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Dynamics of incompressible viscous fluid flow in an elastic tube of varying cross-section

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Abstract. *The pulsating motion of an incompressible viscous fluid in a deformable tube is investigated. The problem is solved analytically by taking the contraction in the pipe.*

Keywords. elastic tube · viscous fluid · wave · pulsating flow · continuity equation.

Mathematics Subject Classification (2010): 76D55

1 Introduction

Let $R = R(x)$ be a tube of semi-infinite variable cross-section and thickness h, and x is the longitudinal coordinate. System of one-dimensional hydroelasticity equations from continuity equations [1 - 5]:

$$\frac{\partial}{\partial x}(Su) + L\frac{\partial w}{\partial t} = 0 \tag{1.1}$$

momentum equation

$$\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(-p + \sigma) \tag{1.2}$$

and consists of the tube's equation of motion for linear viscoelasticity

$$p = \frac{n}{R^2(x)}E^{\nu}w = \rho_* h\frac{\partial^2 w}{\partial t^2} \tag{1.3}$$

When writing equation (1.3), the tube is thin-walled and rigidly attached to the environment. As a result, the tube cannot move along the axis. Classical descriptions of the hydrodynamics of an ideal and viscous Newtonian fluid are unacceptable when describing the

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flow of a medium with long macromolecular assemblies. This fact is of primary importance for many technological processes, such as colloidal solutions, suspensions, emulsions, etc. This includes for this, in order to relate the above equations, we write down the rheological relations of the fluid and assume it to be linear viscoelastic:

$$\mathop{\Pi}_{j=1}^{r}\left(1+\lambda_j\frac{\partial}{\partial t}\right)\cdot\sigma=2\eta\mathop{\Pi}_{j=1}^{s}\left(1+\theta_j\frac{\partial}{\partial t}\right)\cdot e \tag{1.4}$$

In equations (1.1) - (1.4) $u(x,t)$-the flow rate of the liquid, $w(x,t)$- the radial displacement of the pipe walls, $p(x,t)$ is the hydrodynamic pressure, $\sigma(x,t)$ - stress, ρband ρ_*- the density of the liquid and the material of the pipe, $e(x,t)$- the rate of deformation, $S=\pi R^2$- the cross-sectional area, $L=2\pi R(x)$- the length of the pipe circumference, η- the dynamic viscosity coefficient of the liquid, λ_j and $\theta_j$characterize relaxation and retardation. In (1.3) E^ν- is an operator of inherited type.

$$E^\nu=E\left(1-\Gamma^*\right),\Gamma^*w(x,t)=\int_{-\infty}^{l}\Gamma(t-\tau)w(x,\tau)d\tau,$$

where E - the modulus of elasticity, Γ^*- the relaxation operator, $\Gamma(t-\tau)$- the difference kernel of the relaxation. (1.3) is written in the open form as follows:

$$p=\frac{h}{R^2(x)}E\left\{w(x,t)-\int_{-\infty}^{l}\Gamma(t-\tau)w(x,\tau)d\tau\right\} \tag{1.5}$$

If we consider the equality $e=\partial u/\partial x$ in (1.4):

$$\mathop{\Pi}_{j=1}^{r}\left(1+\lambda_j\frac{\partial}{\partial t}\right)\cdot\sigma=2\eta\mathop{\Pi}_{j=1}^{s}\left(1+\theta_j\frac{\partial}{\partial t}\right)\cdot\frac{\partial u}{\partial x} \tag{1.6}$$

Written $R(x)$ as the function $R(x)=R_\infty g(x)$, the function $g(x)$ is second order differentiable. At infinity, the tube has R_∞ constant cross-section.

From here we find that

$$\lim_{x\to\infty}g(x)=1 \tag{1.7}$$

At the same time

$$\lim_{x\to\infty}g'(x)=0,\ \lim_{x\to\infty}g''(x)=0, \tag{1.8}$$

Bars denote differentiation with respect to the x coordinate. For example, this function can be shown as follows:

$$g(x)=1+e^{-\beta x}\ \ (\beta>0), \tag{1.9}$$

which shows that the tube tapers into a conical shape along its length. Then, considering (1.5) and (1.6), we get the following system of closed equations:

$$\frac{\partial u}{\partial t}+2\frac{g'(x)}{g(x)}u+\frac{2}{R_\infty g(x)}\frac{\partial w}{\partial t}=0 \tag{1.10}$$

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\frac{\partial\sigma}{\partial x} \tag{1.11}$$

$$p=\frac{h}{R_\infty^2 g'(x)}E\left(w(x,t)-\int_{-\infty}^{l}\Gamma(t-\tau)w(x,\tau)d\tau\right) \tag{1.12}$$

$$\mathop{\Pi}_{j=1}^{r}\left(\sigma+\lambda_j\frac{\partial\sigma}{\partial t}\right)=2\eta\mathop{\Pi}_{j=1}^{s}\left(\frac{\partial u}{\partial x}+\theta_j\frac{\partial^2 u}{\partial x\partial t}\right) \tag{1.13}$$

All the above-mentioned search functions can be described proportionally $\exp(i\omega t)$to the time product, where given $\omega-$ is the real angular frequency, i- is an imaginary number. Therefore, for the class of returning waves, we write:

$$\begin{aligned} u &= u_1(x)\exp(i\omega t), \\ w &= w_1(x)\exp(i\omega t), \\ p &= p_1(x)\exp(i\omega t), \\ \sigma &= \sigma_1(x)\exp(i\omega t). \end{aligned} \tag{1.14}$$

Here u_1, w_1, p_1, σ_1- are complex functions of coordinates. First, let's write the equation (1.12). If we consider formulas (1.2) and (1.3) in (1.4) and (1.12), we get the following expression:

$$p_1 = \exp(i\omega t) - \frac{h}{R_\infty^2 g(x)} E\left\{ w_1 \exp(i\omega t) - w_1 \int_{-\infty}^{l} \mathit{\Gamma}(t-\tau)e^{i\omega t} d\tau \right\} \tag{1.15}$$

Accepting here $t - \tau = \theta$and after some changes, (1.5) can be written as follows:

$$p_1 = w_1 h \left\{ \frac{E}{R_\infty^2 g^2(x)}(1-\alpha) - \rho_* \omega^2 \right\}$$

here,

$$\alpha = \int_0^\infty \mathit{\Gamma}(\theta)e^{-i\omega\theta} d\theta \tag{1.16}$$

Based on the possible relaxation kernels, the complex value α determined by the formula can be determined analytically or algebraically. After changes in equations (1.10), (1.11) and (1.13) as above, we find that:

$$u_1' + 2\frac{g'(x)}{g(x)}u_1 + 2i\frac{\omega}{R_\infty g(x)}w_1 = 0, \tag{1.17}$$

$$i\omega\rho u_1 = -p_1' + \sigma_1'', \tag{1.18}$$

$$p_1 = k(x)w_1, \tag{1.19}$$

$$\sigma_1 = 2\eta\frac{b}{a}u_1', \tag{1.20}$$

and here

$$a = \mathop{\mathit{\Pi}}_{j=1}^{r}(1 + i\lambda_j\omega), b = \mathop{\mathit{\Pi}}_{j=1}^{s}(1 + i\theta_j\omega), k(x) = h\left\{ \frac{E}{R_\infty^2 g^2(x)}(1-\alpha) - \rho_*\omega^2 \right\} \tag{1.21}$$

Then

$$2\eta\frac{b}{a}u_1'' - k'w_1 - kw_1' - i\omega\rho u_1 = 0. \tag{1.22}$$

From (1.7) it can be seen that

$$w_1 = -\frac{1}{Q_2(x)}u_1' - \frac{Q_1(x)}{Q_2(x)}u_1. \tag{1.23}$$

$$Q_1(x) = 2\frac{g'(x)}{g(x)}, Q_2(x) = 2i\frac{\omega}{R_\infty g(x)}.$$

Now we can find w_1' from (1.23) and by writing it in (1.22) we can get an equation for the function u_1.

$$G_1(x)u_1'' + G_2(x)u_1' + G_3(x)u_1 = 0, \tag{1.24}$$

here,

$$G_1(x) = 2\eta\frac{b}{a} - i\frac{R_\infty}{2\omega}k(x)g(x), \tag{1.25}$$

$$G_2(x) = -\frac{iR_\infty}{2\omega}\left\{(gk)' + kg'\right\}, \tag{1.26}$$

$$G_3(x) = -i\left\{\frac{R_\infty}{2\omega}(kg') + \omega\rho\right\}. \tag{1.27}$$

Let's bring the solution of the problem to the solution of the singular boundary problem for the Sturm-Louisville equation. Let's use the Louisville substitute for this

$$y(x) = u_1 \exp\frac{1}{2}\int\frac{G_2(x)}{G_1(x)}dx \equiv u_1(x)\chi(x), \tag{1.28}$$

Then (1.24) will be written like this

$$y'' + I(x)y = 0 \tag{1.29}$$

The invariant $I(x)$ is found by this formula:

$$I(x) = \frac{G_3}{G_1} - \frac{1}{4}\left(\frac{G_2}{G_1}\right)^2 - \frac{1}{2}\left(\frac{G_2}{G_1}\right)' \tag{1.30}$$

Based on (1.7) and (1.8), we get the following equation

$$\lim_{x\to\infty} k(x) = h\left\{\frac{E}{R_\infty^2}(1-\alpha) - \rho_*\omega^2\right\}.$$

It is clear from here,

$$\lim_{x\to\infty} G_1(x) = 2\eta\frac{b}{a} - i\frac{R_\infty h}{2\omega}\left\{\frac{E}{R_\infty^2}(1-\alpha) - \rho_*\omega^2\right\},$$

$$\lim_{x\to\infty} G_2(x) = 0, \lim_{x\to\infty} G_3(x) = -i\omega\rho.$$

Then, we can write that

$$\lim_{x\to\infty} I(x) = -\frac{i\omega\rho}{2\eta\frac{b}{a} - i\frac{R_\infty h}{2\omega}\left\{\frac{E}{R_\infty^2}(1-\alpha) - \rho_*\omega^2\right\}} = \delta^2 \tag{1.31}$$

Dividing the dispersion equation (1.31) into real and imaginary parts, we get the following:

$$\delta^2 = \mu_0 - i\mu_1. \tag{1.32}$$

Here

$$\mu_0 = \frac{\omega\rho m_3}{4\eta^2 m_1^2 + m_3^2},$$

$$\mu_1 = 2\eta\frac{\omega\rho m_1}{4\eta^2 m_1^2 + m_3^2},$$

$$m_1 = Re\frac{b}{a},$$

$$m_2 = Jm\frac{b}{a},$$

$$m_3 = -2\eta m_2 + \frac{R_\infty h}{2\omega}\left\{\frac{E}{R_\infty^2}(1-\alpha) - \rho_*\omega^2\right\}.$$

From here, we find the square root of complex numbers δ

$$\delta = \pm\left\{\sqrt{\frac{\psi+\mu_0}{2}} - i\sqrt{\frac{\psi-\mu_0}{2}}\right\}, \psi = \sqrt{\mu_0^2+\mu_1^2}$$

Later, the root $Jm\delta < 0$ will be used. It is obvious that,

$$\delta = \delta_0 - i\delta_1.$$

$$\delta_0 = \sqrt{\frac{\psi+\mu_0}{2}}, \delta_1 = \sqrt{\frac{\psi-\mu_0}{2}}.$$

Taking into account that

$$q(x) = 1 - \frac{I(x)}{\delta^2} \tag{1.33}$$

From (1.29) we get the differential equation of the problem:

$$y'' + \delta^2 y = \delta^2 q(x) y. \tag{1.34}$$

The integration condition is applied to the complex potential function $q(x)$ [2]:

$$\int_0^\infty |q(x)|\, dx < +\infty. \tag{1.35}$$

The function $q(x)$ obtained according to the formula (1.33) together with (1.9) satisfies the condition (1.33).

Then let us add the following boundary conditions to equation (1.32) to construct the solution.

$$y(0) = y_0,$$

$$\lim_{x\to\infty} y(x) = 0. \tag{1.36}$$

Condition (1.36) shows the boundedness of the sought condition. Thus, the obtained hydroelasticity problem (1.34), (1.36) was brought to the solution of the Sturm-Louisville singular boundary problem.

Let us bring the solution of the Sturm-Louisville boundary value problem to the solution of the integral equation. Consider the homogeneous equation

$$y'' + \delta^2 y = 0 \tag{1.37}$$

has the following system of fundamental solutions

$$y_1 = e^{-i\delta x},$$

$$y_2 = e^{i\delta x}.$$

(1.37) is treated as an inhomogeneous equation whose right-hand side is known as $\delta^2 q(x)y$, using the variational method, (1.34), (1.36), we bring the solution to the equivalent integral equation.

$$y(x,-\delta) = Ce^{-i\delta x} + \delta \int_x^\infty \sin\delta(\eta - x)q(\eta)y(\eta,-\delta)d\eta, \tag{1.38}$$

$$y_{n+1}(x,-\delta) = Ce^{-i\delta x} + \delta \int_x^\infty \sin\delta(\eta - x)q(\eta)y_n(\eta,-\delta)d\eta$$

To find the number C-, it is necessary to choose it in such a way that it satisfies the boundary conditions (1.34). Therefore, let's write the following integral equation for equation (1.38):

$$f(x,-\delta) = e^{-i\delta x} + \delta \int_x^\infty \sin\delta(\eta - x)q(\eta)y(\tau,-\delta)d\eta \tag{1.39}$$

We denote $f(x,-\delta)$ as its solution and then find C:

$$C = \frac{y_0}{f(0,-\delta)} \tag{1.40}$$

Function

$$y(x,-\delta) = y_0 \frac{f(x,-\delta)}{f(0,-\delta)}$$

(1.34), (1.37) is the solution of the Sturm-Louisville boundary value problem.

Based on the method of successive approximations, the solution of equation (1.39) is written as follows:

$$f(x,-\delta) = \sum_{n=0}^{\infty} \delta^n f_n(x,-\delta), \tag{1.41}$$

Here,

$$f_0(x,-\delta) = e^{-i\delta x} \tag{1.42}$$

$$f_n(x,-\delta) = \int_x^\infty \sin\delta(\eta - x)q(\eta)f_{n-1}(\eta,-\delta)d\eta \quad (n = 1, 2, \dots)$$

Using the formulas (1.17) and (1.20) we find from the current coordinates the functions u, w, p, σ. Let's take into account that,

$$F(x) = \frac{1}{\chi(x)} \frac{f(x,-\delta)}{f(0,-\delta)}$$

Then,

$$u = y_0 F(x)\exp(i\omega t) \tag{1.43}$$

$$w = y_0 \frac{iR_\infty}{\omega} \left\{ \frac{1}{2} g(x)F'(x) + g'(x)F(x) \right\} \exp(i\omega t) \tag{1.44}$$

$$p = y_0 k(x) \frac{iR_\infty}{\omega} \left\{ \frac{1}{2} g(x)F'(x) + g'(x)F(x) \right\} \exp(i\omega t) \tag{1.45}$$

$$\sigma = 2y_0 \eta \frac{b}{a} F'(x)\exp(i\omega t). \tag{1.46}$$

Let's set the pulsating pressure to record the fluid velocity, displacement, pressure and viscoelastic stress as the boundary condition in the pipe cross-section.

$$p(0,t) = p_0 \exp(i\omega t), \tag{1.47}$$

p_0- is a given empirical unit. If we compare (1.47) and (1.45),

$$y_0 = p_0 \frac{\omega}{iR_\infty k(0)\left\{\frac{1}{2}g(0)F'(0) + g'(0)F(0)\right\}},$$

and from formulas (1.43) - (1.46) we get:

$$\begin{aligned} u(x,t) &= -p_0 \frac{i\omega}{R_\infty k(0)} \frac{F(x)}{\frac{1}{2}g(0)F'(0) + g'(0)F(0)} \exp(i\omega t), \\ w(x,t) &= \frac{p_0}{k(0)} \frac{\frac{1}{2}g(x)F'(x) + g'(x)F(x)}{\frac{1}{2}g(0)F'(0) + g'(0)F(0)} \exp(i\omega t), \\ p(x,t) &= p_0 \frac{k(x)}{k(0)} \frac{\frac{1}{2}g(x)F'(x) + g'(x)F(x)}{\frac{1}{2}g(0)F'(0) + g'(0)F(0)} \exp(i\omega t), \\ \sigma(x,t) &= -2ip_0\eta \frac{b}{a} \frac{\omega}{R_\infty k(0)} \frac{F'(x)}{\frac{1}{2}g(0)F'(0) + g'(0)F(0)} \exp(i\omega t). \end{aligned} \tag{1.48}$$

Similarly, pulsating liquid consumption was given as a boundary condition in the cross section of the pipe:

$$Q = Q_0 \exp(i\omega t),$$

Here

$$Q(x,t) = S(x)u(x,t)$$

then,

$$y_0 = \frac{Q_0}{\pi R_\infty^2 g(0)} \frac{1}{F(0)}$$

From here

$$\begin{aligned} u(x,t) &= \frac{Q_0}{\pi R_\infty^2 g(0)} \frac{F(x)}{F(0)} \exp(i\omega t), \\ w(x,t) &= \frac{iQ_0}{\pi R_\infty \omega g^2(0)F(0)} \left\{\frac{1}{2}g(x)F'(x) + g'(x)F(x)\right\} \exp(i\omega t) \\ p(x,t) &= iQ_0 \frac{k(x)}{\pi R_\infty \omega g^2(0)F(0)} \left\{\frac{1}{2}g(x)F'(x) + g'(x)F(x)\right\} \exp(i\omega t), \\ \sigma(x,t) &= 2Q_0 \frac{\eta}{\pi R_\infty^2 g^2(0)} \frac{b}{a} \frac{F(x)}{F(0)} \exp(i\omega t). \end{aligned} \tag{1.49}$$

Formulas (1.48) and (1.49) describe the real parts, and our example is considered solved.

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