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The inverse problem of simultaneous determination two time-dependent coefficients in the equation of motion of waves with surface stress

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Abstract. *We study the classical solution of the nonlinear inverse boundary value problem for the equation of motion of waves with surface stress The essence of the problem is that it is required together with the solution to determine the unknown coefficient. The problem is considered in a rectangular area. To solve the considered problem, the transition from the original inverse problem to some auxiliary inverse problem is carried out. The existence and uniqueness of a solution to the auxiliary problem are proved with the help of contracted mappings. Then the transition to the original inverse problem is made, as a result, a conclusion is made about the solvability of the original inverse problem.*

Keywords. inverse boundary value problem · classical solution · uniqueness · existence · Fourier method · Boussinesq equation.

Mathematics Subject Classification (2010): 35R30

1 Introduction

The theory of inverse problems for the differential equations is a dynamically developing branch of mathematics. Inverse problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc., which puts them among the topical problems. The presence of additional unknown functions in the inverse problems requires some additional redefinition conditions are also given.

The sixth-order Boussinesq equation with double dispersion describes motion of waves on water with a stress surface and was considered by in [15]. Various boundary value problems for the Boussinesq type equation were studied in [2, 4, 6, 7, 11, 13, 14, 16, 17]. Various inverse problems for certain types of partial differential equations have been studied in many works [8-9]. For the Boussinesq equation of the sixth order, inverse problems were considered in [1, 3, 5, 12].

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In this paper we consider the inverse problem for the sixth-order Boussinesq equation with double dispersion, where, along with the solution, an unknown coefficient should also be found.

2 Formulation of the problem and its equivalent form

Let $D_T = \{(x,t):\ \ 0 \le x \le 1,\ \ 0 \le t \le T\}$. and $f(x,t)$, $\varphi(x)$, $\psi(x)$, $h_i(t)$ $(i = 1,2)$ are given functions defined for $x \in [0,1],\ t \in [0,T]$. Consider the following inverse problem: to find a triple $\{u(x,t)\ ,\ a(t), b(t)\}$ of the functions $u(x,t)$, $a(t)$, $b(t)$ satisfying the equation

$$u_{tt}(x,t) - u_{xx}(x,t) - \ u_{ttxx}(x,t) + u_{xxxx}(x,t) + u_{ttxxxx}(x,t) =$$

$$= a(t)u(x,t) + b(t)u_t(x,t) + f(x,t) \tag{2.1}$$

with initial

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \quad (0 \le x \le 1), \tag{2.2}$$

and boundary conditions

$$u(0,t) = u_x(1,t) = \ u_{xx}(0,t) = u_{xxx}(1,t)\ \ = 0 \quad (0 \le t \le T) \tag{2.3}$$

and with additional conditions

$$u(x_i,t) = h_i(t) \quad (0 < x_i < 1\ ,\ i = 1,2; x_1 \ne x_2; 0 \le t \le T), \tag{2.4}$$

Introduce the designation

$$\tilde{C}^{4,2}(D_T) = \left\{u(x,t):\ u(x,t) \in C^2(D_T), u_{xxxx}(x,t), u_{ttxxxx}(x,t \in C(D_T)\ \right\}$$

Definition 2.1 *A triple* $\{u(x,t), a(t), b(t)\}$ *of the functions* $u(x,t) \in C^{4,2}(D_T)$*,*
$a(t) \in C[0,T]$ *and* $b(t) \in C[0,T]$ *satisfying equation* (2.1) *in* D_T*, condition* (2.2) *in* $[0,1]$ *and conditions* (2.3)-(2.4) *in* $[0,T]$ *we call a classical solution to boundary value* (2.1)-(2.4).

We prove the following

Theorem 2.1 *Let* $f(x,t) \in C(D_T)$, $\varphi(x)$, $\psi(x) \in C[0,1]$, $h_i(t) \in C^2[0,T] (i = 1,2)$, $h(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \ne 0$ $(0 \le t \le T)$ *and the matching conditions*

$$\varphi(x_i) = h_i(0)\ ,\ \psi(x_i) = h_i'(0)\ .(i = 1,2)$$

are satisfied. Then the problem of finding a classical solution to problem (2.1)-(2.4) *is equivalent to the problem of determining the functions* $u(x,t) \in C^{4,2}(D_T)$, $a(t) \in C[0,T]$ *and*$b(t) \in C[0,T]$ *from* (2.1)-(2.3) *and*

$$h_i''(t) - u_{xx}(x_i,t) - u_{ttxx}(x_i,t) + u_{xxxx}(x_i,t) +$$

$$+u_{ttxxxx}(x_i,t)\ = a(t)h_i(t) + b(t)h_i'(t) + f(x_i,t)\ (\ i = 1,2;\ \ 0 \le t \le T)\,. \tag{2.5}$$

Proof. Let $\{u(x,t), a(t), b(t)\}$ be a classical solution to problem (2.1)-(2.4). Since $h_i(t) \in C^2[0,T] (i = 1, 2)$, differentiating (2.4) two times over t we get

$$u_t(x_i,t) = h_i'(t)\ ,\ \ u_{tt}(x_i,t) = h_i''(t) \quad (i = 1, 2;\ 0 \le t \le T). \tag{2.6}$$

Taking $x = x_i$ in equation (2.1) we find

$$u_{tt}(x_i,t) - u_{xx}(x_i,t) - u_{ttxx}(x_i,t) + u_{xxxx}(x_i,t) + u_{ttxxxx}(x_i,t) =$$
$$= a(t)u(x_i,t) + b(t)u_t(x_i,t) + f(x_i,t)\ \ (i = 1, 2;\ \ 0 \le t \le T). \tag{2.7}$$

From this considering (2.4) and (2.6) we arrive at (2.5).

Now let's suppose that $\{u(x,t), a(t), b(t)\}$ is a solution of problem (2.1)-(2.3), (2.5). Then from (2.5) and (2.7) we get

$$\frac{d^2}{dt^2}(u(x_i,t) - h_i(t)) = a(t)(u(x_i,t) - h_i(t))\ + b(t)\frac{d}{dt}(u(x_i,t) - h_i(t))$$
$$(i = 1, 2; 0 \le t \le T)\,. \tag{2.8}$$

Considering (2.2) and $\varphi(x_i) = h_i(0)\ ,\ \psi(x_i) = h_i'(0)\ .(i = 1, 2)$we have

$$u(x_i,0) - h_i(0) = \varphi(x_i) - h_i(0) = 0,$$
$$.u_t(x_i,0) - h_i'(0) = \psi(x_i) - h_i'(0) = 0 \quad (i = 1, 2). \tag{2.9}$$

From (2.8), taking into account (2.9), it is clear that condition (2.4) is also satisfied. The theorem is proved.

3 Solvability of the inverse boundary value problem

The first component $u(x,t)$ of the solution $\{u(x,t), a(t), b(t)\}$ to problem (2.1)-(2.3), (2.5) we seek in the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin \lambda_k x \quad \left(\lambda_k = \frac{\pi}{2}(2k-1)\right)\ , \tag{3.1}$$

where

$$u_k(t) = 2\int_0^1 u(x,t) \sin \lambda_k x dx\ \ (k = 1, 2, ...).$$

Then applying the formal Fourier scheme, from (2.1) and (2.2) we obtain

$$(1 + \lambda_k^2 + \lambda_k^4)u_k''(t) + (\lambda_k^2 + \lambda_k^4)u_k(t) == F_k(t; u, a, b)\ \ (0 \le t \le T;\ k = 1, 2, ...) \tag{3.2}$$
$$u_k(0) = \varphi_k,\ \ u_k'(0) = \psi_k\ \ (\ k = 1, 2, ...), \tag{3.3}$$

where

$$F_k(t; u, a, b) = a(t)u_k(t) + b(t)u_k'(t) + f_k(t) \quad , f_k(t) = 2\int_0^1 f(x,t) \sin \lambda_k x\, dx,$$

$$\varphi_k = 2\int_0^1 \varphi(x) \sin \lambda_k x dx, \quad \psi_k = 2\int_0^1 \psi(x) \sin \lambda_k x dx \quad (k = 1, 2, ...).$$

Solving problem (3.2)-(3.3) we find

$$u_k(t)\ =\ \varphi_k \cos \beta_k t\ +\ \frac{1}{\beta_k}\psi_k \sin \beta_k t +$$

$$+\frac{1}{\beta_k(1+\lambda_k^2+\lambda_k^4)}\int_0^t F_k(\tau;u,a,b)\sin\beta_k(t\,-\,\tau)d\tau\ (k\,=\,1,2,...), \qquad (3.4)$$

where

$$\beta_k^2\,=\frac{\lambda_k^2+\lambda_k^4}{1+\lambda_k^2+\lambda_k^4}(k=1,2,...).$$

After substitution of the expression $u_k(t)\ \ (k=1,2,...)$ into (3.1) for the determination of $u(x,t)$ we get

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\varphi_k\cos\beta_k t\,+\,\frac{1}{\beta_k}\psi_k\sin\beta_k t\quad+\right.$$

$$\left.+\frac{1}{\beta_k(1+\lambda_k^2+\lambda_k^4)}\int_0^t F_k(\tau;u,a,)\sin\beta_k(t\,-\,\tau)d\tau\right\}\sin\lambda_k x. \qquad (3.5)$$

Now from (2.5) taking into account (3.1) we have

$$a(t)=\,[h(t)]^{-1}\left\{(h_1''(t)-f(x_1,t))h_2'(t)-(h_2''(t)-f(x_2,t))h_1'(t)+\right.$$

$$\left.+\sum_{k=1}^{\infty}(\lambda_k^2+\lambda_k^4)(u_k''(t)+u_k(t))\left(h_2'(t)\sin\lambda_k x_1-h_1'(t)\sin\lambda_k x_2\right)\right\}, \qquad (3.6)$$

$$b(t)=\,[h(t)]^{-1}\left\{(h_2''(t)-f(x_2,t))h_1(t)-(h_1''(t)-f(x_1,t))h_2(t)+\right.$$

$$\left.+\sum_{k=1}^{\infty}(\lambda_k^2+\lambda_k^4)(u_k''(t)+u_k(t))\left(h_1(t)\sin\lambda_k x_2-h_2(t)\sin\lambda_k x_1\right)\right\}.. \qquad (3.7)$$

Consideration of (3.4) in (3.2) gives

$$(\lambda_k^2+\lambda_k^4)(u_k''(t)+u_k(t))=-u_k''(t)+F_k(t;u,a,b)=$$

$$=\frac{\lambda_k^2+\lambda_k^4}{1+\lambda_k^2+\lambda_k^4}u_k(t)+\left(1-\frac{1}{1+\lambda_k^2+\lambda_k^4}\right)F_k(t;u,a,b)=$$

$$=\frac{\lambda_k^2+\lambda_k^4}{1+\lambda_k^2+\lambda_k^4}u_k(t)+\frac{\lambda_k^2+\lambda_k^4}{1+\lambda_k^2+\lambda_k^4}F_k(t;u,a,b)=\beta_k^2u_k(t)+\beta_k^2F_k(t;u,a,b)=$$

$$=\beta_k^2u_k(t)+\beta_k^2F_k(t;u,a,b)==\beta_k^2\left[\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t+\right.$$

$$\left.+\frac{1}{\beta_k(1+\lambda_k^2+\lambda_k^4)}\int_0^t F_k(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau\right]+\beta_k^2F_k(t;u,a,b),$$

$$k=1,2,...,\ \ 0\le t\le T.$$

To obtain an equation for the second component $a(t),b(t)$ of the solution $\{u(x,t),a(t),b(t)\}$ we put the last relation into (3.6)

$$a(t)=\,[h(t)]^{-1}\left\{(h_1''(t)-f(x_1,t))h_2'(t)-(h_2''(t)-f(x_2,t))h_1'(t)+\right.$$

$$+\sum_{k=1}^{\infty}\beta_k^2\left[\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t+\frac{1}{\beta_k(1+\lambda_k^2+\lambda_k^4)}\int_0^t F_k(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau+\right.$$

$$\left.+F_k(t;u,a,b)\right](h_2'(t)\sin\lambda_k x_1-h_1'(t)\sin\lambda_k x_2)\Bigg\}, \tag{3.8}$$

$$b(t)=[h(t)]^{-1}\Big\{(h_2''(t)-f(x_2,t))h_1(t)-(h_1''(t)-f(x_1,t))h_2(t)+$$

$$+\sum_{k=1}^{\infty}\beta_k^2\left[\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t+\frac{1}{\beta_k(1+\lambda_k^2+\lambda_k^4)}\int_0^t F_k(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau+\right.$$

$$\left.+F_k(t;u,a,b)\right](h_1(t)\sin\lambda_k x_2-h_2(t)\sin\lambda_k x_1)\Bigg\}, \tag{3.9}$$

Thus, solution of problem (2.1)-(2.3), (2.5) is reduced to the solution of system (3.5), (3.8),(3.9) with respect to the unknown functions $u(x,t)$,$a(t)$ and $b(t)$.

To study the problem of the uniqueness of the solution of problem (2.1)-(2.3), (2.5), the following lemma plays an important role.

Lemma. *If* $\{u(x,t),a(t),b(t)\}$ *is arbitrary classical solution of problem* (2.1)-(2.3)*,* (2.5)*, then the function*

$$u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx \ \ (k=1,2,...)$$

satisfies system (3.4) *in* $[0,T]$.

Proof. Let $\{u(x,t),a(t),b(t)\}$ be any solution to problem (2.1)-(2.3), (2.5). Then multiplying both sides of equation (2.1) by the function $2\sin\lambda_k x$ $(k=1,2,...)$, integrating the obtained equality over x from 0 to 1 and using the relations

$$2\int_0^1 u_{tt}(x,t)\sin\lambda_k xdx=\frac{d^2}{dt^2}\left(2\int_0^1 u(x,t)\sin\lambda_k xdx\right)=u_k''(t)(k=1,2,...),$$

$$2\int_0^1 u_{xx}(x,t)\sin\lambda_k xdx=-\lambda_k^2\left(2\int_0^1 u(x,t)\sin\lambda_k xdx\right)=-\lambda_k^2 u_k(t) \ \ (k=1,2,...),$$

$$2\int_0^1 u_{ttxx}(x,t)\sin\lambda_k xdx=-\lambda_k^2\left(2\int_0^1 u_{tt}(x,t)\sin\lambda_k xdx\right)=-\lambda_k^2 u_k''(t) \ \ (k=1,2,...),$$

$$2\int_0^1 u_{xxxx}(x,t)\sin\lambda_k xdx=\lambda_k^4\left(2\int_0^1 u(x,t)\sin\lambda_k xdx\right)=\lambda_k^4 u_k(t) \ \ (k=1,2,...),$$

$$2\int_0^1 u_{ttxxxx}(x,t)\sin\lambda_k xdx=\lambda_k^4\left(2\int_0^1 u(x,t)\sin\lambda_k xdx\right)=-\lambda_k^4 u_k(t) \ \ (k=1,2,...)$$

we obtain that equation (3.2) is satisfied.

Similarly, the fulfilment of (3.3) is obtained from (2.2). Thus $u_k(t)$ $(k=1,2,...)$ is a solution to problem (3.2), (3.3).

As immediately follows from this the function $u_k(t)(k=1,2,...)$ satisfies to system (3.4) on $[0,T]$. Lemma is proved.

It is obvious that if $u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k xdx$ $(k=1,2,...)$ is a solution of system (3.4), then the pair $\{u(x,t),a(t),b(t)\}$ of the functions $u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\lambda_k x$, $a(t)$,and b(t) is a solution to system (3.5), (3.8),(3.9).

This lemma implies the validity of the following

Consequence. *Let system* (3.5), (3.8),(3.9) *have a unique solution. Then problem* (2.1)-(2.3), (2.5) *cannot have more than one solution, i.e. if problem* (2.1)-(2.3), (2.5) *has a solution, then it is unique.*
Now, in order to study problem (2.1)-(2.3), (2.5) consider the following spaces.

1 Denote by $B_{2,T}^{5,4}$ [10] the set of all functions $u\left(x,t\right)$ of the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\lambda_k x\quad\left(\lambda_k=\frac{\pi}{2}(2k-1)\right),$$

Defined on D_T, where each of the functions $u_k(t)\ \in C^1\ [0,T](k=1,2,\ldots)$ and

$$J_T(u)\equiv\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}(\lambda_k^4\left\|u_k'(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}<+\infty.$$

The norm in this space is defined as

$$\left\|u(x,t)\right\|_{B_{2,T}^{5,4}}=J(u).$$

1 By $E_T^{5,4}$ we denote the space of the vector functions $\{u(x,t),a(t),b(t)\}$ such that $u\left(x,t\right)\in B_{2,T}^{5,4}$, $a\left(t\right)\in C[0,T]$,$b\left(t\right)\in C[0,T]$ and equip this space by the norm

$$\left\|z\right\|_{E_T^{5,4}}=\left\|u(x,t)\right\|_{B_{2,T}^{5,4}}+\left\|a(t)\right\|_{C[0,T]}+\left\|b(t)\right\|_{C[0,T]}.$$

Clearly, $B_{2,T}^{5,4}$ and $E_T^{5,4}$ are Banach spaces.

Now we consider in $E_T^{5,4}$ the operator

$$\Phi(u,a,b)=\{\Phi_1(u,a,b),\Phi_2(u,a,b),\Phi_3(u,a,b)\},$$

where

$$\Phi_1(u,a,b)=\tilde{u}(x,t)\equiv\sum_{k=1}^{\infty}\tilde{u}_k(t)\sin\lambda_k x,\Phi_2(u,a,b)=\tilde{a}(t),\Phi_3(u,a,b)=\tilde{b}(t),$$

$\tilde{u}_k(t)\ \ (\ k=1,2,...)$, $\tilde{a}(t)$and $\tilde{b}(t)$ are the right hand sides of (3.4) and (3.8),(3.9) correspondingly.
Obviously

$$\frac{\sqrt{3}}{3}<\beta_k<\sqrt{2},\ \frac{\sqrt{2}}{2}<\frac{1}{\beta_k}<\sqrt{3}.$$

Then we have

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|\tilde{u}_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\leq\sqrt{5}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\varphi_k\right|)^2\right)^{\frac{1}{2}}+\sqrt{15}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\psi_k\right|)^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{15T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\left|f_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\sqrt{15}\ T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{15}\ T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^4\left\|u_k'(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}},\tag{3.10}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^4\left\|\tilde{u}'_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq \sqrt{10}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\varphi_k\right|)^2\right)^{\frac{1}{2}}+\sqrt{5}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\psi_k\right|)^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{5T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\left|f_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\sqrt{5}\,T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{5}\,T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^4\left\|u'_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{3.11}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]}=$$

$$=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(h''_1(t)-f(x_1,t))h'_2(t)-(h''_2(t)-f(x_2,t))h'_1(t)\right\|_{C[0,T]}+\right.$$

$$+2\left\|\left|h'_2(t)\right|+\left|h'_1(t)\right|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\varphi_k\right|)^2\right)^{\frac{1}{2}}+\sqrt{3}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\psi_k\right|)^2\right)+\right.$$

$$+\sqrt{3T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^4\left|f_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\sqrt{3}\,T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{3}\,T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^4\left\|u'_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}.+\left(\sum_{k=1}^{\infty}(\lambda_k\left\|f_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$\left.\left.+\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^4\left\|u'_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}, \tag{3.12}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}=$$

$$=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(h''_2(t)-f(x_2,t))h_1(t)-(h''_1(t)-f(x_1,t))h_2(t)\right\|_{C[0,T]}+\right.$$

$$+2\left\|\left|h_2(t)\right|+\left|h_1(t)\right|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\varphi_k\right|)^2\right)^{\frac{1}{2}}+\sqrt{3}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\psi_k\right|)^2\right)+\right.$$

$$+\sqrt{3T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^4\left|f_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\sqrt{3}\,T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{3}\,T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^4\left\|u'_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}.+\left(\sum_{k=1}^{\infty}(\lambda_k\left\|f_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\ \|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^4\left\|u_k'(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}, \tag{3.13}$$

where

$$\tilde{u'}_k(t)\ =\ -\beta_k\varphi_k\sin\beta_k t\ +\ \psi_k\sin\beta_k t+$$

$$+\frac{1}{1+\lambda_k^2+\lambda_k^4}\int_0^t F_k(\tau;u,a,b)\cos\beta_k(t\ -\ \tau)d\tau\ (k\ =\ 1,2,...).$$

Assume that the data of problem (2.1)-(2.3), (2.5) satisfy the following conditions:

1.$\varphi(x)\in C^4[0,1],\ \varphi^{(5)}(x)\in L_2(0,1),\ \varphi'(0)=\varphi(1)=\varphi'''(0)=\varphi''(1)==\varphi^{(4)}(1)\ =0'$.

2.$\psi(x)\in C^4[0,1],\ \psi^{(5)}(x)\in L_2(0,1),\ \psi'(0)=\psi(1)=\psi'''(0)=\psi''(1)==\psi^{(4)}(1)=0$.

3.$f(x,t)\in\ C(D_T),\ f_x(x,t)\ \in\ L_2(D_T), f(1,t)\ =0\ \ (0\le t\le T)$.

4.$h_i(t)\in C^2[0,T](i=1,2), h(t)\equiv h_1(t)h_2'(t)-h_2(t)h_1'(t)\neq 0\ (0\le t\le T)$.

Then from (3.10)-(3.13) we have

$$\|\tilde{u}(x,t)\|_{B_{2,T}^{5,4}}\le A_1(T)+B_1(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,4}}, \tag{3.14}$$

$$\|\tilde{a}(t)\|_{C[0,T]}\le A_2(T)+B_2(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,4}}, \tag{3.15}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\le A_3(T)+B_3(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,4}} \tag{3.16}$$

where

$$A_1(T)=\sqrt{5}(1+\sqrt{2})\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\sqrt{5}(1+\sqrt{3})\left\|\psi^{(5)}(x)\right\|_{L_2(0,1)}+$$

$$+\sqrt{5T}(1+\sqrt{3})\left\|f_x(x,t)\right\|_{L_2(D_T)}, B_1(T)=\sqrt{5}(1+\sqrt{3})T,,$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(h_1''(t)-f(x_1,t))h_2'(t)-(h_2''(t)-f(x_2,t))h_1'(t)\right\|_{C[0,T]}+\right.$$

$$+2\left\|\left|h_2'(t)\right|+\left|h_1'(t)\right|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\right.$$

$$\left.\left.+\sqrt{3}\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)}+\sqrt{3T}\left\|f_x(x,t)\right\|_{L_2(D_T)}+\left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0.1)}\right]\right\},$$

$$B_2(T)=2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\|\left|h_2'(t)\right|+\left|h_1'(t)\right|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(T+1),$$

$$A_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(h_2''(t)-f(x_2,t))h_1(t)-(h_1''(t)-f(x_1,t))h_2(t)\right\|_{C[0,T]}+\right.$$

$$+2\left\|\left|h_2(t)\right|+\left|h_1(t)\right|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\right.$$

$$\left.+\sqrt{3}\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)}|++\sqrt{3T}\left\|f_x(x,t)\right\|_{L_2(D_T)}+\left\|\left\|f_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0.1)}\right]\Bigg\},$$

$$B_3(T)=2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\|\left|h_2(t)\right|+\left|h_1(t)\right|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(T+1),$$

From inequalities (3.14)-(3.16) we conclude

$$\|\tilde{u}(x,t)\|_{B_{2,T}^{5,4}}+\|\tilde{a}(t)\|_{C[0,T]}+\left\|\tilde{b}(t)\right\|_{C[0,T]}\le$$

$$\le A(T)+B(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,4}}, \tag{3.17}$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T)\,,B(T)=B_1(T)+B_2(T)+B_3(T)\,.$$

So, we can prove the following theorem:

Theorem 3.1 *Let conditions 1-4 be satisfied and*

$$(A(T)+2)^2B(T)<1. \tag{3.18}$$

The problem (2.1)-*(3),*(2.5) *has a unique solution in the ball* $K=K_R(\|z\|_{E_T^{5,4}}\le R=A(T)+2)$ *of the space* $E_T^{5,4}$.

Remark. Inequality (3.18) is satisfied for sufficiently small values of $T+\left\|[h(t)]^{-1}\right\|_{C[0,T]}$.

Proof. In the space $E_T^{5,4}$ consider the equation

$$z=\Phi z, \tag{3.19}$$

where $z=\{u,a,b\}$, the components $\Phi_i(u,a,b)(i=1,2,3)$ of the operator $\Phi(u,a,b)$ are defined by the right hand sides of equations (3.5),(3.8) and (3.9) .

Consider the operator $\Phi(u,a,b)$ in the ball $K=K_R$ from $E_T^{5,4}$. Similarly to (3.13) we obtain that the estimations

$$\|\Phi z\|_{E_T^{5,4}}\le A(T)+B(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,4}}, \tag{3.20}$$

$$\|\Phi z_1-\Phi z_2\|_{E_T^{5,4}}\le\le B(T)R\Big(\|a_1(t)-a_2(t)\|_{C[0,T]}+\|b_1(t)-b_2(t)\|_{C[0,T]}+$$

$$+\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^{5,4}}\Big) \tag{3.21}$$

for the arbitrary $z,\ z_1,\ z_2\in K_R$. Then, from estimates (3.20), (3.21), taking into account (3.18), it follows that the operator Φ acts in the ball and is contractive. Therefore in the ball $K=K_R$ the operator Φ has a single fixed point $\{u,a,b\}$ which is a unique solution to equation (3.19) in the ball $K=K_R$, i.e. $\{u,a,b\}$ is a unique solution to system (3.5),(3.8) and (3.9) in the ball $K=K_R$.

The function $u(x,t)$ as an element of the space $B_{2,T}^{5,4}$ has continuous derivatives $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t)$, $u_t(x,t)$, $u_{tx}(x,t)$, $u_{txx}(x,t)$, $u_{txxx}(x,t)$ in D_T.

As one can easily see from

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5 \left\|u_k''(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq \sqrt{2}\left(\sum_{k=1}^{\infty}(\lambda_k^5 \left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} +$$

$$+\sqrt{2}\left\|\left\|f_x(x,t) + a(t)u_x(x,t) + b(t)u_{tx}(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}.$$

It implies that $u_{tt}(x,t)$,$u_{ttx}(x,t)$,$u_{ttxx}(x,t)$,$u_{ttxxx}(x,t)$,$u_{ttxxxx}(x,t)$ are continuous in D_T.

It is easy to check that equation (2.1) and conditions (2.2), (2.3) and (2.5) are satisfied in the usual sense. Therefore, $\{u(x,t), a(t), b(t)\}$is a solution to problem (2.1)-(2.3), (2.5), and, by virtue of the corollary of Lemma 1, it is unique in the ball $K = K_R$.

The theorem is proved.

Using Theorem 1, we prove the following

Theorem 3.2 *Let all conditions of Theorem 2 be satisfied and*

$$\varphi(x_i) = h_i(0)\,,\ \psi(x_i) = h_i'(0)\ .(i = 1,2).$$

The problem (2.1)-(2.4) *has unique classical solution in the ball* $K = K_R(\|z\|_{E_T^{5,4}} \leq R = A(T) + 2)$ *from* $E_T^{5,4}$.

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