;y)\right) \times$$

$$\times \left[u(\xi_j,t) - \gamma_{1i}^j\right] dt + 2\varepsilon\left(\beta_{1i}^j - \hat{\beta}_{1i}^j\right) \Big\} +$$

$$+\sum_{i=1}^{N_c}\sum_{j=1}^{N_o}\Delta\gamma_{1i}^{j}\Biggl\{\int_{t_0}^{t_f}\Bigl(\psi\left(z_i\left(t\right),t\right)-2\mathcal{R}_1\mathrm{sgn}\left(\breve{g}_1^{i}(t;y)\right)g_1^{i,+}(t;y)\Bigr)\times$$

$$\times\Bigl(\alpha_{1i}^{j}(z_i\left(t\right)-\xi_j)^2+\beta_{1i}^{j}\Bigr)\,dt+2\varepsilon\Bigl(\gamma_{1i}^{j}-\hat{\gamma}_{1i}^{j}\Bigr)\Biggr\}+$$

$$+\sum_{i=1}^{N_c}\sum_{j=1}^{N_o}\Delta\alpha_{2i}^{j}\Biggl\{-\int_{t_0}^{t_f}\Bigl(\varphi_i(t)-2\mathcal{R}_2\mathrm{sgn}\left(\breve{g}_2^{i}(t;y)\right)g_2^{i,+}(t;y)\Bigr)\left(z_i\left(t\right)-\xi_j\right)^2\times$$

$$\times\Bigl[u\left(\xi_j,t\right)-\gamma_{2i}^{j}\Bigr]\,dt+2\varepsilon\Bigl(\alpha_{2i}^{j}-\hat{\alpha}_{2i}^{j}\Bigr)\Biggr\}+$$

$$+\sum_{i=1}^{N_c}\sum_{j=1}^{N_o}\Delta\beta_{2i}^{j}\Biggl\{-\int_{t_0}^{t_f}\Bigl(\varphi_i(t)-2\mathcal{R}_2\mathrm{sgn}\left(\breve{g}_2^{i}(t;y)\right)g_2^{i,+}(t;y)\Bigr)\times$$

$$\times\Bigl[u\left(\xi_j,t\right)-\gamma_{2i}^{j}\Bigr]\,dt+2\varepsilon\Bigl(\beta_{2i}^{j}-\hat{\beta}_{2i}^{j}\Bigr)\Biggr\}+$$

$$+\sum_{i=1}^{N_c}\sum_{j=1}^{N_o}\Delta\gamma_{2i}^{j}\Biggl\{\int_{t_0}^{t_f}\Bigl(\varphi_i(t)-2\mathcal{R}_2\mathrm{sgn}\left(\breve{g}_2^{i}(t;y)\right)g_2^{i,+}(t;y)\Bigr)\times$$

$$\times\Bigl(\alpha_{2i}^{j}(z_i\left(t\right)-\xi_j)^2+\beta_{2i}^{j}\Bigr)\,dt+2\varepsilon\Bigl(\gamma_{2i}^{j}-\hat{\gamma}_{2i}^{j}\Bigr)\Biggr\}+$$

$$+\sum_{j=1}^{N_o}\Delta\xi_j\Biggl\{-\sum_{i=1}^{N_c}\int_{t_0}^{t_f}\Bigl(\psi\left(z_i\left(t\right),t\right)-2\mathcal{R}_1\mathrm{sgn}\left(\breve{g}_1^{i}(t;y)\right)g_1^{i,+}(t;y)\Bigr)\times$$

$$\times\Bigl(\alpha_{1i}^{j}(z_i\left(t\right)-\xi_j)^2+\beta_{1i}^{j}\Bigr)\,u_x\left(\xi_j,t\right)dt-$$

$$-\sum_{i=1}^{N_c}\int_{t_0}^{t_f}\Bigl(\varphi_i(t)-2\mathcal{R}_2\mathrm{sgn}\left(\breve{g}_2^{i}(t;y)\right)g_2^{i,+}(t;y)\Bigr)\times$$

$$\times\Bigl(\alpha_{2i}^{j}(z_i\left(t\right)-\xi_j)^2+\beta_{2i}^{j}\Bigr)\,u_x\left(\xi_j,t\right)dt+$$

$$+\sum_{i=1}^{N_c}\int_{t_0}^{t_f}\Bigl(\psi\left(z_i\left(t\right),t\right)-2\mathcal{R}_1\mathrm{sgn}\left(\breve{g}_1^{i}(t;y)\right)g_1^{i,+}(t;y)\Bigr)\times$$

$$\times2\alpha_{1i}^{j}(z_i\left(t\right)-\xi_j)\Bigl[u\left(\xi_j,t\right)-\gamma_{1i}^{j}\Bigr]\,dt+$$

$$+\sum_{i=1}^{N_c}\int_{t_0}^{t_f}\Big(\varphi_i(t)-2\mathcal{R}_2\mathrm{sgn}\left(\breve{g}_2^i(t;y)\right)g_2^{i,+}(t;y)\Big)\times$$

$$\times 2\alpha_{2i}^j(z_i\left(t\right)-\xi_j)\left[u\left(\xi_j,t\right)-\gamma_{2i}^j\right]dt+2\varepsilon\left(\xi_j-\hat{\xi}_j\right)\Bigg\}-$$

$$-\sum_{i=1}^{N_c}\left(\dot{\varphi}_i(t_f)+a_i\varphi_i(t_f)\right)\Delta z_i(t_f)+\sum_{i=1}^{N_c}\varphi_i(t_f)\Delta\dot{z}_i(t_f)+$$

$$+\sum_{i=1}^{N_c}\Bigg\{\int_{t_0}^{t_f}\Bigg(\ddot{\varphi}_i(t)+a_i\dot{\varphi}_i(t)-b_i\varphi_i(t)-$$

$$-\Big(\psi\left(z_i\left(t\right),t\right)-2\mathcal{R}_1\mathrm{sgn}\left(\breve{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\Big)\left\{\sum_{j=1}^{N_o}2\alpha_{1i}^j(z_i\left(t\right)-\xi_j)\left[u\left(\xi_j,t\right)-\gamma_{1i}^j\right]\right\}-$$

$$-\Big(\varphi_i(t)-2\mathcal{R}_2\mathrm{sgn}\left(\breve{g}_2^i(t;y)\right)g_2^{i,+}(t;y)\Big)\left\{\sum_{j=1}^{N_o}2\alpha_{2i}^j(z_i\left(t\right)-\xi_j)\left[u\left(\xi_j,t\right)-\gamma_{2i}^j\right]\right\}-$$

$$-\psi_x\left(z_i\left(t\right),t\right)\left\{\sum_{j=1}^{N_o}\left(\alpha_{1i}^j(z_i\left(t\right)-\xi_j)^2+\beta_{1i}^j\right)\left[u\left(\xi_j,t\right)-\gamma_{1i}^j\right]\right\}\Bigg)\Delta z_i\left(t\right)dt\Bigg\}+$$

$$+o\left(\|\Delta u(x,t)\|_{L_2[\Omega]},\|\Delta z\left(t\right)\|_{L_2^{N_c}[t_0,t_f]},\|\Delta y\|_{\mathbb{R}^{\mathcal{N}}}\right).$$

Using the well-known results on the solution of the initial–boundary-value problem (2.14), (2.2), (2.5), and the initial-value problem (2.15), (2.7), one can obtain estimates $\|\Delta u(x,t)\|\le k_1\|\Delta y\|$, $\|\Delta z(t)\|\le k_2\|\Delta y\|$. From there it follows that the functional of the problem is differentiable.

Considering that the functions $\psi(x,t)$ and $\varphi_i(t)$, $i=1,2,\dots,N_c$ are arbitrary, we require the conditions (3.13)–(3.17) to be satisfied.

Then it is clear that the components of the gradient of the functional $I_{\mathcal{R}}(y;b,\theta)$ are defined by the formulas:

$$\frac{\partial I(y;b,\theta)}{\partial\alpha_{1i}^j}=-\int_{t_0}^{t_f}\Big(\psi\left(z_i\left(t\right),t\right)-2\mathcal{R}_1\mathrm{sgn}\left(\breve{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\Big)\left(z_i\left(t\right)-\xi_j\right)^2\times$$
$$\times\left[u\left(\xi_j,t\right)-\gamma_{1i}^j\right]dt+2\varepsilon\left(\alpha_{1i}^j-\hat{\alpha}_{1i}^j\right),\qquad(3.27)$$

$$\frac{\partial I(y;b,\theta)}{\partial\beta_{1i}^j}=-\int_{t_0}^{t_f}\Big(\psi\left(z_i\left(t\right),t\right)-2\mathcal{R}_1\mathrm{sgn}\left(\breve{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\Big)\left[u\left(\xi_j,t\right)-\gamma_{1i}^j\right]dt+$$
$$+2\varepsilon\left(\beta_{1i}^j-\hat{\beta}_{1i}^j\right),\qquad(3.28)$$

$$\frac{\partial I(y;b,\theta)}{\partial \gamma_{1i}^{j}} = \int\limits_{t_0}^{t_f} \left(\psi\left(z_i\left(t\right),t\right) - 2\mathcal{R}_1 \mathrm{sgn}\left(\breve{g}_1^i(t;y)\right) g_1^{i,+}(t;y)\right) \times$$

$$\times \left(\alpha_{1i}^{j}(z_i\left(t\right) - \xi_j)^2 + \beta_{1i}^{j}\right) dt + 2\varepsilon\left(\gamma_{1i}^{j} - \hat{\gamma}_{1i}^{j}\right), \tag{3.29}$$

$$\frac{\partial I(y;b,\theta)}{\partial \alpha_{2i}^{j}} = -\int\limits_{t_0}^{t_f} \left(\varphi_i(t) - 2\mathcal{R}_2 \mathrm{sgn}\left(\breve{g}_2^i(t;y)\right) g_2^{i,+}(t;y)\right) (z_i\left(t\right) - \xi_j)^2 \times$$

$$\times \left[u\left(\xi_j,t\right) - \gamma_{2i}^{j}\right] dt + 2\varepsilon\left(\alpha_{2i}^{j} - \hat{\alpha}_{2i}^{j}\right), \tag{3.30}$$

$$\frac{\partial I(y;b,\theta)}{\partial \beta_{2i}^{j}} = -\int\limits_{t_0}^{t_f} \left(\varphi_i(t) - 2\mathcal{R}_2 \mathrm{sgn}\left(\breve{g}_2^i(t;y)\right) g_2^{i,+}(t;y)\right) \left[u\left(\xi_j,t\right) - \gamma_{2i}^{j}\right] dt +$$

$$+ 2\varepsilon\left(\beta_{2i}^{j} - \hat{\beta}_{2i}^{j}\right), \tag{3.31}$$

$$\frac{\partial I(y;b,\theta)}{\partial \gamma_{2i}^{j}} = \int\limits_{t_0}^{t_f} \left(\varphi_i(t) - 2\mathcal{R}_2 \mathrm{sgn}\left(\breve{g}_2^i(t;y)\right) g_2^{i,+}(t;y)\right) \times$$

$$\times \left(\alpha_{2i}^{j}(z_i\left(t\right) - \xi_j)^2 + \beta_{2i}^{j}\right) dt + 2\varepsilon\left(\gamma_{2i}^{j} - \hat{\gamma}_{2i}^{j}\right), \tag{3.32}$$

$$\frac{\partial I(y;b,\theta)}{\partial \xi_j} = -\sum_{i=1}^{N_c} \int\limits_{t_0}^{t_f} \left(\psi\left(z_i\left(t\right),t\right) - 2\mathcal{R}_1 \mathrm{sgn}\left(\breve{g}_1^i(t;y)\right) g_1^{i,+}(t;y)\right) \times \tag{3.33}$$

$$\times \left(\alpha_{1i}^{j}(z_i\left(t\right) - \xi_j)^2 + \beta_{1i}^{j}\right) u_x\left(\xi_j,t\right) dt -$$

$$-\sum_{i=1}^{N_c} \int\limits_{t_0}^{t_f} \left(\varphi_i(t) - 2\mathcal{R}_2 \mathrm{sgn}\left(\breve{g}_2^i(t;y)\right) g_2^{i,+}(t;y)\right) \left(\alpha_{2i}^{j}(z_i\left(t\right) - \xi_j)^2 + \beta_{2i}^{j}\right) u_x\left(\xi_j,t\right) dt +$$

$$+\sum_{i=1}^{N_c} \int\limits_{t_0}^{t_f} \left(\psi\left(z_i\left(t\right),t\right) - 2\mathcal{R}_1 \mathrm{sgn}\left(\breve{g}_1^i(t;y)\right) g_1^{i,+}(t;y)\right) 2\alpha_{1i}^{j}(z_i\left(t\right) - \xi_j) \left[u\left(\xi_j,t\right) - \gamma_{1i}^{j}\right] dt +$$

$$+\sum_{i=1}^{N_c} \int\limits_{t_0}^{t_f} \left(\varphi_i(t) - 2\mathcal{R}_2 \mathrm{sgn}\left(\breve{g}_2^i(t;y)\right) g_2^{i,+}(t;y)\right) 2\alpha_{2i}^{j}(z_i\left(t\right) - \xi_j) \left[u\left(\xi_j,t\right) - \gamma_{2i}^{j}\right] dt +$$

$$+ 2\varepsilon\left(\xi_j - \hat{\xi}_j\right).$$

Taking into account the formula (3.18) from (3.27)–(3.33), we obtain the desired formulas (3.6)–(3.12).

4 Numerical experiments

Numerical experiments were carried out on the example of test problem, in which the parameters and functions involved in problem were as follows:

$$a^2 = 1, \quad \lambda_0 = 0.01, \quad \lambda_1 = \lambda_2 = 0.001, \quad l = 1, \quad t_0 = 0, \quad t_f = 1,$$

$$\mu(x) \equiv 1, \quad U(x) = 30, \quad x \in [0;\ 1], \quad \varepsilon = 0.1, \quad N_c = 2, \quad N_o = 4,$$

$$a_1 = 0.184,\ b_1 = 0.259,\ a_2 = -0.174,\ b_2 = -0.254,\ \xi_j \in [0.05;\ 0.95], \quad j = 1, 2, \ldots, 4,$$

$$\mathrm{B} = [4.8; 5.2], \quad \rho_{\mathrm{B}}(b) = 2.5\left(1 + \cos\left(5(x-5)\,\pi\right)\right),$$

$$\varTheta = [4.75; 5.25], \quad \rho_{\varTheta}(\theta) = 2\left(1 + \cos(4(x-5)\pi)\right),$$

$$0 \le q_1(t) \le 80, \quad 0 \le q_2(t) \le 65, \quad t \in [0;\ 1],$$

$$-3.5 \le \vartheta_1(t) \le 3.5, \quad -3.5 \le \vartheta_2(t) \le 3.5, \quad t \in [0;\ 1].$$

The direct initial–boundary-value (2.14), (2.2), (2.5) and conjugate boundary-value (3.13), (3.14), (3.15) problems of parabolic type were solved using an implicit scheme by the grid method with steps in the spatial variable $h_x = 0.01$ and in the time variable $h_t = 0.001$. To solve direct (2.15), (2.7) and conjugate (3.16), (3.17) initial-value problems, Euler method was used with a step $h_t = 0.001$ [1,2].

The $\delta(\cdot)$ – Dirac delta function was approximated as the following trigonometric everywhere smooth (differentiable) function:

$$\delta_\sigma(x;\eta) = \begin{cases} 0, & |x-\eta| > \sigma, \\ \frac{1}{2\sigma}\left[1 + \cos\left(\frac{x-\eta}{\sigma}\pi\right)\right], & |x-\eta| \le \sigma. \end{cases}$$

In this case, for arbitrary value of $\sigma > 0$ satisfies equality:

$$\int\limits_{\eta-\sigma}^{\eta+\sigma} \delta_\sigma(x;\eta)dx = 1.$$

In test experiments the value of parameter σ of function $\delta_\sigma(x;\eta)$ was set equal to $3h_x$ where h_x is the step of the grid approximation of the segment $x \in [0;1]$. Such a choice of the form of the Dirac δ-function ensures a certain smoothness of the functional $J_{\mathcal{R}}(y)$ with respect to the optimized measuring points location ξ and coordinates of point-wise heat sources $z(t)$.

Let us give a general description of the algorithm for solving the test problem of synthesis of the parameter vector y, which dimension in this case is equal to $\mathcal{N} = N_o(6N_c + 1) = 52$. With the chosen penalty coefficients $\mathcal{R}_1$, $\mathcal{R}_2$ and regularization parameters ε, $\hat{y}$ to implement the procedure (3.5), at each iteration of which, for the current values of the parameters y^k, $k = 0, 1, 2, \ldots$ to be optimized, for all possible values $\theta \in \varTheta$ and $b \in \mathrm{B}$ the following steps are performed:

1. the direct initial–boundary-value problem (2.14), (2.2), (2.5) and initial-value problems (2.15), (2.7), are solved;
2. the conjugate problems (3.13), (3.14), (3.15) and (3.16), (3.17) are solved;
3. components of gradient of the penalty function (3.6)–(3.12) are calculated;
4. into the direction of the projected on the positional constraints (2.13) anti-gradient of the functional, one-dimensional minimization is carried out with respect to $\alpha \ge 0$:

For finding anti-gradient direction step α at each iteration of procedure (3.5), the golden section method was used [16].

These steps are repeated until any stop criterion is met. For example, step α or the difference of the values of the functional (3.3) at two successive iterations is less than a given small value. Further, according to known approaches, using the resulting parameter values y^* we change the regularization parameters ε, $\hat{y}$; in particular, we decrease (divide by ten) ε, and take as $\hat{y}$ the resulting optimal value of the vector y^* and repeat procedure until a stopping criterion is met. The penalty coefficients $\mathcal{R}_1$, $\mathcal{R}_2$ are increased until the optimized values of the parameters y obtained for two consecutive values of the penalty coefficient change by an amount that exceeds the specified required accuracy for solving the entire problem.

Tables 4.1 and 4.2 shows the results of calculations in which two values y_1^0 and y_2^0 were used as the initial vectors for the iterative procedure (3.5) for gradient projection method of the penalty function (3.3), (3.4) . Values of matrices $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ and ξ are given in the following order: $(\alpha_{11}^1, \ldots, \alpha_{1N_c}^1, \ldots, \alpha_{11}^{N_o}, \ldots, \alpha_{1N_c}^{N_o}, \beta_{11}^1, \ldots, \beta_{1N_c}^1, \ldots, \beta_{11}^{N_o}, \ldots, \beta_{1N_c}^{N_o}, \gamma_{11}^1, \ldots, \gamma_{1N_c}^1, \ldots, \gamma_{11}^{N_o}, \ldots, \gamma_{1N_c}^{N_o}, \alpha_{21}^1, \ldots, \alpha_{2N_c}^1, \ldots, \alpha_{21}^{N_o}, \ldots, \alpha_{2N_c}^{N_o}, \beta_{21}^1, \ldots, \beta_{2N_c}^1, \ldots, \beta_{21}^{N_o}, \ldots, \beta_{2N_c}^{N_o}, \gamma_{21}^1, \ldots, \gamma_{2N_c}^1, \ldots, \gamma_{21}^{N_o}, \ldots, \gamma_{2N_c}^{N_o}, \xi_1, \ldots, \xi_{N_o})$.

It can be seen that, as mentioned above, due to the possible multi-extremity of the objective functional, the optimization results obtained from different starting vectors differ in arguments, although the difference is not significant in terms of the functional. Here it is also necessary to take into account (as other specially conducted numerical experiments have shown) that the functional of the problem has a strong ravine structure.

Fig. 4.1 and 4.2 shows the plots of point-wise heat sources trajectories, power respectively to initial vector y_1^0 and for the synthesized optimal vector y_1^*.

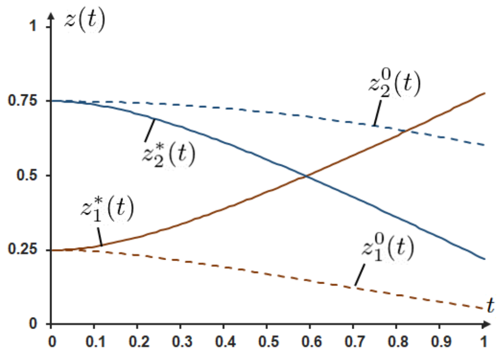

Fig. 4.1. Plots of point-wise sources motion trajectories for initial y_1^0 (- - -) and synthesized optimal vector y_1^* (——).

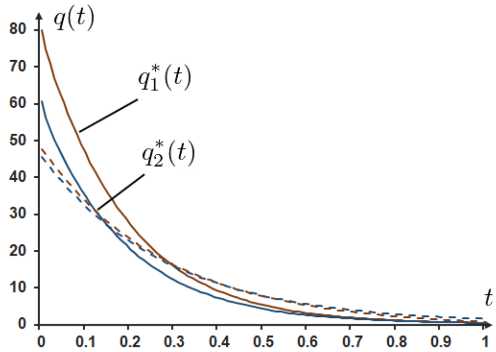

Fig. 4.2. Plots of point-wise sources powers for initial vector y_1^0 (- - -) and synthesized optimal vector y_1^* (——).

Computer experiments were carried out to measurement the heating process at optimal values of the synthesized feedback parameters under the assumption that the measurements are carried out with errors (noise), namely:

$$\tilde{u}_j(t) = [1 + \chi_j(t)]u(\xi_j, t), \quad t \in [t_0, t_f], \quad \xi_j \in [0, l], \quad j = 1, 2, \ldots, N_o.$$

Here $\chi_j(t)$, $j = 1, 2, \ldots, N_o$ for each t is a random variable uniformly distributed on the segment $[-\zeta, \zeta]$. In the experiments performed, the values of ζ were chosen as 0.01; 0.03; 0.05, which corresponded to measurement errors of 1%, 3% and 5% of the measured values.

In table 4.3 shows the results obtained when solving the synthesis of feedback parameters in the presence of an error in the measurements. As can be seen from the comparison

Table 4.1 Solutions of the test problem obtained for initial vector y_1^0 using of procedure (3.5) **for the gradient projection method for the functional** (3.3)**,** (3.4)**.**

N	$y_1 = (\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \xi)$							$J_{\mathcal{R}}(y)$
	0.03518	0.08541	0.05315	-0.05637	-0.26985	-0.11157	-0.02281	0.00650	
	-0.51182	-0.57293	-0.78061	-0.78465	-0.76613	-0.80055	-0.65828	-0.72692	
	26.0254	26.8899	29.4461	28.5202	27.6437	28.5494	26.8635	27.7104	
0	-0.00213	-0.00000	0.00149	0.00597	0.00657	0.00115	-0.00069	0.00204	5.1499
	0.00709	0.00808	0.00212	0.00609	0.01766	0.01536	0.00657	0.01996	
	25.7799	24.9359	29.9999	26.6238	28.3120	27.4680	28.3120	29.1560	
	0.09434	0.36489	0.71126	0.93199					
	-0.10490	-0.11134	-0.15685	-0.19407	-0.13427	-0.11269	-0.11378	-0.13051	
	-0.31767	-0.31766	-0.31767	-0.31766	-0.30976	-0.30976	-0.33353	-0.33352	
	30.00569	30.00578	30.00655	30.00736	30.00521	30.00493	31.00528	31.00551	
1	-0.06743	-0.07591	-0.06949	-0.05782	0.07246	0.07681	0.06550	0.05190	1.7586
	0.02491	0.02492	0.02491	0.02492	-0.01623	-0.01624	0.07246	-0.01623	
	29.99877	29.99926	29.99889	29.99869	29.99913	29.99867	29.99905	29.99904	
	0.27089	0.46873	0.80772	0.95000					
	0.03316	0.08698	0.03728	-0.07231	-0.27267	-0.11134	-0.02100	0.00637	
	-0.49054	-0.56866	-0.82530	-0.80955	-0.77727	-0.83669	-0.65307	-0.74264	
	28.9121	29.2129	30.0812	29.7261	29.4246	29.7656	29.2025	29.4876	
2	-0.00210	0.00044	0.00387	0.01123	-0.00187	-0.00176	-0.00076	0.00161	0.3562
	0.01771	0.01851	0.01368	0.01690	-0.00049	-0.00231	-0.00187	0.00133	
	28.6458	28.3750	30.0000	28.9165	29.4584	29.1876	29.4584	29.7293	
	0.18274	0.41080	0.64419	0.87122					
	-0.14555	-0.09502	-0.16813	-0.19548	-0.12544	-0.12519	-0.07036	-0.11151	
	-0.58800	-0.58662	-0.58700	-0.58536	-0.54044	-0.53960	-0.51331	-0.51185	
	30.07640	30.07446	30.07699	30.07872	29.95758	29.95985	30.97161	30.97209	
3	-0.05245	-0.05857	-0.05058	-0.04315	0.05762	0.05883	0.04821	0.04085	0.0039
	0.01976	0.02043	0.02025	0.02097	-0.01008	-0.01097	0.05762	-0.00954	
	29.91602	29.98079	29.91922	29.89254	29.96091	29.88954	29.95553	29.96324	
	0.49206	0.41197	0.83733	0.82616					
	-0.21977	-0.04087	-0.05581	-0.02319	-0.0666	-0.13475	0.03527	-0.03495	
	-0.77779	-0.77377	-0.77466	-0.76968	-0.71843	-0.71603	-0.4673	-0.46295	
	30.17152	30.16512	30.16483	30.16413	29.81274	29.82145	30.84935	30.84929	
4	-0.02234	-0.02227	-0.01885	-0.01962	0.02276	0.02255	0.0183	0.02342	0.0001
	-0.02931	-0.02722	-0.02786	-0.02584	0.02643	0.02339	0.02276	0.02800	
	29.75672	29.94563	29.76523	29.68732	29.88733	29.67877	29.87166	29.89432	
	0.82655	0.24571	0.65390	0.485480					

of the obtained values of the feedback parameters, approximately they differ in proportion to the errors of the measurements.

5 Conclusions

An approach to feedback control of the motion and power of lumped point-wise heat sources in systems with distributed parameters is proposed. The problem of control of moving point-wise sources used for heating the rod is considered. Power and motion controls on the movement of point-wise sources are determined in the form of proposed dependencies on the results of measurements. The differentiability of the functional with respect to the feedback parameters is shown, formulas for the gradient of the functional with respect to the synthesized parameters are obtained. The formulas make it possible to solve the problem of point-wise source control synthesis using efficient first-order numerical optimization methods and available standard software packages.

Table 4.2 Intermediate iterations of gradient projection method of the functional (3.3)**,** (3.4) **for procedure** (3.5) **for initial vector y_2^0.**

N	$y_2 = (\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \xi)$							$J_{\mathcal{R}}(y)$
	-0.11768	-0.00264	-0.04027	-0.01600	-0.00502	-0.08881	-0.01202	-0.04145	
	-0.25363	-0.25363	-0.25363	-0.25363	-0.35932	-0.35933	-0.39109	-0.39109	
	28.48576	27.94587	28.12459	29.12458	28.98575	29.02154	28.89745	28.88752	
0	0.00259	-0.00002	0.00121	0.00029	-0.00008	-0.00138	-0.00017	-0.00059	2.6475
	0.00939	0.00939	0.00939	0.00939	-0.00565	-0.00565	-0.00008	-0.00565	
	28.87549	28.32548	29.12589	29.10258	28.45782	29.12898	28.98547	29.91257	
	0.88887	0.25326	0.55078	0.41375					
	-0.23781	-0.00575	-0.06449	-0.02774	-0.01321	-0.13047	-0.02436	-0.06894	
	-0.45807	-0.45681	-0.45689	-0.45682	-0.55728	-0.55734	-0.64390	-0.64391	
	28.98574	28.96324	29.84572	30.05701	29.03132	28.93307	29.03513	29.03548	
1	0.00146	-0.00034	0.00007	-0.00054	0.00023	-0.00012	0.00020	0.00012	1.5728
	0.01346	0.01346	0.01346	0.01346	0.00068	0.00068	0.00023	0.00068	
	29.27872	29.19297	28.95689	29.95876	28.932568	28.93268	29.92698	29.32587	
	0.95000	0.29361	0.45185	0.38077					
	-0.18353	-0.00408	-0.05577	-0.02300	-0.00906	-0.11677	-0.01889	-0.05866	
	-0.37173	-0.37163	-0.37164	-0.37163	-0.48362	-0.48366	-0.55264	-0.55264	
	29.92060	29.91582	29.91624	29.81589	29.92294	29.32415	29.42534	28.92557	
2	0.00087	-0.00024	0.00019	-0.00041	-0.00003	-0.00103	-0.00008	-0.00040	0.1139
	0.01157	0.01157	0.01157	0.01157	-0.00414	-0.00414	-0.00003	-0.00414	
	29.92872	29.91273	29.79368	29.73697	29.93268	29.24789	29.23268	29.12593	
	0.94415	0.27610	0.50028	0.39812					
	-0.23144	-0.00547	-0.06351	-0.02714	-0.01317	-0.12994	-0.02425	-0.06899	
	-0.44875	-0.44789	-0.44796	-0.44791	-0.55487	-0.55496	-0.65599	-0.65601	
	30.06281	30.04656	30.04761	30.04679	30.03938	30.04122	30.04451	30.04490	
3	0.00114	-0.00032	0.00005	-0.00054	0.00025	-0.00003	0.00024	0.00018	0.0025
	0.01303	0.01303	0.01303	0.01303	0.00111	0.00111	0.00025	0.00111	
	29.93268	29.91245	29.93698	29.97581	29.96598	29.98754	29.99568	29.99584	
	0.95000	0.29154	0.45284	0.37971					
	-0.34519	-0.01498	-0.08148	-0.04085	-0.04823	-0.15460	-0.04299	-0.09072	
	-0.60453	-0.60331	-0.60336	-0.60331	-0.90687	-0.90697	-0.56974	-0.56966	
	30.11458	30.08816	30.08919	30.08834	29.73687	29.74135	30.79480	30.79543	
4	0.01156	0.00218	0.00282	0.00208	-0.00016	-0.00071	-0.00036	-0.00047	0.0001
	0.00582	0.00925	0.00933	0.00930	-0.00709	-0.00769	-0.00016	-0.00768	
	30.00022	30.00017	30.00017	30.00017	29.99992	29.99992	29.99991	29.99992	
	0.94973	0.44209	0.45865	0.44977					

Note that the proposed approach to synthesis leads to the problem of parametric optimal control of a process described by loaded differential equations with ordinary and partial derivatives.

The proposed approach to the control of point-wise sources with feedback can be used in systems for automatic control and regulation of lumped sources for many other technological processes and technical objects.

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Table 4.3 Solutions of the test problem obtained from the initial vector y_1^0 with measurement errors of 1%, 3%, 5%.

$\chi(t)$	$y_1^* = (\alpha_1^*, \beta_1^*, \gamma_1^*, \alpha_2^*, \beta_2^*, \gamma_2^*, \xi^*)$								$J_{\mathcal{R}}(y^*)$
	-0.21979	-0.04087	-0.05581	-0.02319	-0.06662	-0.13476	0.03530	-0.03494	
	-0.77788	-0.77386	-0.77474	-0.76976	-0.71852	-0.71611	-0.46726	-0.46292	
	30.17143	30.16503	30.16472	30.16403	29.81266	29.82137	30.84932	30.84925	
1%	-0.02238	-0.02227	-0.01887	-0.01963	0.02277	0.02264	0.01830	0.02344	0.000011
	-0.02946	-0.02737	-0.02801	-0.02598	0.02687	0.02381	0.02277	0.02844	
	29.75674	29.94565	29.76523	29.68733	29.88734	29.67878	29.87167	29.89433	
	0.82669	0.24539	0.65407	0.48551					
	-0.21977	-0.04083	-0.05579	-0.02316	-0.06664	-0.13475	0.03533	-0.03493	
	-0.77760	-0.77356	-0.77437	-0.76941	-0.71856	-0.71613	-0.46698	-0.46262	
	30.17145	30.16498	30.16451	30.16388	29.81254	29.82121	30.84941	30.84928	
3%	-0.02226	-0.02227	-0.01881	-0.01961	0.02276	0.02249	0.01830	0.02340	0.000013
	-0.02905	-0.02695	-0.02759	-0.02558	0.02621	0.02316	0.02276	0.02778	
	29.75698	29.94596	29.76522	29.68751	29.88743	29.67891	29.87177	29.89437	
	0.82629	0.24562	0.65379	0.48532					
	-0.21979	-0.04083	-0.05582	-0.02318	-0.06668	-0.13476	0.03526	-0.03495	
	-0.77778	-0.77376	-0.77465	-0.76967	-0.71870	-0.71630	-0.46747	-0.46313	
	30.17142	30.16501	30.16472	30.16403	29.81267	29.82138	30.84931	30.84925	
5%	-0.02228	-0.02227	-0.01882	-0.01961	0.02276	0.02257	0.01830	0.02342	0.000015
	-0.02912	-0.02703	-0.02767	-0.02565	0.02655	0.02350	0.02276	0.02812	
	29.75672	29.94563	29.76523	29.68732	29.88733	29.67877	29.87166	29.89432	
	0.82688	0.24589	0.65495	0.48599					

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