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The mixed problem for the uniformly elliptic equation in generalized weighted Morrey spaces

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Abstract. *In this paper we obtain generalized weighted Sobolev – Morrey estimates with weights from the Muckenhoupt class A_p by establishing boundedness of several important operators in harmonic analysis such as Hardy – Littlewood operators and Calderon –Zygmund singular integral operators in generalized weighted Morrey spaces. As a consequence, a priori estimates for the weak solutions mixed boundary problem uniformly elliptic equations of higher order in generalized weighted Sobolev – Morrey spaces in a smooth bounded domain $\Omega \subset R^n$ are obtained.*

Keywords. unifomly, elliptic equations, higher order, mixed boundary problem

Mathematics Subject Classification (2010): 35J30

1 Introduction

Later we define weighted Morrey spaces $L_{p.k}(\omega)$. Before Guliyev [15] give a concept of the generalized weighted Morrey spaces $M_{p,\phi}(\omega)$ which could be viewed as extension of both $M_{p,\varphi}$ and $L_{p,k}(\omega)$, study the boundedness of the classical operators and their commutators in spaces $M_{p,\phi}(\omega)$ was studied (see, also [18, 25]).

The reason to study continuity properties of these integrals in various functional spaces is that they permit to investigate the regularity of solutions to linear elliptic and parabolic partial differential equations and systems in terms of the data of the corresponding problems. The method, associated to the names of A. Calderon and A. Zygmund (see [2, 3]) uses explicit representation formula for the highest-order derivatives of the solution in terms of singular integrals acting on the known right-hand side plus another one acting on the very same derivatives. This last term appears in a commutator which norm can be made small enough if the coefficients have small oscillation over small balls. This way, suitable "integral continuity" of the principal coefficients ensure boundedness of the commutator and therefore validity of the corresponding a priori estimate. The Sarason class of functions

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with vanishing mean oscillation verifies this requirement although they could be discontinuous. Their good behavior on small balls allows to extend the classical theory of elliptic and parabolic equations and systems with continuous coefficients to operators with discontinuous coefficients (see [5, 6]). A vast number of works are dedicated to boundary value problems for linear elliptic and parabolic operators with *VMO* coefficients in the framework of Sobolev and Sobolev-Morrey spaces (see [7, 8, 16, 17, 20, 21, 23, 28]).

Later these results are extended on the generalized weighted Morrey spaces, which is obtained the boundedness of the Calderon-Zygmund operators from one generalized weighted Morrey space $M_{p,\varphi_1}(\omega)$ to another $M_{p,\varphi_2}(\omega)$ (see [15, 19]), if the pair functions (φ_1, φ_2) satisfy the following condition

$$\int_r^\infty \frac{\underset{t<s<\infty}{ess\inf}\ \varphi_1(x,s)\omega(B(x,s))^{\frac{1}{p}}}{\omega(B(x,s))^{\frac{1}{p}}}\frac{dt}{t} \leq C\varphi_2(x,r), \tag{1.1}$$

where C does not depend on x and r.

Let $W_p^{2m}(\omega)$ be the standard notation for Sobolev spaces. In [1] for the solutions of uniformly elliptic equations in a smooth domain ω the following a priori estimate

$$\|u\|_{W_p^{2m}(\Omega)} \leq C\|f\|_{L_p(\Omega)} \tag{1.2}$$

were obtained. In [27] on a bounded domain ω with smooth boundary $\partial\omega$ for the Laplace equation with weight be $\omega(x)$ belonging to the Muckenhoupt class A_p (see [4]) was proved the following a priori estimate

$$\|u\|_{W_p^2(\Omega,\omega)} \leq C\|f\|_{L_p(\Omega,\omega)}$$

Weighted estimates for a wide class of singular integral operators has been obtained for weights in the class of Muckenhoupt A_p. Therefore, it is a natural question whether analogous weighted a priori estimates can be proved for the derivativies of solutions elliptic equations. In [11] the previous results of [6] (also[12-14]) for powers of the Laplacian operator with homogeneous Drichlet boundary conditions were extended to weighted Sobolev spaces, i.e.,it is proved that

$$\|u\|_{W_p^{2m}(\Omega,\omega)} \leq C\,\|f\|_{L_p(\Omega,\omega)}.$$

For $\omega \in A_p$, where the constant C depends on Ω, m, n, ω.
In [20, 24], Guliyev, Gadjiev and Galandarova study the boundedness of the sublinear operators generated by Calderon-Zygmund operators in local generalized Morrey spaces. By using these results they prove the solvability of the Dirichlet boundary value problem for a polyharmonic equation in modified local generalized Sobolev-Morrey spaces and obtain a priori estimates for the solutions of the Dirichlet boundary value problems for the uniformly elliptic equations in modified local generalized Sobolev-Morrey spaces defined on bounded smooth domains.

Let we consider the homogeneous mixed problem

$$Lu = f \quad \text{in}\ \Omega$$

where $L = \sum\limits_{|\alpha|\leq 2m} a_\alpha D^\alpha$ – is uniformly elliptic. $\partial\Omega = \Gamma_1 \cup \Gamma_2$ in Γ_1 we are giving Dirichlet condition, in Γ_2 we are giving Newman boundary condition, $\Gamma_1 \cap \Gamma_2 = \emptyset$

There exists a constant γ such that

$$\gamma^{-1}\omega(x)\,|\xi\,|^2 \leq \sum_{|\alpha|\leq 2m} a_\alpha(x)\,\xi_\alpha\xi_\beta \leq \gamma\omega(x)\,|\xi|^2,$$

a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^n$ and matrix $a_\alpha(x)$ is real symmetrical matrix.

We define $l_1 > \max_j(2m-j)$ and $l_0 =$. If $a_\alpha \in C^{l_1+1}\left(\overline{\Omega}\right)$.

$|\alpha| \le 2m$, $b_\alpha \in C^{l_1+1}(\partial\Omega)$, $0 \le j \le m-1$, and $\partial\Omega \in C^{l_1+m+1}$, then we have Green function G_m and Poisson kernels K_j for $0 \le j \le m-1$ exist whenever $l_1 > 2(l_0+1)$ for $n=2$ and $l_1 > \frac{3}{2} l_0$ for $n \ge 3$.

Moreover, whenever they are defined, Green function and Poisson kernels of the operator L with these boundary conditions satisfy the estimates (1.7),(1.8), (1.9), (1.10) and (1.11) (see [9] and [10]). Then the following result is valid for weak solution of problem (1.1).

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and the coefficients of operators L and B_j satisfy the conditions $a_\alpha \in C^{l_1+1}\left(\overline{\Omega}\right)$, $|\alpha| \le 2m$, $b_\alpha \in C^{l_1+1}(\partial\Omega)$, $0 \le j \le m-1$. If $\omega \in A_p(\Omega)$, $f \in M_{p,\varphi}(\Omega, \omega)$, φ satisfies the condition* (2.1) *and $u(x)$ is a weak solution of* (1.1)*, then there exists a constant c depending only on n, m, w and Ω such that*

$$\|u\|\, W_0^{2m} M_{p,\,\varphi}(\Omega,\omega) \le C \, \|f\| \, M_{p,\varphi}(\Omega,\omega) \,. \tag{1.3}$$

The proof Theorem 1.1 is a consequence of the above estimates of the

Green function and Lemma 1.4. Corollary 1.1 implies that the operators M and T^* are bounded in $M_{p,\varphi}(\Omega,\omega)$. Therefore statement of the Theorem 1.1 and estimate (1.3) are immediately consequence of inequalities in Lemma 1.4 and Corollary 1.1. Thus the theorem is proved.

From Theorems 1.2 and 1.1, and estimates in Lemma 1.4 we get the following corollary.

Corollary 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and the coefficients of operators L and B_j satisfy the conditions $a_\alpha \in C^{l_1+1}\left(\overline{\Omega}\right)$, $|\alpha| \le 2m$, $b_\alpha \in C^{l_1+1}(\partial\Omega)$, $0 \le j \le m-1$. If $\omega \in A_p(\Omega)$, $f \in M_{p,\varphi_1}(\Omega,\omega)$ the pai (φ_1, φ_2) satisfies the condition* (1.17) *and $u(x)$is a weak solution of* (1.1)*, then there exists a constant C depending only on n, m, ω and Ω such that*

$$\|u\|\, W_0^{2m} M_{p,\varphi_2}(\Omega,\omega) \le C \, \|f\| \, M_{p,\varphi_1}(\Omega,\omega) \,. \tag{1.4}$$

In a bounded domain Ω with smooth boundary $\partial\Omega$. The solution of (1.2) is given by

$$u(x) = \int_\Omega G_m(x,y)\, f(y)\, dy \tag{1.5}$$

where $G_m(x,y)$ is the Green function of the operator in Ω which can be written as

$$G_m(x,y) = \Gamma(x-y) + h(x,y), \tag{1.6}$$

where $G_m(x,y)$ is a fundamental solution, $h(x,y)$ satisfies

$$(-\Delta_x)^m h(x,y) = 0, \; x \,\epsilon\, \Omega,$$

$$\left(\frac{\partial}{\partial v}\right)^j h(x,y) = -\left(\frac{\partial}{\partial v}\right)^j \Gamma(x-y), \; x \,\epsilon \Gamma_1, \; 0 \le j \le m-1,$$

for each fixed $y \,\epsilon\, \Omega$. Then

$$h(x,y) = -\sum_{j=0}^{m-1} \int_{\partial\Omega} K_j(y,p) \left(\frac{\partial}{\partial v}\right)^j \Gamma(P-x)\, ds,$$

where $K_j(y,p)$ are the Poisson kernels, ds denotes the surface measure on $\partial\Omega$. We have the known estimates of the Green function $G_m(x,y)$ and the Poisson kernels $K_j(x,y)$

$$|D_x^\alpha G_m(x,y)| \le C_4,\ for\ |\alpha| < 2m-n, \tag{1.7}$$

$$|D_x^\alpha G_m(x,y)| \le C_5 \log\left(\frac{2diam(\Omega)}{|x-y|}\right),\ for\ |\alpha| < 2m-n, \tag{1.8}$$

$$|D_x^\alpha G_m(x,y)| \le C_6 |x-y|^{2m-n-|\alpha|},\ for\ |\alpha| < 2m-n, \tag{1.9}$$

$$|D_x^\alpha G_m(x,y)| \le C_6 \frac{1}{|x-y|^{"}} \cdot \min\left\{1, \ \frac{d(y)}{|x-y|}\right\}^m,\ for\ |\alpha| = 2m, \tag{1.10}$$

$$|\,K_j(x,y)| \le\ C_7 \frac{dx}{|x-y|^{n-j+m-1}},\quad for\ 0\ \le j \le m-1, \tag{1.11}$$

where $d(x) = dist\ (x,\ \partial\Omega)$ (see [10, 11]).

Also we give known results by point wise estimates.

Lemma 1.1 *([20]). Let $u(x)$ be the solution of the problem* (2.1) *and $|\alpha| \le 2m-n$. Then there exists a constant C depending on $n,\ m$ and Ω for all $x\epsilon\,\Omega$ such that*

$$|D^\alpha u(x)| \le CM\ f(x).$$

Lemma 1.2 *([11]). Let $f,\ g$ be measurable functions on, $\Omega,\ |\alpha| = 2m$ and $D = \{(x,y)\,\epsilon\,\Omega\times\Omega:\ |x-y| > d(x)\}$. Then there exists a constant C depending on n, and Ω such that*

$$\int_D |D^\alpha G_m(x,y)\,f(y)\,g(x)|\,dxdy \le C\left(\int_D Mf(y)\,|g(y)|\,dy + \int_D Mg(y)\,|f(y)|\,dy\right).$$

In order to see how to estimate $D_x^\alpha h(x,y)$ in $\Omega\diagdown D$, we consider separately the functions $h(x,y)$ and $\Gamma\ (x,y)$ involved in $G_m(x,y)$.

Lemma 1.3 *([11]). If $|\alpha| > 2m-n+1$, then there exists a constant C such that*

$$|D_x^\alpha h(x,y)| \le C d^{2m-n-|\alpha|}(x)\quad for\ \ |x-y| \le d(x). \tag{1.12}$$

Let T be a Calderon-Zygmund singular integral operator, briefly a Calderon-Zygmund operator is a linear operator bounded from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to functions in $L_1^{loc}(\mathbb{R}^n)$, represented for such functions by

$$T\ f(x) = \int_{\mathbb{R}^n} K(x,y)\,f(y)\,dy,$$

here $K\ (x,y)$ is a continuous function which satisfies the standard estimates.
It follows from the previous lemmas that for each $x\epsilon\,\Omega$ and $|\alpha| > 2m-n+1$ we have $D_x^\alpha h(x,y)$ is bounded uniformly in a neighborhood of x and so

$$D_x^\alpha \int_\Omega h(x,y)\,f(y)\,dy = \int_\Omega D_x^\alpha h(x,y)\,f(y)\,dy. \tag{1.13}$$

On the other hand, although $D_x^\alpha \Gamma(x,y)$ is a singular kernel for $|\alpha| = 2m$,,
taking β such that $|\beta| = 2m-1$, we have that

$$D_x^\alpha \int_\Omega D_x^\alpha \Gamma(x-y)\,f(y)\,dy = T\ f(x) + a(x)\,f\ \ (x), \tag{1.14}$$

where $a(x)$ is a bounded function and T is a Calderon-Zygmund operator given by

$T\ f(x) = \lim_{\varepsilon\to 0} T_\varepsilon f(x)$ with $T_\varepsilon f(x) = \int_{\mathbb{R}^n \diagdown B(x,\varepsilon)} D_x^\alpha \Gamma(x-y)\,f(y)\,dy.$

We will also make use of the maximal singular operator $T^* f(x) = \sup|T_\varepsilon f(x)|$. Here and in what follows we consider f defined in $\mathbb{R}^n$ extending the original f by zero.

Lemma 1.4 *([11]). Let $g(x)$ be a measurable function on Ω and $|\alpha| = 2m$.Then there exists a constant C depending only on n, m and Ω such that*

$$\int_\Omega |D^\alpha u(x)\, g(x)|\, dx \le C \left(\int_\Omega T^* f(x)\, |g(x)|\, dx + \int_\Omega Mf(x)\, |g(x)|\, dx + \right.$$

$$\left. + \int_\Omega Mg(x)\, |f(x)|\, dx + \int_\Omega |f(x)|\, |g(x)|\, dx \right)$$

We define the generalized weighted Morrey spaces.

Let $1 \le p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and ω be nonnegative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(\omega)$ the generalized weighted Morrey spaces, the space of all functions $f \in L^{loc}_{p,\omega}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_{p,\varphi}(\omega)} = \sup_{x\in\mathbb{R}^n,\ r>0} \varphi^{-1}(x,r)\,,\ \omega(B(x,r))^{-\frac{1}{P}} \|f\|_{L_{p,\omega}(B(x,r))},$$

where $L_{p,\omega}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,\omega}(B(x,r))} \equiv \left\|f_{XB(x,r)}\right\|_{L_{p,\omega}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^P\, \omega\ (y)\, dy \right)^{\frac{1}{P}}$$

Furthermore, by $WM_{p,\varphi}(\omega)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL^{loc}_{p,w}(\mathbb{R}^n)$ for wnich

$$\|f\|_{WM_{p,\varphi(\omega)}} = \sup_{x\in\mathbb{R}^n,\ r>0} \varphi^{-1}(x,r)\,, \omega\,(B(x,r))^{-\frac{1}{P}} \|f\|_{WL_{p,\omega}(B(x,r))} < \infty,$$

where $WL_{p,\omega}(B(x,r))$ denotes the weak $L_{p,\omega}$-space of measurable functions f for which

$$\|f\|_{WL_{p,\omega(B(x,r))}} \equiv \left\|f_{XB(x,r)}\right\| \left\|f_{XB(x,r)}\right\|_{WL_{p,\omega(\mathbb{R}^n)}} =$$

$$= \sup_{t>0} \left(\int_{y\in B(x,r):|f(y)|>t} \omega(y)\, dy \right)^{\frac{1}{P}}$$

Remark 1.1. 1. If $\omega = 1$, then $M_{p,\varphi}(\Omega,1) = M_{p,\varphi}(\Omega)$ is the generalized Morrey space.

2. If $\varphi(x,r) = \omega(B(x,r))^{\frac{k-1}{P}}$, then $M_{p,\varphi}(\Omega,\omega) = L_{p,k}(\omega)$ is the weighted Morrey space.

3. If $(x,r) = \vartheta(B(x,r))^{\frac{k}{P}}\ \omega(B(x,r))^{-\frac{1}{P}}$, then $M_{p,\varphi}(\Omega,\omega) = L_{p,k}(\vartheta,\omega)$ is the two weighted Morrey space.

4. If $\omega = 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{P}}$ with $0 < \lambda < n$, then $M_{p,\varphi}(\omega) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(\omega) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

5. If $\varphi(x,r) = \omega(B(x,r))^{-\frac{1}{P}}$, then $M_{p,\varphi}(\Omega,\omega) = L_{p,\omega}(\mathbb{R}^n)$ is the weighted Lebesgue space.

For any bounded domain Ω we define $M_{p,\varphi}(\Omega)$ taking $f \in L_{p,\omega}(\Omega)$ and Ω_r instead of $B(x,r)$ in the norm above and $\Omega_r = \Omega \cap B(x,r)$. The generalized weighted Sobolev-Morrey space $W^m_{p,\varphi,\omega}(\Omega)$ consists of all Sobolev functions $u \in W^m_P(\Omega)$ with distributional derivatives $D^s u \in M_{p,\varphi}(\Omega,\omega)$, which vanishing to zero in $\Gamma_1$endowed with the norm

$$\|u\|_{W_0^{2m} M_{p,\varphi}(\Omega,\omega)} = \sum_{0 \le |s| \le m} \|D^s u\|_{M_{p,\varphi}(\Omega,\omega)}.$$

Suppose that T_0 represents a linear or a sublinear operator, which satis- fies, for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \neq supp\, f$

$$|T_0 f(x)| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy \tag{1.15}$$

where C is independent of f and x.
For a function b, suppose that the commutator operator $T_{0,\,b}$ represents a linear or a sublinear operator, which satis_es that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \neq supp\, f$

$$|T_{0,\,b} f(x)| \leq C \int_{\mathbb{R}^n} |b(x) - b(y)| \frac{|f(y)|}{|x-y|^n} dy \tag{1.16}$$

where C is independent of f and x.

Theorem 1.2 *([15]). Let $1 \leq p < \infty$, $w \in A_p$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \frac{\underset{t<s<\infty}{ess\inf}\ \varphi_1(x,s)\omega(B(x,s))^{\frac{1}{P}}}{\omega(B(x,s))^{\frac{1}{P}}} \frac{dt}{t} \leq C\varphi_2(x,r), \tag{1.17}$$

where C does not depend on x and r. Let T_0 be a sublinear operator satisfying condition (1.15) *bounded on $L_{p,\omega}(\mathbb{R}^n)$ $for\ p > 1$,, and bounded from $L_{1,\,\omega}(\mathbb{R}^n)$ to $WL_{1,\omega}(\mathbb{R}^n)$. Then the operator T_0 is bounded from $M_{p,\varphi_1}(\omega)$ to $M_{p,\varphi_2}(\omega)$ $for\ p > 1$ and from $M_{1,\varphi_1}(\omega)$ to $WM_{1,\varphi_2}(\omega)$.*

Theorem 1.3 *([15]). Let $1 < p < \infty$, $\omega \in A_p$, $b \in BMO(\mathbb{R}^n)$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \left(1 + ln\frac{t}{r}\right) \frac{\underset{t<s<\infty}{\text{ess inf}} \varphi_1(x,s)\,\omega(B(x,s))^{\frac{1}{P}}}{\omega(B(x,s))^{\frac{1}{P}}} \frac{dt}{t} \leq C\varphi_2(x,r), \tag{1.18}$$

where C does not depend on x and r. Let $T_{0,b}$ be a sublinear operator satisfying condition (1.16) *bounded on $L_{p,\omega}(\mathbb{R}^n)$. Then the operator $T_{0,b}$is bounded from $M_{p,\varphi_1}(\omega)$ to $M_{p,\varphi_2}(\omega)$.*

For $\varphi_1(x,r) = \varphi_2(x,r) \equiv \omega(B(x,r))^{\frac{k-1}{P}}$, from Theorems 1.2 and 1.3 we have the following results.

Theorem 1.4 *([26]). Let $1 \leq p < \infty$, Let $1 < k < 1$ and $w \in A_p$. Let also T_0 be a sublinear operator satisfying condition* (1.15) *bounded on $L_{p,\omega}(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_{1,\omega}(\mathbb{R}^n)$ to $WL_{1,\omega}(\mathbb{R}^n)$. Then the commutator of sublinear operator T_0 is bounded on $L_{p,k}(\omega)$ $for\ p > 1$, and bounded from $L_{1,k}(\omega)$ to $WL_{1,k}(\omega)$.*

Theorem 1.5 *([26]). $1 < p < \infty$, $Let\ 1 < k < 1$, $b \in BMO\ (\mathbb{R}^n)$ and $\omega \in A_p$. Let also $T_{0,b}$ be a sublinear operator satisfying condition* (1.16) *bounded on $L_{p,\omega}(\mathbb{R}^n)$. Then the sublinear commutator operator $T_{0,b}$ is bounded on $L_{p,k}(\omega)$.*

Corollary 1.2 *Note that from Theorem 1.4 we get that for the maximal commutator operator M_b, the commutator of Calderon-Zygmund operators $[b,\ T]$ and the commutator of maximal singular operators $[b,\ T^*]$ are bounded on generalized weight Morrey space, i.e. are bounded from $M_{p,\varphi_1}(\Omega,\omega)$ to $M_{p,\varphi_2}(\Omega,\omega)$.*

We recall the definition of A_p class for $1 < p < \infty$. A non-negative locally integrable function $\omega(x)$ belongs to A_p if there exists a constant C_{11} such that

$$\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right) \left(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{p-1}}(x)\,dx\right)^{p-1} \leq C_{11}$$

for all cube $Q \subset \mathbb{R}^n$.

2 Main result

We can now state and prove our main result.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and φ satisfy the condition*

$$\int_r^\infty \frac{\underset{t<s<\infty}{ess\inf} \ \varphi_1(x,s)\omega(B(x,s))^{\frac{1}{p}}}{\omega(B(x,s))^{\frac{1}{p}}} \frac{dt}{t} \le C\varphi_2(x,r), \tag{2.1}$$

where C does not depend on x and r. If $\omega \in A_p(\Omega)$, $f \in M_{p,\varphi}(\Omega,\omega)$ and $u\ (x)$ a weak solution of (2.1)*, then there exists a constant C depending only $n,\ m,\ \omega$ and Ω such that*

$$\|u\|\, W_0^{2m} M_{p,\varphi}(\Omega,\omega) \leqq C\,\|f\|\, M_{p,\varphi}(\Omega,\omega).$$

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