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On the velocity profile of steady flow of the mixed fluids

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Abstract. *This paper considers the formulas for velocity profile, pressure drop, shear stress for steady-state flow of a two-component fluid in a vertical circular tube. Also volumetric and mass flow rates formulas are derived for the case. In particularly an analogue of Poiseuille's formula is proposed for the two-component viscous fluid. The velocity is a continuous function at the interface between two fluids but the radial derivative suffers a break on it.*

Keywords. laminar flow · incompressible flow · Newtonian fluid · velocity profile · shear stress · volumetric flow rate

Mathematics Subject Classification (2010): 76D05, 76T06, 76D07, 35Q30

1 Introduction

The flow of a mixed fluid in circular pipes is the subject of this paper. Velocity profile, shear stress and flow rate equation for a two-component mixture of incompressible viscous fluids are determined at different values of radius, length and pressure drop on the pipe. Note, this questions the first time is studied by Hagen (1839) and Poiseuille (1840). Accordingly, the fluid velocity in a pipe changes from zero at the pipe surface because of the no-slip condition to a maximum at the pipe center [2]. Precisely [11, 12], the velocity profile is given by

$$v = \frac{R^2}{4\mu}\frac{\Delta p}{L}\left(1 - \frac{r^2}{R^2}\right), \tag{1.1}$$

where $\frac{\Delta p}{L}$ is pressure gradient across length of tube, shear stress is given by:

$$\tau = -\mu\frac{dv}{dr} \tag{1.2}$$

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where $\frac{dv}{dr}$ is gradient of velocity component along shear direction with respect to displacement along normal direction, which decreases away from center line towards wall. Setting (1.1) into (1.2) it follows that

$$\tau = \frac{r}{2}\frac{\Delta p}{L}, \tag{1.3}$$

volumetric flow rate is given by:

$$Q = \frac{\pi R^2}{8\mu}\frac{\Delta p}{L} \tag{1.4}$$

Notable contributions to the subject were made from late 17th century through first half of 20th century by Newton, Bernoulli, Euler, d'Alembert, Lagrange,
Rayleigh, Reynolds, Navier, Stokes and Prandtl which matured the physical understanding of the various fluid flow types and flow phenomena, and produced realistic mathematical solutions to the fluid flow problems. There are a lot of engineering applications of this studies to the life areas including ships, sub-marines, swimming, aircraft, balloons, waterfalls, windmills, respiratory systems, blood supply systems, irrigation and water supply systems, jets, rockets and spacecraft. In short, almost every object on earth is either a fluid or it moves through a fluid. In later half of 20th century, advanced tools of Computational Fluid Dynamics (CFD) and Direct Numerical Simulation (DNS) became available for solving advanced flow problems [6, 8]. The Laser-Doppler measurement is used to determine the velocity profile [6]. We refer to [7, 10] for the unsteady flow profile in a circular pipe. Davey and Drazin (1969) numerically analyzed the stability of Poiseuille flow in a circular pipe [4].

In this paper, we estublish a velosity profile results of the two-component mixed viscous liquid over the cross section of vertical pipe. We show also the pressure drop, share strain and volumetric and mass flow rate result for the case based on the obtained general formula of velocity expression. The obtained results agreed with the cited works in formulas (1.1)-(1.4). We write the kinetic energy expression using the velocity profile for the case. Such a results are actual in the study of the rheology of the fluid. Note that, density of components of fluid are determined by their viscosity as a function $\rho = f(\mu)$ which is a nonlinear function in general. In particular the proportional dependences of type $\rho = C\ \mu$ or $\rho = C_1\ (\mu + C_2)$ are actual. However, there are liquids which has less density, than water which have more greater viscosity, e.g. so are the money, ice mixture, olive oil. For those more suitable the inverse dependence$\rho = \frac{C}{\mu}$ or $\rho = \frac{C_1}{\mu + C_2}$ and $\rho = \frac{C_1\mu}{\mu + C_2}$.

2 Mathematical model

In this article, we will consider the one-dimensional axes-parallel flow of a mixed fluid in a vertical pipe. The equation of such motion is expressed by the Navye-Stokes equation [1, 3, 9]. Our results refer to steady motion in an arbitrary symmetric pipe. But sometimes we will use the terminology "well" instead of the word "pipe". This study is relevant for oil and gas wells. We assume center of spatial axes x, y, z situated at the mouth of pipe (well) and the x -axe goes towards to the bottom of the pipe (well), and the remaining rectangular coordinates y, z are in a plane parallel to the cross section of the pipe.

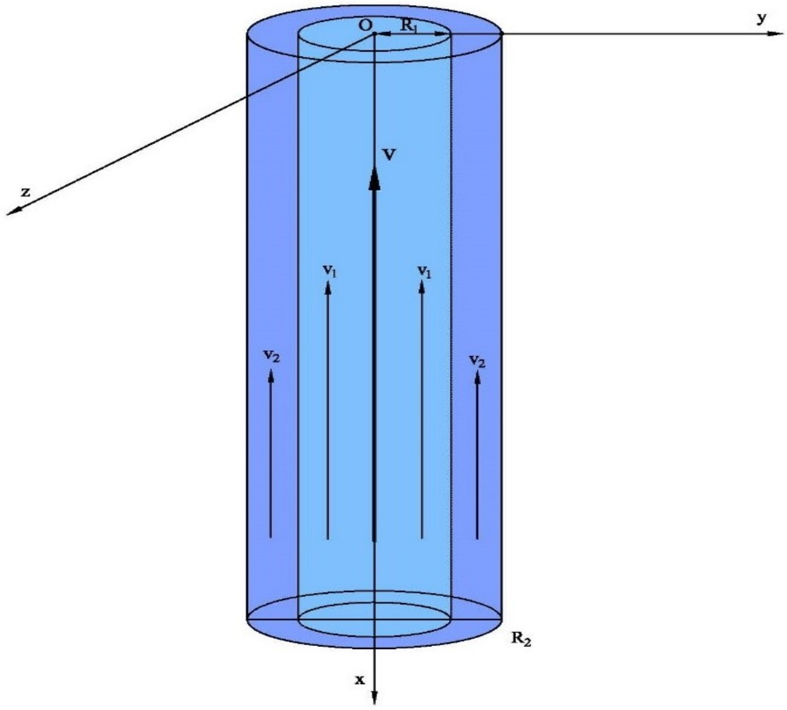

Fig. 1. Mixture flow in the pipe

The given motion equations for the considered processes is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho(r)}\frac{\partial p}{\partial x}=-\frac{1}{r\rho(r)}\frac{\partial}{\partial r}\left(r\mu(r)\frac{\partial v}{\partial r}\right)+g\,,\quad 0<r<R \tag{2.1}$$

and

$$\frac{\partial}{\partial x}\left(\rho(r)v(r)\right)=0\,,\quad 0<r<R$$

for the continuity equation. Since we consider the stable regime of the liquid flow it is assumed that $\frac{\partial v}{\partial t}=0$ and $\frac{\partial v}{\partial x}=0$ at the equation (2.1). Therefore,

$$\begin{cases} \frac{\partial p}{\partial x}=-\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\left(r\right)\frac{\partial v}{\partial r}\right)+g\rho(r),\ 0<r<R \\ v\left(R\right)=0 \end{cases} \tag{2.2}$$

is the system of equations which due to the study. The right and left sides of the equation are functions of x and r respectively. This means both sides are equal to the same constant. Therefore,

$$\begin{cases} -\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\left(r\right)\frac{\partial v}{\partial r}\right)+g\rho\left(r\right)=\lambda\,, \\ \frac{\partial p}{\partial x}=\lambda,\ \ 0<r<R \end{cases} \tag{2.3}$$

3 The velosity and pressure profiles

Now consider the following problem for the case. Given: the depth and radius of the well, viscosity and density of the mixture (arranged by the distance from the well center r) wellhead and bottomhole pressures of well:

$$H; p\left(0\right), p\left(H\right), p\left(H\right) > \ p\left(0\right); \qquad \mu\left(r\right),\ \rho(r), R, \quad 0 < r < R\,.$$

Let us find the distribution of velocity and pressure:

$$v\left(r\right), 0 < r < R; p\left(x\right), 0 < x < H.$$

in dependence of r. Note the axe ox is ordered from the wellhead to the bottomhole and the coordinats y, z are situated on a parallel plane to cross-section of the pipe.

Integrate the second equation (2.3) over $(0,\ x)$,

$$p(x) - p(0) = \lambda x, \quad 0 < x < H.$$

From here,

$$p\left(H\right) - p\left(0\right) = \lambda H,$$

which means

$$\lambda = \frac{p(H) - p(0)}{H}.$$

Using the above expression of $p(x)$we get the pressure distribution,

$$p(x) = p(0) + \frac{p(H) - p(0)}{H}x, \quad 0 < x < H. \tag{3.1}$$

Now, pass to the first equation (2.3):

$$\frac{\partial}{\partial r}\left(r\mu\left(r\right)\frac{\partial v}{\partial r}\right) = -\left(\lambda - g\rho(r)\right)r,\ \ 0 < r <\ R$$

Integrate this equation over the interval $(0,\ z)$:

$$z\mu\left(z\right)\frac{\partial v}{\partial r}\left(z\right) = -\int_0^z \left(\lambda - g\rho(r)\right)rdr, \quad 0 < z <\ R.$$

Then

$$\frac{\partial v}{\partial r}\left(z\right) = -\frac{1}{z\mu\left(z\right)}\int_0^z \left(\lambda - g\rho(r)\right)rdr, \quad 0 < z <\ R.$$

Integrating again this equation over $(\xi,\ R)$and using that $v(R) = 0$ we get

$$v\left(\xi\right) = \int_\xi^R \frac{1}{z\mu\left(z\right)}\int_0^z \left(\lambda - g\rho(r)\right)rdr, \qquad 0 < \xi <\ R.$$

Inserting the value of λ founded above we get the velocity profile formula for the process

$$v\left(\xi\right) = \int_\xi^{R_2} \frac{1}{z\mu\left(z\right)}\int_0^z \left(\frac{\mathrm{p}\left(\mathrm{H}\right) - \mathrm{p}\left(0\right)}{H} - g\rho(r)\right)rdr,\ 0 < \xi <\ R. \tag{3.2}$$

There, $\int_0^R \frac{dz}{z\mu(z)}\ <\ \infty$

4 The velocity profile of two component mixture

We use the formula (3.2) to determine the two-component mixture flow setting that

$$\begin{cases} \mu(r) = \mu_1, & 0 < r < R_1 \\ \mu(r) = \mu_2, & R_1 < r < R_2 \end{cases}$$

and

$$\begin{cases} \varrho(r) = \varrho_1, & 0 < r < R_1 \\ \varrho(r) = \varrho_2, & R_1 < r < R_2 \end{cases},$$

$$\begin{cases} v(r) = v_1(r), & 0 < r < R_1 \\ v(r) = v_2(r), & R_1 < r < R_2 \end{cases}$$

Using the formula (3.2) find the velocity and pressure distribution over the well depth as the following

$$v_2(\xi) = \int_{\xi}^{R_2} \left[\int_0^{R_1} (\lambda - g\rho_1)\, r dr + \int_{R_1}^{z} (\lambda - g\rho_2)\, r dr \right] \frac{dz}{z\mu_2}, \quad R_1 \le \xi < R_2. \tag{4.1}$$

Simplify this expression as

$$v_2(\xi) = \frac{(\lambda - g\rho_1)\, R_1^2}{2\mu_2} \ln \frac{R_2}{\xi} + \frac{(\lambda - g\rho_2)\left(R_2^2 - \xi^2\right)}{4\mu_2} - $$
$$- \frac{(\lambda - g\rho_2)\, R_1^2}{2\mu_2} \ln \frac{R_2}{\xi}, \quad R_1 \le \xi < R_2.$$

Here $\lambda = \frac{\mathrm{p(H)-p(0)}}{H}$ using it, for $v_2(\xi)$ we get the expression

$$v_2(\xi) = \frac{\left(\frac{p(H)-p(0)}{H} - g\rho_1\right) R_1^2}{2\mu_2} \ln \frac{R_2}{\xi} + \frac{\left(\frac{p(H)-p(0)}{H} - g\rho_2\right)\left(R_2^2 - \xi^2\right)}{4\mu_2} -$$
$$- \left(\frac{p(H) - p(0)}{H} - g\rho_2\right) R_1^2 \frac{1}{2\mu_2} \ln \frac{R_2}{\xi}, \; R_1 \le \xi < R_2. \tag{4.2}$$

By the same manner for $v_1(r)$ we find the expression

$$v_1(\xi) = \int_{\xi}^{R_1} \frac{dz}{z\mu_1} \int_0^{z} (\lambda - g\rho_1)\, r dr +$$
$$+ \int_{R_1}^{R_2} \frac{dz}{z\mu_2} \left[\int_0^{R_1} (\lambda - g\rho_1)\, r dr + \int_{R_1}^{z} (\lambda - g\rho_2)\, r dr \right], \quad 0 < \xi \le R_1 \tag{4.3}$$

Integrating the terms given there we find that

$$v_1(\xi) = \frac{(\lambda - g\rho_1)\left(R_1^2 - \xi^2\right)}{4\mu_1} + \frac{(\lambda - g\rho_1)\, R_1^2}{2\mu_2} \ln \frac{R_2}{R_1} + \frac{(\lambda - g\rho_2)\left(R_2^2 - R_1^2\right)}{4\mu_2} -$$
$$- \frac{(\lambda - g\rho_2)\, R_1^2}{2\mu_2} \ln \frac{R_2}{R_1}, \qquad 0 < \xi \le R_1 \; . \tag{4.4}$$

Finally, using $\lambda = \frac{\mathrm{p(H)-p(0)}}{H}$ we get that

$$v_1(\xi) = \frac{\left(\frac{\mathrm{p(H)-p(0)}}{H} - g\rho_1\right)\left(R_1^2 - \xi^2\right)}{4\mu_1} + \frac{\left(\frac{\mathrm{p(H)-p(0)}}{H} - g\rho_1\right)R_1^2}{2\mu_2}\ln\frac{R_2}{R_1} +$$

$$+\frac{\left(\frac{p(H)-p(0)}{H} - g\rho_2\right)\left(R_2^2 - R_1^2\right)}{4\mu_2} - \frac{\left(\frac{\mathrm{p(H)-p(0)}}{H} - g\rho_2\right)R_1^2}{2\mu_2}\ln\frac{R_2}{R_1},$$

$$0 < \xi \le R_1. \tag{4.5}$$

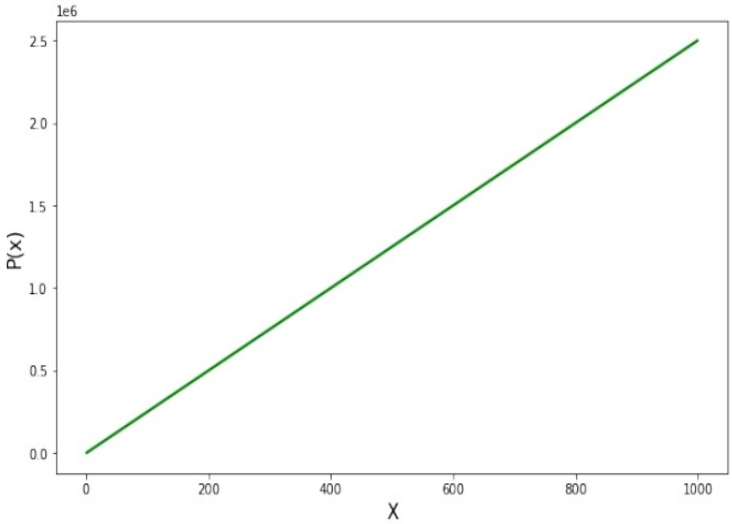

Fig. 2. Dependence of pressure on depth

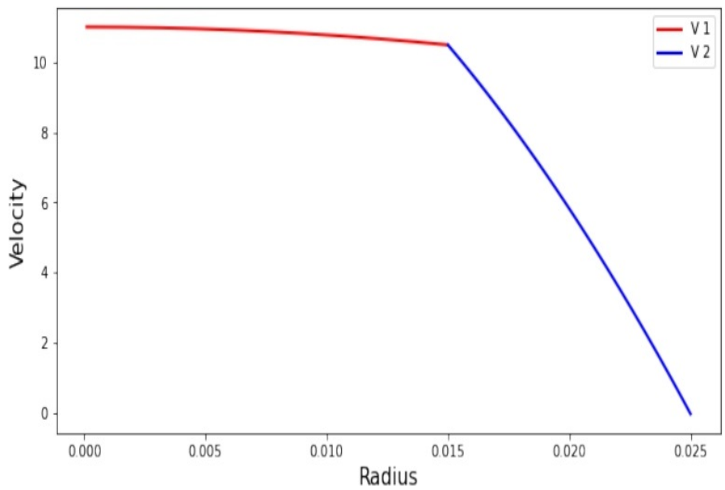

Fig. 3. Dependence of velocity on radius

The volumetric rate

In order to apply formula (3.2) first assume that,

$$\lambda - \rho g \sim \lambda - \overline{\rho} g, \quad \lambda = \frac{p\left(H\right) - p(0)}{H},$$

where $\overline{\rho} = \frac{1}{\pi R^2} \int_0^R \rho(r) r dr$. Denote

$$\Lambda = \lambda - \overline{\rho} g.$$

Then for the volumetric rate of the flow we get the formula

$$Q = 2\pi\Lambda \int_0^R v(r) r dr = 2\pi\Lambda \int_0^R \left(\int_r^R \frac{z dz}{\mu\left(z\right)} \right) r dr =$$
$$= 2\pi\Lambda \int_r^R \frac{z dz}{\mu\left(z\right)} \left(\int_0^z r dr \right) = \pi\Lambda \int_0^R \frac{z^3 dz}{\mu\left(z\right)}.$$

Using this formula we have

$$Q = \pi\Lambda \int_0^{R_1} \frac{z^3 dz}{\mu_1} + \pi\Lambda \int_{R_1}^{R_2} \frac{z^3 dz}{\mu_2} = \pi\Lambda \left(\frac{R_1^4}{8\mu_1} - \frac{R_1^4}{8\mu_2} + \frac{R_2^4}{8\mu_2} \right).$$

Therefore,

$$Q = \pi \left[\frac{R_2^4}{8\mu_2} + \left(\frac{1}{\mu_1} - \frac{1}{\mu_2} \right) \frac{R_1^4}{8} \right] \left(\frac{p\left(H\right) - p(0)}{H} - \overline{\rho} g \right).$$

From here we get the expression for the volumetric rate of flow

$$Q_1 = \pi \left[\frac{R_2^4}{8\mu_2} + \left(\frac{1}{\mu_1} - \frac{1}{\mu_2} \right) \frac{R_1^4}{8} \right] \frac{p\left(H\right) - p(0)}{H}$$

if to neglect the the gravity. Also for the case of the equal components $\mu_1 = \mu_2$ we come to the known Poiseuill formula

$$Q_2 = \frac{\pi\ \Delta p\ R^4}{8H\mu}.$$

Mass rate of the flow

Using the formula (3.2) we have the following expression for the mass rate. Applying here the formula (3.2) we get the expression for mass rate of the flow

$$\hat{Q} = 2\pi\Lambda \int_0^R \rho(r) v(r) r dr$$

Here for the two-component liquid flow we have the expression

$$\hat{Q} = 2\pi\Lambda \int_0^{R_1} \rho_1 r dr \left[\int_r^{R_1} \frac{z}{\mu_1} dz + \int_{R_1}^{R_2} \frac{z}{\mu_2} dz \right] + 2\pi\Lambda \int_{R_1}^{R_2} \rho_2 r dr \left[\int_{R_1}^{R_2} \frac{z}{\mu_2} dz \right]$$

$$= 2\pi\Lambda \int_0^{R_1} \rho_1 r dr \left[\left. \frac{z^2}{2\mu_1} \right|_r^{R_1} + \left. \frac{z^2}{2\mu_2} \right|_{R_1}^{R_2} \right] + + 2\pi\Lambda \int_0^{R_1} \rho_2 r dr \left[\left. \frac{z^2}{2\mu_2} \right|_r^{R_2} \right].$$

Therefore, for the mass rate of mixture we get the formula

$$\hat{Q} = \pi\Lambda \left[\frac{\rho_1 - \rho_2}{4\mu_2} R_1^2 R_2^2 + \frac{\rho_2}{8\mu_2} R_2^4 + \left(\frac{\rho_1}{8\mu_1} - \frac{\rho_1}{4\mu_2} + \frac{\rho_2}{8\mu_2} \right) R_1^4 \right].$$

5 The energy expression

In this paragraph we derive the energy expression dependent on viscosity and density distribution of the mixture. For that denote

$$G(z) = \int_0^z (\lambda - \rho(y)g)ydy,$$

then the energy is writen as

$$E = \int_0^R \rho(r)\, v(r)^2 rdr$$

Using Fubinii's formula we have

$$E = \int_0^R \frac{G(z)dz}{z\mu(z)} \left(\int_0^z \rho(r)\, v(r)\, rdr \right). \tag{5.1}$$

Inserting (3.2) into (5.1) we get

$$E = \int_0^R \frac{G(z)dz}{z\mu(z)} \left(\int_0^z \frac{G(\xi)d\xi}{\xi\mu(\xi)} \left(\int_0^\xi \rho(r)rdr \right) \right) +$$

$$+ \int_0^R \frac{G(z)dz}{z\mu(z)} \left(\int_z^R \frac{G(\xi)d\xi}{\xi\mu(\xi)} \left(\int_0^z \rho(r)rdr \right) \right). \tag{5.2}$$

and integrating it by parts,

$$E = \frac{1}{2} \int_0^R \left(\int_z^R \frac{G(\xi)\, d\xi}{\xi\mu(\xi)} \right)^2 \rho(z)\, zdz. \tag{5.3}$$

Therefore, we get the formula for the energy expression:

$$E = \int_0^R \left(\int_\xi^R \frac{G(z)\, dz}{z\mu(z)} \right)^2 \rho(\xi)\, \xi d\xi.$$

Some application information.
$P(0) = 0.3MPa$
$P(H) = 2.8MPa$
$\rho_1 = 235kq/m^3$
$\rho_2 = 240kq/m^3$
$\mu_1 = 16.5mPa\ san$
$\mu_2 = 1mPa\ san$
$g = 10m/san^2$
$H = 1000m$
$R_1 = 15mm$
$R_2 = 25mm$

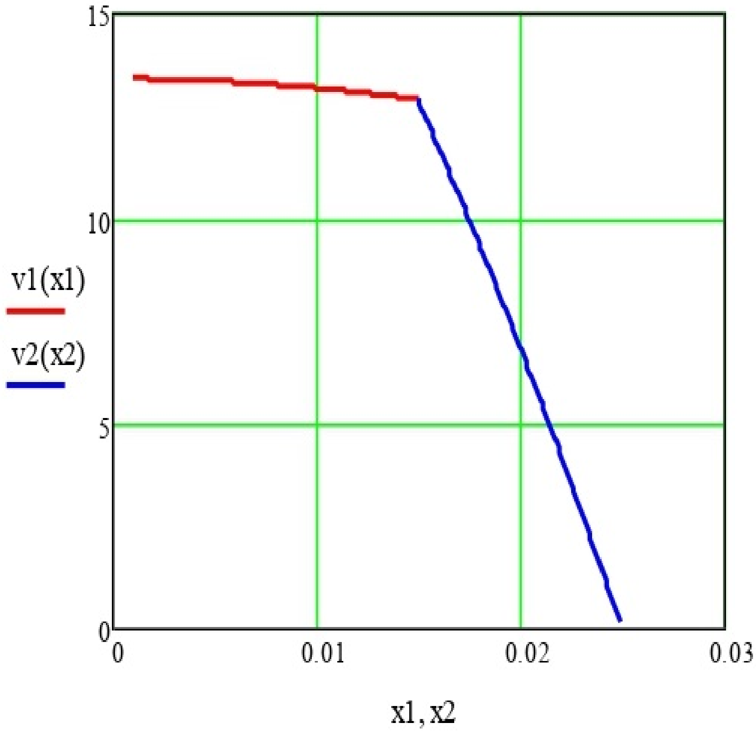

Fig. 4. Dependence of velocity on radius

$$\lambda = \frac{p(H) - p(0)}{H}$$

$$\lambda = 2.5 \times 10^3$$

$$v_1(x_1) = \frac{(\lambda - g\rho_1)\left(R_1^2 - {x_1}^2\right)}{4\mu_1} + \frac{(\lambda - g\rho_1) R_1^2}{2\mu_2}\ln\frac{R_2}{R_1} + \frac{(\lambda - g\rho_2)\left(R_2^2 - R_1^2\right)}{4\mu_2} -$$

$$-\frac{(\lambda - g\rho_2) R_1^2}{2\mu_2}\ln\frac{R_2}{R_1},$$

$$x_1 = 0.001,\ 0.0011..,\ 0.015$$

$$v_2(x_2) = \frac{(\lambda - g\rho_1) R_1^2}{2\mu_2} ln\frac{R_2}{x_2} + \frac{(\lambda - g\rho_2)\left(R_2^2 - {x_2}^2\right)}{4\mu_2} - \frac{(\lambda - g\rho_2) R_1^2}{2\mu_2} ln\frac{R_2}{x_2}$$

$$x_2 = \ 0.015,\ 0.0151..,\ 0.0249$$

$$v_1(0.015) = 12.873 \qquad v_2(0.015) = 12.873$$

Table 1. The values of velocities

$v_1\,(x_1)$	$v_2\,(x_2)$
13.382	12.873
13.382	12.761
13.381	12.648
13.381	12.535
13.38	12.421
13.38	12.308
13.379	12.194
13.378	12.08
13.377	11.965
13.377	11.85
13.376	11.735
13.375	11.62
13.374	11.504
13.373	11.389
13.372	11.272
13.371	11.156
13.369	11.039
13.368	10.922
13.367	10.805
13.366	10.687
13.364	10.569
13.363	10.451
13.361	10.333
13.36	10.214
13.358	10.095
13.357	9.975
13.355	9.855
13.354	9.735
13.352	...
13.35	
13.348	
...	

6 Conclusions

This paper has a theoretical content and relates to the actual topic of the steady flow of the incompressible viscous complex liquid over circular tube. Two component flow of different density and viscosity liquids is taken into the consideration. On the contact surface the equal velocity and equal shear stress conditions are assumed. Also the velocity profile, pressure distribution over the tube length, the shearing stress expression, also near the boundary, the mass and volumetric flow rate formulas are derived. Those are actual in the plane of studies of fluid transportation problems or rheological investigations, the thickness of boundary layer etc. In the case of well exploitations of the old wells the additional heating ingredients are included. In other words, considering not complex liquids with its heated content, it is clear that, not all volume of liquid is heated. Therefore, we have to consider partially heated liquid content which brings us to the subject of present paper. Its content has to parts the hot and not hot parts. As it is well known, some liquids have the property of varying density and viscosity from the temperature. For the case we fall to the case of subject present paper.

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References

1. Batchelor, G.K. An introduction to fluid dynamics. *Cambridge University Press*. ISBN 9780521663960, 2000.
2. Cengel, Y.A. and Cimbala, J.M. Flow in Pipes,*Fluid Mechanics: Fundamentals and Applications*, 1st ed., 2006: https://www.uio.no/studier/emner/matnat/math/MEK4450/h11/undervis ningsmateriale/modul-5/Pipeflow_intro.pdf. Accessed: June 05, 2016.
3. White, F.M. Fluid Mechanics, *New York: McGraw-Hill,*7th ed., 2011.
4. Davey, A. and Drazin P. "The stability of poiseuille flow in a pipe," *Journal of Fluid Mechanics*, **36**, 209-218, 1969.
5. Draad, A.A., Kuiken, G., and Nieuwstadt, F. Laminar–turbulent transition in pipe flow for Newtonian and non-Newtonian fluids,*Journal of Fluid Mechanics*, **377**, 267-312, 1998.
6. Enayet, M., Gibson, M, Taylor, A. and Yianneskis, M. Laser-Doppler measurements of laminar and turbulent flow in a pipe bend,*International Journal of Heat and Fluid Flow*, **3**, 213-219, 1982.
7. Lyne, W. Unsteady viscous flow in a curved pipe,*Journal of Fluid Mechanics*, **45**, 13-31, 1971.
8. Najmi, J. and Ali Shah, S.I. Analysis of velocity profile for laminar flow in a round pipe. 5-th *International Conference on Aerospace Science & Engineering* (ICASE 2017), Institute of Space Technology, Islamabad, Pakistan.
9. Panton, R.L. Incompressible Flow, 4th ed. *New Jersey: Wiley*, 2013.
10. Pillai, A.V. and Manu, K.V. Analytical Solutions for Unsteady Pipe Flows with Slip Boundary Condition, *Journal of Applied Fluid Mechanics*, **13**(3), 1015-1026, 2020. Available online at www.jafmonline.net, ISSN 1735-3572, EISSN 1735-3645. DOI: 10.29252/jafm.13.03.30800
11. Stokes, G.G. On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids. *Transactions of the Cambridge Philosophical Society*. **8**: 287–341, 1845.
12. Sutera, S.P. and Skalak, R. The history of Poiseuille's law,*Annual Review of Fluid Mechanics*, **25**(1), 1-20, 1993.