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A boundary Harnack inequality for degenerate singular nonlinear parabolic equations

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Abstract. *We study a boundary Harnack inequality for solutions to degenerate singular nonlinear parabolic equations in case* $\frac{2n}{n+1} < p < 2$ *in time-independent cylinder whose base is* $C^{1,1}$*-regular. The corresponding estimates valid for the heat equation, that this type of inequalities can not, in general, be expected to hold in the degenerate case* $2 < p < \infty$.

Keywords. nonlinear parabolic equation · degenerate · singular · boundary Harnack inequality.

Mathematics Subject Classification (2010): 35J25

1 Introduction

The study of boundary estimates, and boundary Harnack inequalities for nonlinear harmonic functions, $p \neq 2,\ 2 < p < \infty$ in nonsmooth domains, have been advanced, see [1]. These estimates have been used to solve several problems concerning boundary regularity for the nonlinear parabolic operators. Let $\omega(x)$-is a Mackenhoupt weight function, see [2]. We consider boundary estimates for solutions nonlinear parabolic operator of the type

$$u_t - div\left(\omega(x)\,|Du|^{p-2}\,Du\right) = 0, \tag{1.1}$$

in domains of the form $Q_T = \Omega \times (0,T) \subset R^n \times R$, where $\Omega \subset R^n,\ n \geq 2$, is a bounded domain.

It is well-known, that solutions to the nonlinear parabolic equations exhibit quite different behaviors in the parameter regimes $2 < p < \infty$, degenerate case and $1 < p < 2$, singular case. In the degenerate case the phenomenon of finite speed propagation is present and in the singular case solutions will go extinct. Furthermore, the singular case divided into the regimes super-critical case $\dfrac{2n}{n+1} < p < 2$ and sub-critical case $1 < p \leq \dfrac{2n}{n+1}$. We also note that when $p = 2$ then the our operator coincides with the heat operator.

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Let $(x_0, t_0) \in \partial\Omega \times (0, T)$ and $r < \min\{r_0, C(T, t_0)\}$, where r_0 some constant and C constant depending at T, t_0. Let

$$A_r(x_0, t_0) = (x_0 + 2M_1 e_n, t_0),\ A_2^{\pm}(x_0, t_0) = (x_0 + 2M_1 e_n, t_0 + 2_r^2),$$

where e_n is the unit vector pointing in the positive x_n-direction and defined through the local coordinate system and constants M_1 and r_0 connected with domain. The following result is due to [1]: There exist constants $1 \le C_1, C_2 < \infty$ and $\sigma,\ 0 < \sigma < 1$, such that

$$\left|\frac{u(x,t)}{v(x,t)} - \frac{u(y,s)}{v(y,s)}\right| \le C_1 \frac{u(A_r(x_0,t_0))}{v(A_r(x_0,t_0))}\left(\frac{d_p((x,t),(y,s))}{r}\right)^{\sigma} \tag{1.2}$$

holds whenever $(x, t),\ (y, s) \in (\Omega \times (0, T)) \cap C_{r/4}(x_0, t_0)$ and $0 < r < r_0/C_2$. Where $C_r(x, t) =$
$= B(x, r) \times (t - r^2, t + r^2)$ whenever $(x, t) \in R^{n+1}, r > 0$; $d_p((x, t), (y, s)) = \max\left\{|x - y|,\ |t - s|^{1/2}\right\}$
and $d_p(x, t)$-is the parabolic distance from (x, t) to $\partial\Omega \times (0, T)$.

An important feature of this result (1.2) is that the statement is both forward and backward in time-something considering the time-lag generally in the parabolic Harnack inequality. However, the fact that u and v both vanish continuously on a large portion of $\partial\Omega \times (0, T)$ enables one to establish Harnack inequality.

Also are consider of paper [3]-[8].

2 Degenerate singular version

We consider degenerate singular case for nonlinear parabolic equation and we emphases that we here only consider p in the rage

$$\frac{2n}{n+1} < p < 2 \tag{2.1}$$

to ensure the validity of Harnack solutions may go extinct, a phenomena of infinite propagation in space and in the singular range the equation exhibits elliptic features as seen from the forward-backward Harnack inequality valid for positive solutions to the singular nonlinear parabolic equations in case $\frac{2n}{n+1} < p < 2$. Let $Q_T = \Omega \times (0, T)$, where $\Omega \subset R^n$ a bounded domain and $T > 0$, and p as in (2.1). We suppose that u is a nonnegative and continuous weak solution to (1.1) in Q_T. Let $(\widetilde{x}_0, \widetilde{t}_0) \in Q_T,\ u(\widetilde{x}_0, \widetilde{t}_0) > 0$.

Our main result is a version the result (1.2) valid for p in the range (2.1) and in the setting of time-independent $C^{1,1}$-regular cylinders.

To formulate our result assume the existence of certain intrinsic parameters associated to u and v. For given $u, (\widetilde{x}_0, \widetilde{t}_0)$ and r we let Γ_u denote the set of all values of $\Lambda_u,\ 0 < \Lambda_u < \infty$, for which the following three restrictions hold. Firstly

$$(\Lambda_u)^{2-p}(10r)^p < t_0 < T - (\Lambda_u)^{2-p}(10r)^p. \tag{2.2}$$

Secondly, let $Q_{r,\varphi}^{\lambda,+}(x_0, t_0)$ and $\Lambda_{r,\varphi}^{\lambda}(x_0, t_0)$ some cubes which choose corresponding with property of domain Ω, where $|t - t_0| < \lambda^{2-p} r^p$. Let u is assumed to be a nonnegative solution to (1.1) in $Q_{10r,\varphi}^{\Lambda_u,+}$, continuous on the closure of this set and vanishing continuously in $Q_{10r,\varphi}^{\Lambda_u}(x_0, t_0)$.

Thirdly, we assume that

$$\sup_{Q_{5r,\varphi}^{\Lambda_u,+}(x_0 t_0)} \omega u \le \Lambda_u. \tag{2.3}$$

Corresponding we define Γ_v. Let u and v be two functions which are nonnegative and continuous in a neighborhood of $(\widetilde{x}_0, \widetilde{t}_0)$ and let

$$\theta_u = \omega u(\widetilde{x}_0, \widetilde{t}_0) \quad \text{and} \quad \theta_v = \omega v(\widetilde{x}_0, \widetilde{t}_0) \tag{2.4}$$

are positive. Also let

$$\begin{aligned} \theta_u^{2-p}(10r)^p < t_0 < T - \theta_u^{2-p}(10r)^p \\ \theta_v^{2-p}(10r)^p < t_0 < T - \theta_v^{2-p}(10r)^p. \end{aligned} \tag{2.5}$$

Assuming (2.5) and using the fact that $(\widetilde{x}_0, \widetilde{t}_0) \in Q_{5r,\varphi}^{\Lambda_u,+}(x_0, t_0)$, $(\widetilde{x}_0, \widetilde{t}_0) \in Q_{5r,\varphi}^{\Lambda_u,+}(x_0, t_0)$, we see that any such Λ_u and Λ_v must satisfy $\theta_u \le \Lambda_u$ and $\theta_v \le \Lambda_v$. In the following we assume that $u, v, \left(\widetilde{x}_0, \widetilde{t}_0\right), r, T$ are such that (2.5) holds and

$$\Gamma_u \neq \varnothing \quad \text{and} \quad \Gamma_v \neq \varnothing. \tag{2.6}$$

Based on (2.6) we in the following let Λ_u and Λ_v denote the smallest values of Λ_u and Λ_v for which (2.3), and the corresponding statement for Λ_v, hold. We can assume, that

$$\sup_{Q_{5r,\varphi}^{\Lambda_u,+}} \omega u = \Lambda_u, \quad \sup_{Q_{5r,\varphi}^{\Lambda_v,+}} \omega u = \Lambda_v \tag{2.7}$$

If $\Omega \subset R^n$ is bounded domain and $1 \le p < \infty$, then by $W_p^1(\Omega)$, we denote the space of equivalence classes of functions f with distributional gradient $Df = (f_{x_1}, ..., f_{x_n})$, both of which are p-th power integrable on Ω. Let

$$\|f\|_{W_{p,\omega}^1(\Omega)} = \|f\|_{L_{p,\omega}(\Omega)} + \|Df\|_{L_{p,\omega}(\Omega)}$$

be the norm in $W_{p,\omega}^1(\Omega)$, where $\|\cdot\|_{L_{p,\omega}(\Omega)}$ denotes the usual weight Lebesgue p-norm in Ω. Given $t_1 < t_2$ we denote by $L_p\left(t_1, t_2, W_{p,\omega}^1(\Omega)\right)$ the space of functions such that for almost every $t_1 < t < t_2$, the function $x \to u(x,t)$ belongs to $W_{p,\omega}^1(\Omega)$ and

$$\|u\|_{L_p\left(t_1, t_2, W_{p,\omega}^1(\Omega)\right)} =$$

$$= \left(\int_{t_1}^{t_2} \int_{\Omega} \omega(x) \; \left(|u(x,t)|^p + |Du(x,t)|^p\right) dxdt \right)^{1/p} < \infty.$$

Now we give definition of weak solutions to (1.1). We say that a function u is a weak super-solution (subsolution) to (1.1) in an open set $Q_T \in R^{n+1}$ if, whenever $Q'_T = \Omega \times (t_1, t_2) \in Q_T$ with $\Omega \subset R^n$ and $t_1 < t_2$, then $u \in L_p\left(t_1, t_2; W_{p,\omega}^1(\Omega)\right)$ and

$$\int_{Q'_T} \left[\left(\omega \left|Du\right|^{p-2} Du, D\varphi\right) - u\varphi_t\right] dxdt \ge (\le) \; 0 \tag{2.8}$$

for all nonnegative $\varphi \in C_0^\infty\left(Q'_T\right)$. A weak solution is a distributional solution satisfying (2.8) with equality and without sign restrictions for test functions. Let given continuous boundary data $h : Q_T \to R$. We consider the problem (1.1) with boundary condition $u = h$ on ∂Q_T, here $\partial Q_T = \left(\overline{\Omega} \times \{0\}\right) \cup (\partial\Omega \times [0,T])$ denotes the parabolic boundary of Q_T.Thus we have problem

$$u_t - div\left(\omega(x) \left|Du\right|^{p-2} Du\right) = 0 \text{ in } \; Q_T \tag{2.9}$$

$$u = h \quad \text{on } \; \partial Q_T.$$

3 Main result

Now we giving main result in this paper.

Theorem 3.1 *Let* $Q_T = \Omega \times (0,T),\ \Omega \subset R^n$ *is a bounded* $C^{1,1}$*-regular domain,* $T > 0$*. Let* p *as in (2.1)be fixed and we consider problem (2.9). Let* $\left(\widetilde{x}_0, \widetilde{t}_0\right),\ (x_0,t_0)$ *and* r *be as above and* u, v *be weak solutions to (2.9). Also satisfy (2.4), (2.5), (2.6) and (2.7). In addition assume that there exist* $\lambda_u, \lambda_v,\ 1 \le \lambda_u < \infty,\ 1 \le \lambda_v < \infty$ *such that*

$$\theta_u \le \Lambda_u \le \lambda_u \theta_u,\ \ \theta_v \le \Lambda_v \le \lambda_v \theta_v.$$

Then there exist constants $C_1, C_2,\ 1 \le C_1 C_2 < \infty$, *and* $\sigma,\ 0 < \sigma < 1$, *such that*

$$\left| \frac{\omega(x)u(x,t)}{v(x,t)} - \frac{\omega(y)u(y,s)}{v(y,s)} \right| \le$$

$$\le C_1 \frac{\theta_u}{\theta_v} \left(\frac{|x-y|}{r} + \left(\frac{1}{\theta_{uv}}\right)^{\frac{2}{p-1}} \left(\frac{|s-t|^{1/p}}{r^p} \right) \right)^{\sigma}$$

holds whenever $(x,t),\ (y,s) \in Q^{\theta_{uv},+}_{r/C_2,\varphi}(x_0,t_0)$*, where* $\theta_{uv} = \min\{\theta_u, \theta_v\}$.

Proof. Note that in case $p = 2$, Theorem 3.1 coincides with the linear result (1.2). Indeed, in when $p = 2$ we see that if both u and v vanish on a sufficiently large portion of the lateral boundary, ensure the validity of the forward-backward in time Harnack inequalities

$$\sup_{Q^{1,+}_{5r,\varphi}(x_0,t_0)} \omega u \le c\theta_u, \quad \sup_{Q^{1,+}_{5r,\varphi}(x_0,t_0)} \omega v \le c\theta_v,$$

for some $C,\ 1 < C < \infty$, independent of r and $\left(\widetilde{x}_0, \widetilde{t}_0\right)$, then

$$\Lambda_u = \sup_{Q^{1,+}_{5r,\varphi}(x_0,t_0)} \omega u \le C\theta_u,\ \ \Lambda_v = \sup_{Q^{1,+}_{5r,\varphi}(x_0,t_0)} \omega v \le C\theta_v.$$

Hence, the statement of Theorem 2.1 reduces to

$$\left| \frac{\omega(x)u(x,t)}{v(x,t)} - \frac{\omega(y)u(y,s)}{v(y,s)} \right| \le$$

$$\le C_1 \frac{u\left(\widetilde{x}_0, \widetilde{t}_0\right)}{v(x_0,t_0)} \left(\frac{d_p\left((x,t),(y,s)\right)}{r} \right)^{\sigma},$$

whenever $(x,t),(y,s) \in Q^{1,+}_{5r,\varphi}(x_0,t_0)$.

Later we doing intrinsic sculling parameters. A crucial ingredient in the regularity theory for the problem (2.9) is the use DiBenedetto's intrinsic geometry when deriving local estimates. We see that $\Lambda_u,\ \Lambda_v,\ \lambda_u$ and λ_v serve as intrinsic scaling parameters. Concerning the conditions in (2.7), focusing on u, assuming $(x_0,t_0) = (0,0), r = 1,\ \left(\widetilde{x}_0, \widetilde{t}_0\right) = (\ln, 0)$ that if we define

$$\widetilde{u}(x,t) = u\left(x, t\Lambda_u^{2-p}\right)/\Lambda_u \ \text{ for } \ (x,t) \in Q^{1,+}_{5r,\varphi}(0,0)$$

then, by construction

$$\sup_{Q^{1,+}_{5r,\varphi}(0,0)} \omega\widetilde{u} = 1.$$

We can describe scaling properties of weak solutions of the problem (2.9) and formulate a boundary gradient estimate. Also to prove of Theorem we use a barrier type argument at the boundary and here the assumption regular domain is important.

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