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On mathematical modeling of the damageability of a cylindrically isotropic thick pipe in a complex stress state

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Abstract. *This paper investigates the scattered fracture process of a thick pipe with uniform pressure at the inner boundary of the pipe. The process of destruction of a cylindrically isotropic tube under the impact of internal pressure, assuming that the tube material behind the fracture front completely loses its bearing capacity and the acting external pressure is transferred to a new boundary surface, which is the moving fracture front. In the present work, this assumption is omitted, considering that the pipe material behind the fracture front significantly reduces its bearing capacity, even if in a small way, it retains it. Thus, the picture of the fracture process is as follows: at some point in time the inner surface layer of the pipe is fractured. Then the fracture zone increases encompassing an annular zone. This fracture zone extends to cover the entire area of the pipe, at which point the pipe fractures and loses its load-bearing capacity. This expansion can, depending on the correlation of the process parameters, occur at a finite rate or, at some stage, this rate can become infinitely high and the corresponding time will determine the time of complete destruction of the pipe.*
The damage process is described by the kinetic equation. The problem is solved taking into account the residual strength of the pipe material behind the fracture front. The numerical calculation has been carried out and the curves of the fracture front movement depending on the measure of the residual strength behind the fracture front have been plotted.

Keywords. stress intensity · damageability · fracture · fracture front

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1 Introduction

The specifics of calculating the strength of bodies in a heterogeneous stress state consists in the difference in the fracture time of its individual parts. Expanding destruction parts changes the interface between the fractured and non-fractured parts. This moving boundary surface is called the fracture front and was first introduced by L.M. Kachanov [2]. As noted above, such a situation occurs for an inhomogeneous stress state with the structure. The strength theories are unsuitable for investigating the fracture of such bodies, because the material element for which the strength criterion is fulfilled is considered to be completely collapsed and completely lost its resistance to loading. The description of further behavior of such an element under continued loading within the framework of strength theories is impossible. Theories of damageability or theories of dispersed fracture offer great opportunities here.

One of the ways to analyze the fracture of a body in an inhomogeneous stress state is the technique, based on the concept of the fracture front. In this case, in addition to the constitutive equations and the fracture criterion, additional assumptions not derived from the deformation and fracture model are required.

Since in heterogeneous stress states the stress levels at different points are different, the degrees of damage at these points also differ accordingly. The equations that relate stresses to strains-defining equations-at each point will be valid until the corresponding fracture criterion is met for that point. From this point onwards, the given material particle is unable to perform its functional duty, to carry a certain load, and to collapse. As a consequence, there is a redistribution of stresses in the body, which further leads to the destruction of the adjacent material particle. Over time, the collapsed part of the body increases until the entire structure loses its load-bearing capacity.

Thus, two stages of diffuse fracture are distinguished. The first stage, called the latent fracture stage or incubation period, extends up to the time when a fractured area first forms in the body, which may consist of at least one point. Thereafter, this area of the body increases in size. The movement of the fracture front, which characterizes the increase in the fractured area, occurs up to the point in time when the entire body structure, the load-bearing capacity, completely fails. The time period from before is called the fracture propagation stage. Determining the time moment requires additional assumptions. For example, the condition that the velocity of the fracture front travels to infinity is possible. However, this condition is not always acceptable, because for some structures, the velocity of the fracture front during the entire stage of fracture propagation remains finite.

2 Problem statement

The equation of motion of the destruction front is determined by the kinetic equation.

The article adopts the model [4, 5] which considers destruction as a critical stage of material deformation, as the basic model of a damaged body. The convenience in using this theory is that the same operator characterizing the process of damage accumulation is included as in the kinetic equation.

We will take the failure criterion in the form, also following the works [4, 5]:

$$(1 + \mathrm{M}^*)\,\sigma_i = \sigma_0 \tag{2.1}$$

where σ_i is the stress intensity, which for a thick pipe under plane deformation has the form:

$$\sigma_i = \sqrt{2}p\frac{a^2b^2}{b^2 - a^2} \cdot \frac{1}{r^2} \tag{2.2}$$

Here *a* and *b* are the inner and outer radii of the pipe, respectively, *r* is the current radius of the pipe, *p* is the indentation on the inner surface of the pipe created by the filler.

Initially, the pipe consists of a single non-permitted material. The maximum value of stress intensity is reached on the inner surface of the pipe, where the most intensive process of damage accumulation takes place. This process leads to the initiation of a destruction zone there at the moment of time t_0, determined on the basis of the destruction criterion [3]:

$$(1+\mathrm{M}^*)\,\sigma_{i\max} = \sigma_{?o}. \tag{2.3}$$

Denoting $a/b = \beta_0$ and $\frac{\sigma_{?o}}{\sqrt{2}p} = g$ and taking into account (2.2) in (2.1), we get:

$$\int_0^{t_0} M(\tau)d\tau = g\left(1-\beta_0^2\right)-1 \tag{2.4}$$

Let us give an explicit form for the initial destruction time for two types of nuclei $M(t)$

$$M(t) = m; \qquad t_0 = \frac{1}{m}\left(g(1-\beta_0^2)-1\right) \tag{2.5}$$

$$M(t) = mt^{-\alpha};\ 0 <\ \alpha < 1; \qquad t_0 = \left\{\frac{1-\alpha}{m}\left[g\left(1-\beta_0^2\right)-1\right]\right\}^{\frac{1}{1-\alpha}} \tag{2.6}$$

Further, the boundary of the destruction zone - the destruction front will move towards the outer surface of the pipe. The fracture zone itself is an annular zone. The material of this fracture zone retains its bearing capacity, but to a much lesser extent than the original material before the fracture front. We assume that at the fracture front, the pipe material abruptly changes its instantaneous rheological characteristics in this problem, the value of the shear modulus G. Let us take for G_1 the shear modulus of the pipe material ahead of the fracture front, and G_0 behind the fracture front. Let's introduce the notation:

$$\chi = \frac{G_0}{G_1} \tag{2.7}$$

It's obvious that $\chi < 1$.

The stress state at an arbitrary point in time is defined as for a two-layer pipe with different elastic characteristics.

For the stress intensity in the pipe area in front of the fracture front, which will provide index 1, according to the known formulas [2], assuming the material as incompressible, we obtain:

$$\sigma_{\mathrm{i}}^{(1)} = \sqrt{2}q\frac{k^2b^2}{b^2-k^2}\cdot\frac{1}{r^2} \tag{2.8}$$

where q is the pressure at the destruction front, *k* is the radius of the fracture front.

For radial displacements of the points of the region behind and in front of the destruction front, under the condition of incompressibility of the material, again, according to [1], we have:

$$\mathrm{u}^{(1)} = \frac{1}{2G_1}\left(1+M^*\right)\frac{k^2b^2q}{b^2-k^2}\cdot\frac{1}{r^2};u^{(0)} = \frac{a^2k^2(p-q)}{2G_0(k^2-a^2)}\cdot\frac{1}{r} \tag{2.9}$$

From the condition of continuity of displacements at the fracture front

$$u^{(1)}\Big|_{r=k} = u^{(0)}\Big|_{r=k} \tag{2.10}$$

we get:

$$\frac{\mathrm{b}^2k(t)\mathrm{q(t)}}{\mathrm{b}^2-k^2(t)} + \int_0^t M(t-\tau)\frac{\mathrm{b}^2k^2(\tau)\mathrm{q}(\tau)}{\mathrm{b}^2-k^2(\tau)}\cdot\frac{1}{k(t)}d\tau = \frac{1}{\chi}\frac{a^2k(t)}{k^2(\tau)-a^2}\left(p-q(t)\right); \tag{2.11}$$

Introducing the dimensionless quantity $\beta(t) = k(t)/b$ relative to this dimensionless radial coordinate of the fracture front, we obtain the following nonlinear integral equation:

$$\frac{\beta^2(t)\tilde{q}(t)}{1-\beta^2(t)} + \int_0^t M(t-\tau)\frac{\beta^2(\tau)\tilde{q}(\tau)}{1-\beta^2(\tau)}d\tau = \frac{1}{\chi}\frac{\beta_0^2\beta^2(t)(\tilde{p}-\tilde{q}(t))}{\beta^2(t)-\beta_0^2}. \qquad (2.12)$$

The stress intensity formula (2.8) in the dimensionless radial coordinate has the form:

$$\tilde{\sigma}_i = \sqrt{2}\tilde{q}(t)\frac{\beta^2(\tau)}{1-\beta^2(t)}\frac{1}{\beta^2(t)} \qquad (2.13)$$

The kinetic equation for the development of the fracture front was adopted as follows:

$$\frac{d\beta(t)}{dt} = \varphi(\varepsilon_i) \qquad (2.14)$$

where $\varphi(\varepsilon_i)$ are the strain intensity functions, $\beta(t)$ is the radius of the fracture front.

Stress at a point in time t causes elastic deformation. Therefore, the total deformation $\tilde{\sigma}_i = \sigma_i/E$ at the moment of time is the sum of this deformation and the deformation that has arisen due to the stresses acting before the moment of time t,

$$\varphi(\varepsilon_i) = \tilde{\sigma}_i + \int_0^t M(t-\tau)\tilde{\sigma}_i(\tau)d\tau \qquad (2.15)$$

Then the expression obtained by (2.15) is substituted into equation (2.14) and we obtain,

$$\frac{d\beta(t)}{dt} = \tilde{\sigma}_i + \int_0^t M(t-\tau)\tilde{\sigma}_i(\tau)d\tau. \qquad (2.16)$$

Taking into account formula (2.13) in equation (2.16), we obtain,

$$\frac{\beta^2(t)}{\sqrt{2}}\frac{d\beta(t)}{dt} = \frac{\tilde{q}(t)}{1-\beta^2(t)} + \int_0^t M(t-\tau)\frac{\tilde{q}(\tau)}{1-\beta^2(\tau)}\beta^2(\tau)d\tau \qquad (2.17)$$

Thus, we have a system of two nonlinear integral equations (2.12), (2.17) with respect to the radial coordinate $\beta(t)$ of the destruction front and the pressure $\tilde{q}(t)$on it. Note that if t_0 is the time of initial destruction, that is, the destruction of the inner surface of the pipe$\beta = \beta_0$, determined according to formula (2.5), then in system (2.11), (2.16) for $\tau \leq t_0$ we should no assume $\beta(\tau) = \beta_0;\ \tilde{q}(\tau) = \tilde{p}$. The integral equation (2.17) makes sense only for $t > t_0$.

Equations (2.12), (2.17) can be reduced to solving one non-linear integral equation. To do this, using the identity of the structure of the integral terms of equations (2.12) and (2.17), excluding them, we obtain the following explicit representation of the pressure dependence at the fracture front on its radial coordinate:

$$\tilde{q}(t) = \tilde{p} - \frac{\chi(\beta^2(t)-\beta_0^2)}{\sqrt{2}\beta_0^2}\frac{d\beta(t)}{dt} \qquad (2.18)$$

Taking into account thus representation in integro-differential equation (2.17), we obtain the following nonlinear integral equation:

$$\frac{\beta^2(t)}{\sqrt{2}}\frac{d\beta(t)}{dt} = \frac{1}{1-\beta^2(t)}\left[\tilde{p} - \frac{\chi(\beta^2(t)-\beta_0^2)}{\sqrt{2}\beta_0^2}\frac{d\beta(t)}{dt}\right] +$$

$$+\int_0^t M(t-\tau)\frac{\beta^2(\tau)}{1-\beta^2(\tau)}\left\{\tilde{p}-\frac{\chi(\beta^2(\tau)-\beta_0^2)}{\sqrt{2}\beta_0^2}\frac{d\beta(\tau)}{d\tau}\right\}d\tau \tag{2.19}$$

The solution of equation (2.19) determines the nature of the expansion of the annular destruction zone $\beta = \beta(t)$. Further, according to formula (2.18), the pressure at the destruction front is determined. It should be noted that the solution of the integral equation is valid as lounge as the pressure calculated by formula (2.19) $\mathrm{q(t)}$ is positive. Its equality to zero or its negativity means a violation of the continuity of the material with the formation of an arc crack along the destruction front.

In order to elucidate the qualitative picture of the destruction process, let us take as the damage operator kernel: $\mathrm{M(t-\tau) = m = const}$ then, introducing the dimensionless time $\theta = mt$ and $\eta = m\tau$, equation (2.19) will take the form

$$\frac{\beta^2(\theta)}{\sqrt{2}}\frac{d\beta(\theta)}{d\theta} = \frac{1}{1-\beta^2(\theta)}\left[\tilde{p}-\frac{\chi(\beta^2(\theta)-\beta_0^2)}{\sqrt{2}\beta_0^2}\frac{d\beta(\theta)}{d\theta}\right]+$$

$$+\int_0^t \frac{\beta^2(\eta)}{1-\beta^2(\eta)}\left\{\tilde{p}-\frac{\chi(\beta^2(\eta)-\beta_0^2)}{\sqrt{2}\beta_0^2}\frac{d\beta(\eta)}{d\eta}\right\}d\tau \tag{2.20}$$

Differentiating with respect to the dimensionless time θ, we obtain the following differential equation:

$$A(\beta)\frac{d^2\beta}{d\theta^2}+B(\beta)\left(\frac{d\beta}{d\theta}\right)^2+C(\beta)\frac{d\beta}{d\theta} = \frac{\beta^2}{1-\beta^2}\tilde{P} \tag{2.21}$$

$$\begin{cases} A(\beta) = \frac{\beta^2\beta_0^2+\chi(\beta^2-\beta_0^2)}{\sqrt{2}\beta_0^2} \\ B(\beta) = \frac{2\beta\left(\beta_0^2(1-\beta^2)^2+\chi\beta^2(1-\beta^2)+\chi(\beta^2-\beta_0^2)\right)}{\sqrt{2}\beta_0^2(1-\beta^2)^2} \\ C(\beta) = \frac{\chi\beta^2(\beta^2-\beta_0^2)(1-\beta^2)-2\sqrt{2}\tilde{P}\beta\beta_0^2}{\sqrt{2}\beta_0^2(1-\beta^2)^2} \end{cases} \tag{2.22}$$

The initial condition for it will be condition (2.8):

$$\beta|_{\theta=\theta_0} = \beta_0; \quad \theta_0 = g\left(1-\beta_0^2\right)-1; \quad \frac{d\beta}{d\theta}|_{\theta=\theta_0} = 0 \tag{2.23}$$

The resulting Cauchy problem (2.21), (2.23) was solved numerically method for the values of input parameters: $\beta_0 = 0,5;\quad g = 3,9;\ 4,7;\ 5,9$ and $\chi = 0;\ 0,2;\ 0,4;\ 0,6$. The curves of movement of destruction front for there values of the parameter of residual strength χ depending on the parameter g are given in Fig 1, 2.

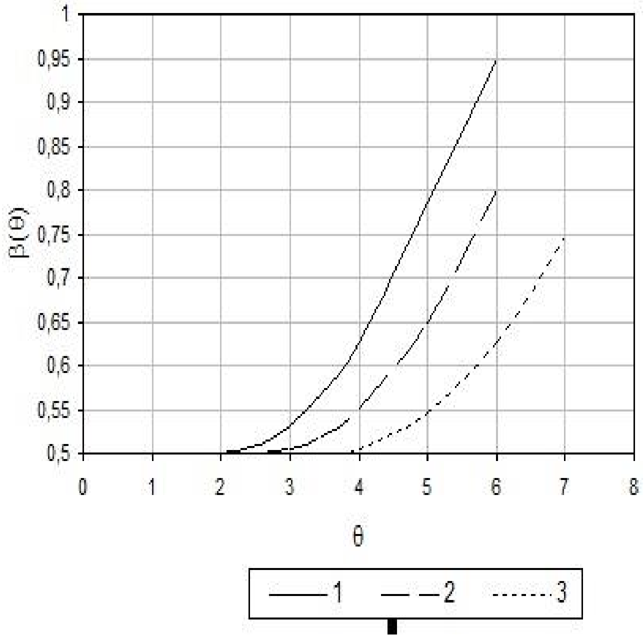

Fig. 1. Fracture front motion curves for the damage kernel $M(t-\tau) = m = Const$ for $\chi = 0,01$:1.$g = 3,9$, 2. $g = 4,7$, 3. $g = 5,9$

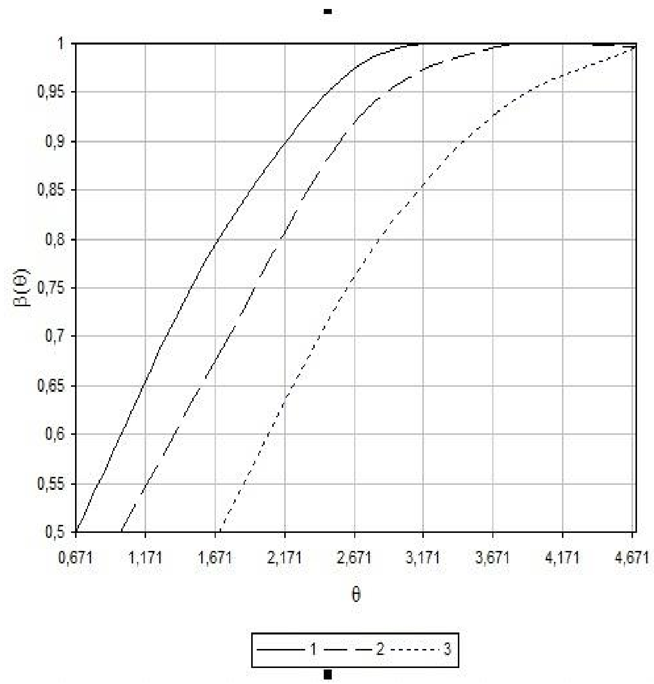

Fig. 2. Fracture front motion curves for the damage kernel $M(t-\tau) = (t-\tau)^{-\alpha}$. $\chi = 0,2$; $\alpha = 0,25$: 1. $g = 3,9$, 2. $g = 4,7$, 3. $g = 5,9$

Fig. 3 and 4 show the fracture front movement curves for different values of the residual strength parameter as a function of the parameter g.

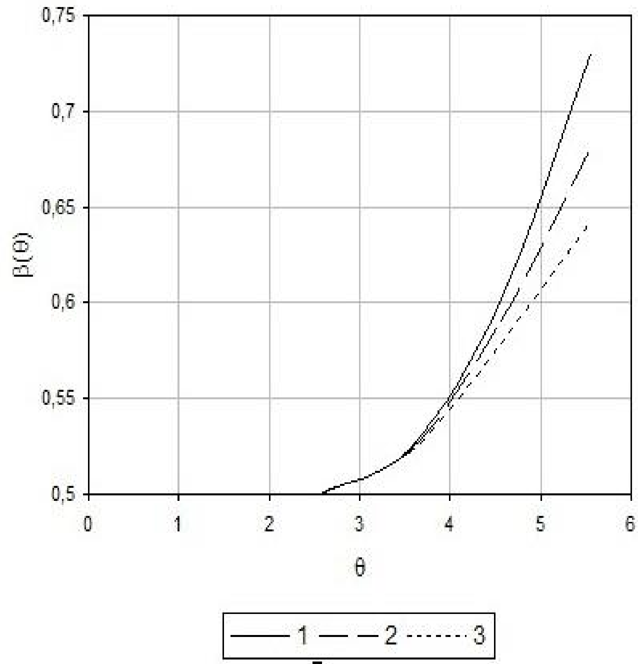

Fig. 3. Fracture front motion curves for the damage kernel $M(t-\tau) = m = Const$ for $g = 4,7$:1. $\chi = 0$, 2. $\chi = 0,2$, 3. $\chi = 0,6$

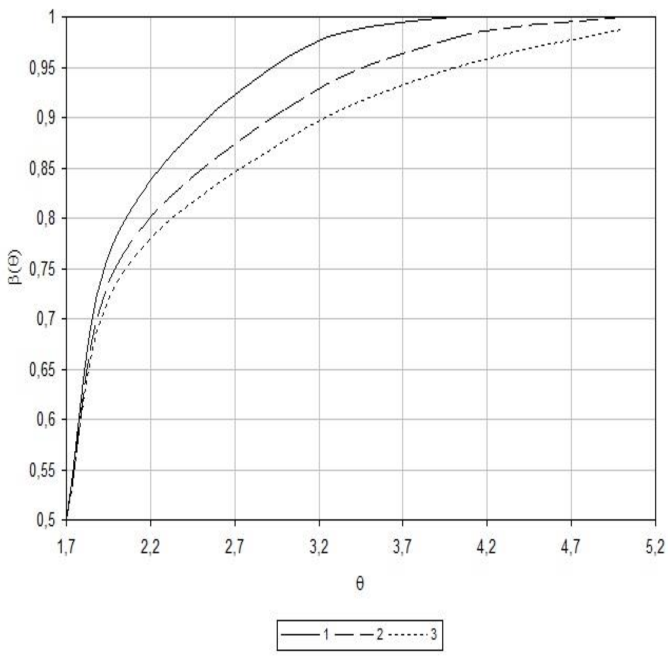

Fig. 4. Fracture front motion curves for the damage kernel $M(t-\tau) = (t-\tau)^{-\alpha}$.

For $g = 5,9$; $\alpha = 0,25$:1. $\chi = 0$,2,2. $\chi = 0,4$,3. $\chi = 0,6$

As follows from the graph, the destruction front moves at a decreasing speed. The calculations also showed that the presence of residual strength behind the destruction front has little effect on the nature of the fracture front movement, but it strongly affects the time of onset of delaminating.

3 Conclusions

A kinetic equation is derived for the radial coordinate of the destruction front, taking into account the process of damage to the material of the pipe itself.

Explicit formulas for contact pressures at the destruction front are obtained. An analysis of the relationship between critical situations of delaminating at the destruction front and an analysis of destruction due to the accumulation of a critical amount of damage is given.

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