Tran. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mechanics, **42** (8), 25-35 (2022)

On the influence of the inhomogeneous residual stresses on the dispersion of axisymmetric longitudinal waves in the hollow cylinder

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Received: 02.06.2022 / Revised: 14.09.2022 / Accepted: 07.10.2022

Abstract. *The paper studies the influence of the axisymmetric residual stresses generated by the contacting of the longitudinal cuts on the dispersion curves of the axisymmetric waves propagating in this cylinder. The investigation is carried out using the three-dimensional linearized theory of elastic waves in initially stressed bodies. The corresponding eigenvalue problem is solved using the discrete analytic method developed in the earlier work of the author. Numerical results on the influence of the magnitude of the residual stresses on the propagation curves of the lowest, first and second modes are presented and discussed. In particular, based on these results, it is found that new low-lying modes appear as a result of the residual stresses and the dispersion curves of these modes approach the dispersion curve of the first mode as the cut angle increases. Moreover, it is found that the influence of residual stresses on the dispersion curves of the first mode is not only quantitative but also qualitative in character.*

Keywords. axisymmetric residual stresses · axisymmetric waves · dispersion curves · hollow cylinder

Mathematics Subject Classification (2010): 74H55

1 Introduction

The dynamical problems as well as the problems related to axisymmetric wave dispersion in bi-material inhomogeneously prestressed cylindrical systems have been studied in recent years in works [1 - 7]. However, in these works it was assumed that the inhomogeneous initial stresses (i.e., the prestresses) in the mentioned systems are caused by external forces or temperature changes acting in the initial state, i.e., before the beginning of the dynamic processes in these systems. Moreover, the initial loading is assumed to persist in all subsequent dynamic processes.

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It is known that there are also cases when the cylinders have initial stresses without initial loading, which can be called residual stresses and which occur in the cylinders as a result of the processes used in the manufacture of the cylinders. One type of residual stress state in the cylinders appears as a result of the following procedures. It is assumed that the cylinder has a cut as shown in Fig. 1 and this cut is closed by applying some external forces or moments and is ”contacted” through welding or in other ways. After contacting, the external forces and moments are removed and as a result of these procedures, the cylinder has inhomogeneous residual stresses, the values of which are determined through the expressions given in the monograph [8]. Note that such contacting methods are used in the manufacture of thick-walled cylinders used in various branches of modern industry, especially oil platforms used for exploration and production of oil from the sea and sea beds. In order to provide a scientific basis for the application of non-destructive methods for flaw detection by ultrasonic waves in these cases, a fundamental theoretical investigation of how these inhomogeneous residual stresses affect wave propagation in such layered cylinders is required. Namely, the subject of the present work is devoted to such problems and to the study of the influence of the aforementioned type of axisymmetric inhomogeneous residual stresses on the dispersion of the axisymmetric longitudinal wave in the hollow cylinder within the framework of the so-called 3D linearized theory of elastic waves in bodies with initial stresses.

2 Mathematical formulation of the problem

Let us assume that the wall of the cylinder in the natural state has a cut marked by the central angle α, as shown in Fig. 1a', and let us assume that$\alpha \ll 1$. After removing the cut by touching their ends (Fig. 1a), we obtain the hollow cylinder shown in Fig. 1b. According to the monograph [8], the cylinder has the following residual stresses as a result of contact:

$$\sigma^0_{rr} = -4\frac{M}{N}\left[R^2(R+h)^2\frac{1}{r^2}\log\left(\frac{(R+h)}{R}\right) + (R+h)^2\log\left(\frac{r}{R+h}\right) + R^2\log\left(\frac{r}{R}\right)\right];$$

$$\sigma^0_{\theta\theta} = -4\frac{M}{N}\left[-R^2(R+h)^2\frac{1}{r^2}\log\left(\frac{(R+h)}{r}\right) + (R+h)^2\log\left(\frac{r}{(R+h)}\right) + \right.$$

$$\left. R^2\log\left(\frac{r}{R}\right) + (R+h)^2 - R^2\right]; \sigma^0_{zz} = \nu(\sigma^0_{rr} + \sigma^0_{\theta\theta}). \tag{2.1}$$

In (2.1) the following notation is used.

$$M = -\frac{\alpha\mu}{4\pi(1-\nu)}\left(\left((R+h)^2 - R^2\right)^2 - 4(R+h)^2 R^2\left(\log\frac{(R+h)}{R}\right)^2\right)\frac{1}{2\left((R+h)^2 - R^2\right)};$$

$$N = \left((R+h)^2 - R^2\right)^2 - 4(R+h)^2R^2\left(\log\frac{(R+h)}{R}\right)^2. \tag{2.2}$$

In relations in (2.1), and below, the upper index 0 means that the stresses are the residual ones. Moreover, in (2.1) the following notation is used: ν is Poisson's ratio, μ and λ are the Lame constants of the cylinder material.

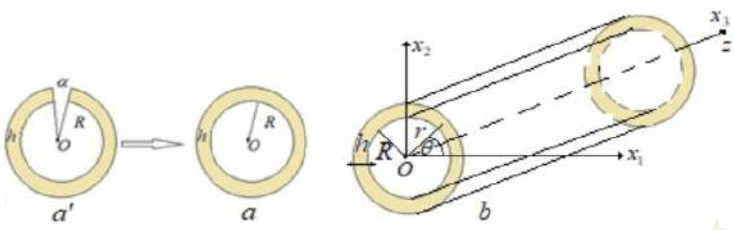

Fig. 1. The sketch of the cut, cut angle and the hollow cylinder

Thus, we want to investigate how the residual stresses acting in the hollow cylinder shown in Fig. 1b and determined by expressions (2.1) and (2.2) affect the dispersion of axially symmetric longitudinal waves propagating in this cylinder. Assuming that the material of the cylinder is moderately stiff, we perform this investigation in the framework of the second version of the small initial deformation of the linearized 3D theory of elastic waves in bodies with initial stresses. For this purpose, we use the cylindrical $Or\theta z$and Cartesian $Ox_1x_2x_3$ coordinate systems associated with the central axis of the cylinder under consideration and determine the position of the points of this cylinder with the Lagrangian coordinates in these coordinate systems. We note that the expressions in (2.1) and (2.2) are also written in this cylindrical coordinate system.

Let us write according to the references [9] and [10] the complete system of equations and relations of the scope of application of the second version of the small initial deformation of the linearized 3D theory of elastic waves in bodies with initial stresses.

The equations of motion:

$$\frac{\partial t_{rr}}{\partial r}+\frac{\partial t_{zr}}{\partial z}+\frac{1}{r}(t_{rr}-t_{\theta\theta})=\rho\frac{\partial^2 u_r}{\partial t^2},\frac{\partial t_{rz}}{\partial r}+\frac{1}{r}t_{rz}+\frac{\partial t_{zz}}{\partial z}=\rho\frac{\partial^2 u_z}{\partial t^2}, \tag{2.3}$$

where

$$t_{rr}=\sigma_{rr}+\sigma^0_{rr}(r)\frac{\partial u_r}{\partial r},t_{rz}=\sigma_{rz}+\sigma^0_{rr}(r)\frac{\partial u_z}{\partial r},t_{\theta\theta}=\sigma_{\theta\theta}+\sigma^0_{\theta\theta}(r)\frac{u_r}{r},$$

$$t_{zr}=\sigma_{zr}+\sigma^0_{zz}(r)\frac{\partial u_r}{\partial z},t_{zz}=\sigma_{zz}+\sigma^0_{zz}(r)\frac{\partial u_z}{\partial z}. \tag{2.4}$$

The elasticity and strain-displacement relations:

$$\sigma_{(ij)}=\lambda(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{zz})+2\mu\varepsilon_{(ij)},(jj)=rr;\ \theta\theta;\ zz,\sigma_{rz}=2\mu\varepsilon_{rz}. \tag{2.5}$$

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\varepsilon_{\theta\theta}=\frac{u_r}{r},\varepsilon_{zz}=\frac{\partial u_z}{\partial z},\varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right). \tag{2.6}$$

The equations (2.3) - (2.6) are the complete system of field equations in the framework in which the present investigations are carried out. Note that the conventional notation is used in these equations.

We add to these equations the following boundary and contact conditions

$$t_{rr}|_{r=R+h}=0,\ t_{rz}|_{r=R+h}=0,\ t_{rr}|_{r=R}=0,\ t_{rz}|_{r=R}=0. \tag{2.7}$$

This completes the formulation of the problem.

3 Method of solution

For the solution to the formulated problem we attempt to employ the discrete-analytical method developed and employed in the papers[1-7], according to which, the interval $[R, R+h]$ is divided into N number of subintervals or sublayers. The thickness of the sublayers of the region $[R, R+h]$ is equal to h/N_2 and in the $n-th$ sub-layer, the relation $(R+(n-1)h/N) \leq r \leq (R+nh/N)$ takes place, where $1 \leq n \leq N$. Thus, after the foregoing division, the inhomogeneous initial stresses, determined by the expressions (2.1), and (2.2) in each of the foregoing $n-th$ sublayers, are taken as constants, the values of which are determined by the following relations:

$$\sigma_{rr}^{0}(r) \approx \sigma_{rr}^{0}(r_n), \sigma_{\theta\theta}^{0}(r) \approx \sigma_{\theta\theta}^{0}(r_n), \sigma_{zz}^{0}(r) \approx \sigma_{zz}^{0}(r_n),$$
$$r_n = R+(n-1)h/N + h/(2N). \tag{3.1}$$

The above division of the interval $[R, R+h]$ into subintervals requires the formulation of the contact conditions between these subintervals and the paraphrasing of the corresponding boundary conditions on the sub-interfaces with the new notation for the upper indices. We assume that the mentioned contact conditions are perfect, and in order to reduce the size of the paper, the mathematical expressions for them are not written here. Although, these expressions are explicitly presented in the papers [1 - 7]. By direct verification, it is found that the number of contact and boundary conditions mentioned is equal to $4N$ and, as in the investigations in the works [1 - 7], the concrete values of the number N is determined from the convergence requirement of the numerical results.

Thus, if we take the assumptions formulated in (3.1) and substitute the expressions in (2.4) into the equations (2.3), we obtain the following equations of motion which are satisfied within each sublayer.

$$\frac{\partial \sigma_{rr}^{n}}{\partial r} + \sigma_{rr}^{0}(r_n)\frac{\partial^2 u_r^n}{\partial r^2} + \frac{\partial \sigma_{zr}^{n}}{\partial z} + \sigma_{zz}^{0}(r_n)\frac{\partial^2 u_r^n}{\partial z^2} + \frac{1}{r}(\sigma_{rr}^{n} - \sigma_{\theta\theta}^{n}) +$$
$$\sigma_{rr}^{0}(r_n)\frac{1}{r}\frac{\partial u_r^n}{\partial r} - \sigma_{\theta\theta}^{0}(r_n)\frac{u_r^n}{r^2} = \rho\frac{\partial^2 u_r^n}{\partial t^2},$$
$$\frac{\partial \sigma_{rz}^{n}}{\partial r} + \sigma_{rr}^{0}(r_n)\frac{\partial^2 u_z^n}{\partial r^2} + \frac{1}{r}\sigma_{rz}^{n} + \sigma_{rr}^{0}(r_n)\frac{1}{r}\frac{\partial u_z^n}{\partial r} + \frac{\partial \sigma_{zz}^{n}}{\partial z} +$$
$$\sigma_{zz}^{0}(r_n)\frac{\partial^2 u_z^n}{\partial z^2} = \rho\frac{\partial^2 u_z^n}{\partial t^2}, m = 1,2. \tag{3.2}$$

At the same time, within each sublayer, the relations in (2.5) and (2.6) remain valid without change, and in this way we obtain the complete system of equations consisting of (3.2), (2.5), and (2.6). As in the paper [1], we use the classical Lame decomposition (see, e.g., the monograph [11]) to solve these equations, which can be presented for the axisymmetric problems as follows.

$$u_r^n = \frac{\partial \Phi^n}{\partial r} + \frac{\partial^2 \Psi^n}{\partial r \partial z}, u_z^n = \frac{\partial \Phi^n}{\partial z} - \frac{\partial^2 \Psi^n}{\partial r^2} - \frac{\partial \Psi^n}{r \partial r}. \tag{3.3}$$

By the usual procedure, we obtain the following equations for the potentials Φ^n and Ψ^n from equations (3.2), (2.5) and (2.6).

$$(1+\frac{\sigma_{rr}^{0}(r_n)}{\lambda+2\mu})\frac{\partial^2 \Phi^n}{\partial r^2} + (1+\frac{\sigma_{\theta\theta}^{0}(r_n)}{\lambda+2\mu})\frac{\partial \Phi^n}{r\partial r} + (1+\frac{\sigma_{zz}^{0}(r_n)}{\lambda+2\mu})\frac{\partial^2 \Phi^n}{\partial z^2} = \frac{1}{(c_1)^2}\frac{\partial^2 \Phi^n}{\partial t^2},$$
$$(1+\frac{\sigma_{rr}^{0}(r_n)}{\mu})\frac{\partial^2 \Psi^n}{\partial r^2} + (1+\frac{\sigma_{\theta\theta}^{0}(r_n)}{\mu})\frac{\partial \Psi^n}{r\partial r} + (1+\frac{\sigma_{zz}^{0}(r_n)}{\mu})\frac{\partial^2 \Psi^n}{\partial z^2} = \frac{1}{(c_2)^2}\frac{\partial^2 \Psi^n}{\partial t^2}, \tag{3.4}$$

where $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$ are the speed of dilatation and distortion wave propagation velocities, respectively in the cylinder material.

As we consider the harmonic wave propagation along the $O\,z$ axis, we can present the sought functions Φ^n, u_r^n, σ_{rr}^n, $\sigma_{\theta\theta}^n$ and σ_{zz}^n with the multiplying $\cos(kz - \omega t)$ as well the sought functions Ψ^n, u_z^n and σ_{rz}^n with the multiplying $\sin(kz - \omega t)$ and, by denoting the amplitudes of these quantities with the same symbols, we obtain the equation:

$$\frac{d^2\Phi^n}{d(r_2)^2} + \frac{\alpha_1(r_n)}{r_2}\frac{d\Phi^n}{dr_2} + \Phi^n = 0, \frac{d^2\Psi^n}{d(r_1)^2} + \frac{\alpha'(r_n)}{r_1}\frac{d\Psi^n}{dr_1} + \Psi^n = 0, \qquad (3.5)$$

for the amplitudes Φ^n and Ψ^n, where

$$\alpha'(r_n) = \frac{1 + \sigma_{\theta\theta}^0(r_n)/\mu}{1 + \sigma_{rr}^0(r_n)/\mu}, \beta(r_n) = \frac{1 + \sigma_{zz}^0(r_n)/\mu}{1 + \sigma_{rr}^0(r_n)/\mu},$$

$$r_1^n = kr\sqrt{\frac{c^2}{(c_2)^2(1{+}\sigma_{rr}^0(r_n)/\mu)} - (\beta(r_n))^2},$$

$$c = \omega/\kappa, \alpha_1(r_n) = \frac{1 + \sigma_{\theta\theta}^0(r_n)/(\lambda{+}2\mu)}{1 + \sigma_{rr}^0(r_n)/(\lambda + 2\mu)},$$

$$\beta_1(r_n) = \frac{1 + \sigma_{zz}^0(r_n)/(\lambda{+}2\mu)}{1 + \sigma_{rr}^0(r_n)/(\lambda + 2\mu)},$$

$$r_2^n = kr\sqrt{\frac{c^2}{(c_1)^2(1{+}\sigma_{rr}^0(r_n)/(\lambda + 2\mu))} - (\beta_1(r_n))^2}. \qquad (3.6)$$

According to [12], the solution to the equations in (3.5) are found as follows.

$$\Phi^n = A_1^n(r_2)^{\gamma_1(r_n)}E_{\gamma_1(r_n)}(r_2^n) + A_2^n(r_2)^{\gamma_1(r_n)}F_{\gamma_1(r_n)}(r_2^n), \qquad (3.7)$$

$$\Psi^n = B_1^n(r_1)^{\gamma(r_n)}E_{\gamma(r_n)}(r_1^n) + B_2^n(r_1)^{\gamma(r_n)}F_{\gamma(r_n)}(r_1^n), \qquad (3.8)$$

where

$$\gamma_1(r_n) = (1 - \alpha_1(r_n))/2, \gamma(r_n) = (1 - \alpha'(r_n))/2,$$

$$E_{\gamma_1(r_n)}(r_2^n) = \begin{cases} J_{\gamma_1(r_n)}(r_2^n) \text{ if } (r_2^n)^2/r^2 > 0 \\ I_{\gamma_1(r_n)}(r_2^n) \text{ if } (r_2^n)^2/r^2 < 0 \end{cases},$$

$$F_{\gamma_1(r_n)}(r_2^n) = \begin{cases} Y_{\gamma_1(r_n)}(r_2^n) \text{ if } (r_2^n)^2/r^2 > 0 \\ K_{\gamma_1(r_n)}(r_2^n) \text{ if } (r_2^n)^2/r^2 < 0 \end{cases},$$

$$E_{\gamma(r_n)}(r_1^n) = \begin{cases} J_{\gamma(r_n)}(r_1^n) \text{ if } (r_1^n)^2/r^2 > 0 \\ I_{\gamma(r_n)}(r_1^n) \text{ if } (r_1^n)^2/r^2 < 0 \end{cases},$$

$$F_{\gamma(r_n)}(r_1^n) = \begin{cases} Y_{\gamma(r_n)}(r_1^n) \text{ if } (r_1^n)^2/r^2 > 0 \\ K_{\gamma(r_n)}(r_1^n) \text{ if } (r_1^n)^2/r^2 < 0 \end{cases}. \qquad (3.9)$$

In (3.9), $J_\delta(x)$ and $I_\delta(x)$ ($Y_\delta(x)$ and $K_\delta(x)$) are the Bessel and modified Bessel functions of the first (second) kind. Moreover, in (3.7) and (3.8)A_1^n, A_2^n, B_1^n, and B_2^n are unknown constants.

Thus, using (3.7) - (3.9), we obtain the expressions from (2.4) and (2.5) for the displacements and stresses for each sublayer, and substituting these expressions into the above-mentioned contact and boundary conditions, we obtain the homogeneous system of linear algebraic equations in terms of the unknown constants A_1^n , A_2^n, B_1^n, and B_2^n $(n =$

$1, 2, ..., N$). Equating the determinant of the coefficient matrix to zero (denote it by (a_{qp})), we obtain the dispersion equation. Formally, this equation can be written as follows.

$$\det(a_{qp}(c, kR, \alpha, h/R, \lambda/\mu)) = 0, q; p = 1, 2, ..., 4N) \tag{3.10}$$

In order to reduce the size of the paper, we omit here the explicit expressions for the components a_{qp}, which follow easily from the corresponding expressions presented and discussed above.

This concludes the consideration of the discrete-analytic solution method used in the present investigation.

4 Numerical results and discussions

Assume that the material of the cylinder is Steel with the mechanical constants $\mu = 79 \times 10^9\, Pa$, $\lambda = 94.4 \times 10^9\, Pa$, $\rho = 7790 kg/m^3$, and $c_2 = 3184\, m/s$, where μ and λ are the Lame constants, ρ is a density, and c_2 is a shear wave propagation velocity in the Steel.

In the present numerical investigations, the most important parameter is the angle Alfa, which enters the dispersion equation (3.10) and characterizes the magnitude of residual stresses. Consequently, through the parameter α we will characterize the influence of the magnitude of residual stresses on the dispersion curves. In addition, the ratio h/R, which also enters the dispersion equation (3.10), characterizes the geometry of the cylinder on the dispersion curves. Numerical results illustrating the influence of these two parameters on the dispersion curves are presented and analyzed below. In addition, the ratio λ/μ, which enters the dispersion equation (3.10), characterizes the influence of the material properties of the cylinder on the dispersion curves. The numerical results, which are not presented here, show that the influence of the ratio λ/μ on the dispersion curves is insignificant. Therefore, we assume here that the material of the cylinder is steel and fix the value of the ratio λ/μ.

In the context of this study, we consider the dispersion curves in terms of the lowest, first, and second modes. By the lowest modes, we mean the modes where the dispersion curves appear as a consequence of the residual stresses and these dispersion curves disappear with the decrease of the angle α. At the same time, the wave propagation velocity on these dispersion curves is lower than that on the corresponding dispersion curves related to the first mode and obtained for the case when the residual stresses are not present in the cylinder.

So, let us consider the dispersion curves shown in Figs. 2, 3 and 4, which are plotted for the cases when $h/R = 0.15$, 0.20 and 0.25 for different values of the angle α. Note that these curves relate to the aforementioned lowest and first modes.

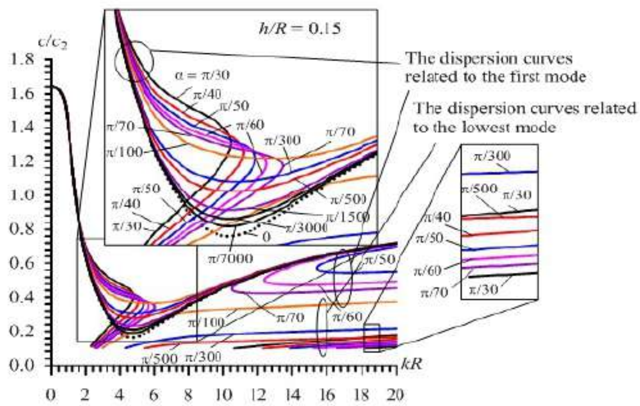

Fig. 2. Dispersion curves obtained for the lowest and first modes in the case when$h/R = 0.15$

The analysis of the graphs shown in Figs. 2, 3, and 4 shows that under relatively small values of the angle α, (i.e, in the cases when $\alpha \leq \pi/100$ under $h/R = 0.15$, in the case when $\alpha \leq \pi/70$ under $h/R = 0.20$, and in the case $\alpha \leq \pi/40$ under $h/R = 0.25$) the residual stresses in the cylinder causes to increase the wave propagation velocity in the first mode. At the same time in these cases, as a result of the mentioned residual stresses it is appear the additional dispersion curves of the lowest mode. What is more, for each α it is appear several dispersion curves of the lowest modes. Let us call the "maximum" dispersion curve of the lowest modes on which the wave propagation velocity is maximum among all lowest dispersion curves. The results shows that an increase in the values of angle α causes to approach (from below) the "maximum" dispersion curve to the dispersion curve of the first mode related to the cylinder in which residual stresses are absent.

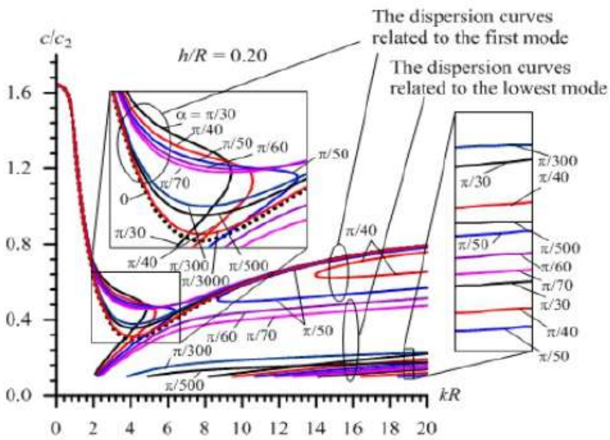

Fig. 3. Dispersion curves obtained for the lowest and first modes in the case when $h/R = 0.20$

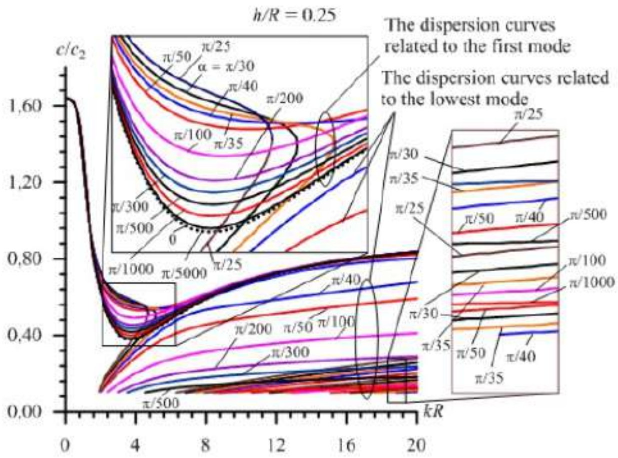

Fig. 4. Dispersion curves obtained for the lowest and first modes in the case when $h/R = 0.25$

As a result of this increase, there is a value for the α (denote this value by $\alpha*$) below which the "maximum" dispersion curve of the lowest mode at a certain value of the dimensionless wave number kR (denote it by $(kR)*$) touches the dispersion curve of the first mode obtained for the cylinder with residual stresses. And after this touch, further increase of the values of α leads to the appearance of two branches (left and right branches) of the first mode. The left branch exists in the region $0 < kR < (kR)'$, but the right branch exists in the region $(kR)'' < kR < \infty$ and $(kR)' < (kR)''$. Also, the results show that there exists the following relations:

$$d(c/c_2)/d(kR)|_{kR=(kR)'} = \infty and\ d(c/c_2)/d(kR)|_{kR=(kR)''} = \infty. \quad (4.1)$$

As well as, the results show that the difference $((kR)'' - (kR)')$ increase with angl α. Besides all this, it follows from the results that the value of $\alpha*$ increase with h/R.

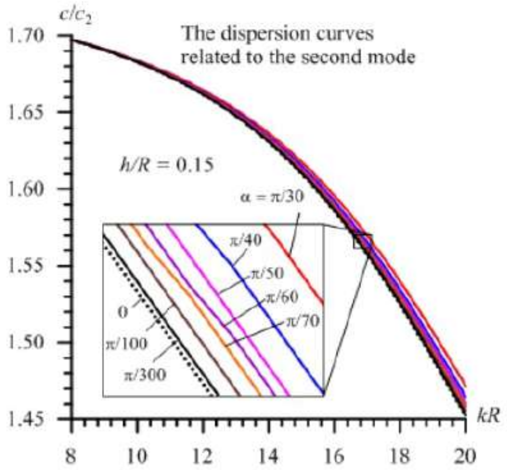

Fig. 5. Dispersion curves obtained for the second mode in the case when $h/R = 0.15$

Thus, from the results presented in Figs. 2, 3, and 4, it is clear that the influence of the residual stresses determined by expressions (2.1) and (2.2) on the dispersion curves with respect to the lowest and first modes is not only quantitative but also qualitative. However, the influence of these residual stresses on the dispersion curves of the second mode, shown in Figs. 5, 6 and 7 for the cases $h/R = 0.15$, 0.20 and 0.25, respectively, has only a quantitative character. As can be seen from Figs. 5, 6 and 7, an increase in the values of the angle α leads to a monotonic increase in the values of the wave propagation velocity on the dispersion curves of the second mode. However, this increase is more pronounced in the cases where $h/R > 8.0$. Therefore, the parts of the second mode dispersion curves that relate to this case are shown here.

The numerical results, which are not shown here, indicate that the magnitude of the influence of the residual stresses considered on the dispersion curves associated with the third and subsequent higher modes is insignificant, so they are not considered here.

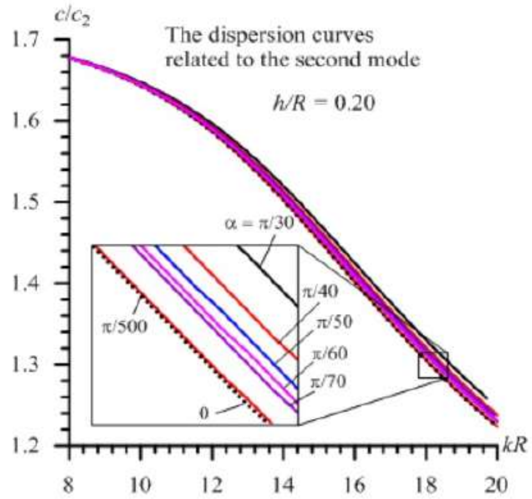

Fig. 6. Dispersion curves obtained for the second mode in the case when $h/R = 0.20$

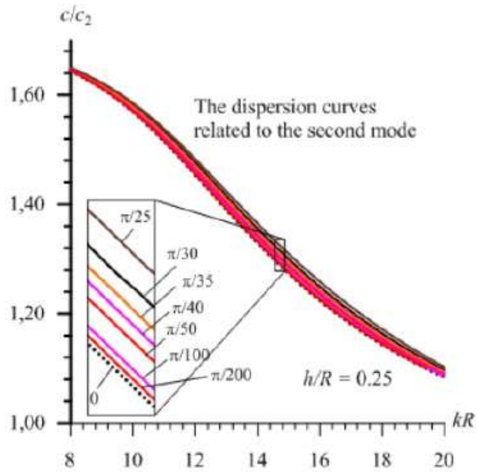

Fig. 7. Dispersion curves obtained for the second mode in the case when $h/R = 0.25$

Note that all numerical results discussed above were obtained for the case where $N = 25$, i.e., for the case where the cylinder is divided into 25 sub-cylinders of thickness h/N. To illustrate the convergence of the obtained numerical results with respect to this N number, we consider the results shown in Fig. 8, obtained in the most unfavorable case with respect to the sense of numerical convergence, i.e., in the case where $h/R = 0.25$ and for the parts of the dispersion curves related to the left branch of the first and ”maximum” lowest modes. Note that these results show not only the numerical convergence of the results but also the formation of the left branches of the first mode. Thus, from the results in Fig. 8, it appears that the convergence of the obtained numerical results is satisfactory for the cases considered here.

This concludes the analysis of the presented numerical results.

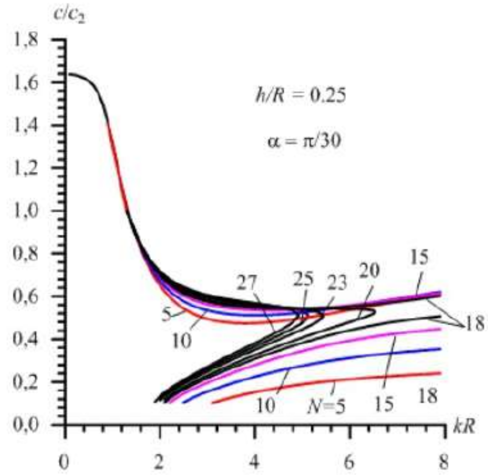

Fig. 8. Convergence numerical results with respect to the number N indicating the number of sublayers into which is divided the cylinder

5 Conclusions

Thus, in the present paper, the influence of the axisymmetric residual stresses generated by the contacting of the contacting of cuts shown in Fig. 1 on the dispersion curves of the axisymmetric waves propagating in this cylinder is investigated. The investigation is carried out using the three-dimensional linearized theory of elastic waves in initially stressed bodies. The corresponding eigenvalue problem is solved using the discrete analytic method developed in earlier work of the author. Numerical results on the influence of the magnitude of the residual stresses on the propagation curves of the lowest, first and second modes are presented and discussed. In the context of this discussion, the mentioned quantity is characterized by the cut angle (Fig. 1).

Based on these results, it is found that new low-lying modes appear as a result of the residual stresses and the dispersion curves of these modes approach the dispersion curve of the first mode as the cut angle increases. Moreover, it is found that the influence of residual stresses on the dispersion curves of the first mode is not only quantitative but also qualitative in character, while this influence on the dispersion curves of the second mode is only quantitative in character.

Note that further detailed conclusions on the studied influences can be found in the text of the paper.

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