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The influence of the material properties of the inhomogeneously pre-stressed hollow cylinder contained compressible inviscid fluid on the dispersion of the axisymmetric waves propagating

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Abstract. *The influence of the inhomogeneously pre-stressed hollow cylinder material properties on the dispersion of the axially symmetric waves propagating in that in the case where this cylinder contains compressible inviscid fluid is studied. It is considered the dispersion curves related to the first and second modes which also exist in the corresponding empty cylinder. Investigations are made by utilizing the so-called three-dimensional linearized theory of elastic waves in bodies with initial stresses under describing the motion of the cylinder and by utilizing the linearized Navier-Stokes equations under describing the flow of the fluid. The corresponding eigenvalue problem is solved by employing the discrete-analytical method which for the related problems has been developed in the earlier papers by the author. Concrete numerical results (dispersion curves) are presented and discussed for the cases when the material of the cylinder is Steel, Aluminum, Lucite, and Soft rubber. Water is taken as a fluid contained in the cylinder. As a result of the comparison of these results, it is made corresponding conclusions on the influence of the cylinder material properties on the axisymmetric wave propagation velocity. In particular, it is established that for the metal type materials the influence of the initial stresses on the wave propagation velocity is more considerable in the low wavelength cases, however, for the non-metal type materials in the high wavelength cases.*

Keywords. inhomogeneously pre-stressed cylinder · axisymmetric waves dispersion · material properties · compressible fluid

Mathematics Subject Classification (2010): 35Q30, 76D05, 74B15, 74J05

1 Introduction

Fundamental theoretical investigations on dynamics of the "hollow cylinder +fluid" systems are necessary for controlling and safety of the liquid transportations in hollow cylinders

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(which are taken as models of the pipes) in chemical, mechanical, and other branches of modern engineering. A brief review of the related investigations was made in the paper [1] in which it is also studied the influence of the inhomogeneous pre-stresses in the hollow cylinder on the dispersion of the axisymmetric waves propagating in the "hollow cylinder +fluid" system. At the same time, in the paper [1] the three-dimensional linearized theory of elastic waves in bodies with initial stresses [2, 3] is employed for describing the motion of the cylinder and the fluid flow is written by utilizing the linearized Navier-Stokes equations [4] for the compressible inviscid fluids. However, in the paper [1], concrete numerical results are obtained and analyzed for the case where the material of the cylinder is steel and the fluid is water.

In the paper [5] the investigations carried out in the paper [1] are developed for the study of how the fluid properties influence the velocity of the axisymmetric waves propagating in the inhomogeneous pre-stressed hollow cylinder containing this fluid. For this purpose, Glycerin, Water, and Kerosene are selected as fluids contained in the hollow cylinder and for each of these fluids, the dispersion curves are obtained and compared with each other.

In the present paper, we attempt to develop the investigations started in the papers [1, 5] and to study how the mechanical properties of the cylinder material influence the velocity of the axisymmetric waves propagating in the inhomogeneous pre-stressed hollow cylinder containing the inviscid compressible fluid. For this purpose Steel, Aluminum, Lucite, and Soft-rubber are taken as cylinder material, and Water is taken as a fluid contained in the cylinder. Numerical results are presented for the first, and second modes for each selected material case. As in the papers [1, 5], these results are obtained for various values of the fluid pressure which causes the inhomogeneous initial stresses in the cylinder.

2 Formulation of the problem

Consider the inhomogeneously pre-stressed hollow cylinder with infinite length containing compressible inviscid barotropic fluid and associate the cylindrical $Or\theta z$and Cartesian $Ox_1x_2x_3$ ($x_3 = z$) (Fig. 1) systems of coordinates with the central axis of the cylinder. Introduce the Lagrange and Euler coordinates for describing the motion of the cylinder and fluid respectively, and distinguish the initial and perturbed states in the hydro-elastic system under consideration Due to the smallness of the deformations and perturbations, the difference between the Lagrange and Euler coordinates will be ignored.

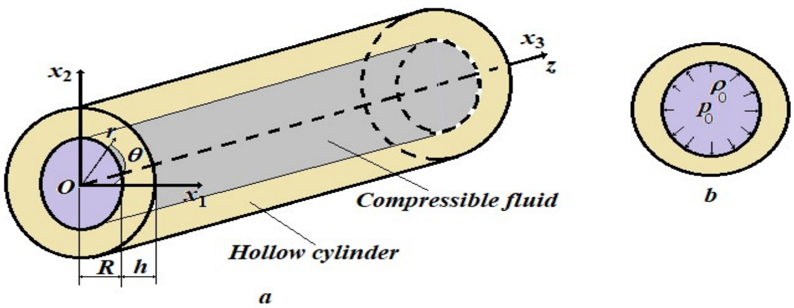

Fig. 1 The sketch of the hydro-elastic system under consideration

Assume that in the initial state the fluid pressure $p_0$acts on the interior of the cylinder and causes a static stress-strain state in this cylinder and it is called the "initial stress-strain state". Also, we assume that in the initial state the fluid in the interior of the cylinder is rest.

According to the well-known Lame problem, discussed almost in all textbooks related to the theory of elasticity (see, for instance, the book [6]), the mentioned initial stresses can be presented as follows.

$$\sigma_{rr}^{0}=\frac{p_0}{(1+h/R)^2-1}\left(1-\frac{R^2}{r^2}\left(1+\frac{h}{R}\right)^2\right),$$

$$\sigma_{\theta\theta}^{0}=\frac{p_0}{(1+h/R)^2-1}\left(1+\frac{R^2}{r^2}\left(1+\frac{h}{R}\right)^2\right),$$

$$\sigma_{zz}^{0}=\nu(\sigma_{rr}^{0}+\sigma_{\theta\theta}^{0}). \tag{2.1}$$

In (2.1) the conventional notation is used and the upper index "0" indicates the belonging of the related quantities to the initial stress state.

We suppose that after appearing of the initial stress state determined by expressions in (2.1) in the cylinder, the hydro-elastic system gets a certain dynamical perturbation as a result of which the longitudinal axisymmetric waves propagate in that. The aim of the present paper is to investigate how the material properties of the cylinder influence the dispersion of the mentioned waves in the presence of the foregoing initial inhomogeneous stresses in the cylinder. For this purpose, we attempt to use the 3D linearized theory of elastic waves in initially stressed bodies for describing the motion of the cylinder and the linearized Navier-Stokes equations for describing the flow of the inviscid compressible barotropic fluid.

Thus, according to the monographs [2, 3, 4, and 7], we write 3D linearized equations and corresponding relations describing the motion of the cylinder:

The equations of motion:

$$\frac{\partial t_{rr}}{\partial r}+\frac{\partial t_{zr}}{\partial z}+\frac{1}{r}(t_{rr}-t_{\theta\theta})=\rho\frac{\partial^2 u_r}{\partial t^2},\ \frac{\partial t_{rz}}{\partial r}+\frac{1}{r}t_{rz}+\frac{\partial t_{zz}}{\partial z}=\rho\frac{\partial^2 u_z}{\partial t^2}, \tag{2.2}$$

where

$$t_{rr}=\sigma_{rr}+\sigma_{rr}^{0}(r)\frac{\partial u_r}{\partial r},\ t_{rz}=\sigma_{rz}+\sigma_{rr}^{0}(r)\frac{\partial u_z}{\partial r},\ t_{\theta\theta}=\sigma_{\theta\theta}+\sigma_{\theta\theta}^{0}(r)\frac{u_r}{r},$$

$$t_{zr}=\sigma_{zr}+\sigma_{zz}^{0}(r)\frac{\partial u_r}{\partial z},\ t_{zz}=\sigma_{zz}+\sigma_{zz}^{0}(r)\frac{\partial u_z}{\partial z}. \tag{2.3}$$

The elasticity relations:

$$\sigma_{(jj)}=\lambda(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{zz})+2\mu\varepsilon_{(jj)},\ (jj)=rr;\ \theta\theta;\ zz,\ \sigma_{rz}=2\mu\varepsilon_{rz}. \tag{2.4}$$

The strain-displacement relations:

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\ \varepsilon_{\theta\theta}=\frac{u_r}{r},\ \varepsilon_{zz}=\frac{\partial u_z}{\partial z},\ \varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right). \tag{2.5}$$

In (2.2) and (2.3) the notation t_{rr}, t_{rz}, $t_{\theta\theta}$, t_{zr} and t_{zz} shows the components of the non-symmetric Kirchhoff stress tensor and the other notation used in (2.2) – (2.5) is conventional.

We write also the linearized Navier-Stokes equations for describing the flow of the fluid contained in the cylinder. According to [4], these equations are:

The linearized continuity equation:

$$\frac{\partial \rho'}{\partial t}+\rho_0\left(\frac{\partial V_r}{\partial r}+\frac{V_r}{r}+\frac{\partial V_z}{\partial z}\right)=0; \tag{2.6}$$

Linearized equations of the fluid flow:

$$\frac{\partial V_r}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial r}, \frac{\partial V_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z}. \tag{2.7}$$

The linearized state equation

$$p' = a_0^2\rho', a_0^2 = \left(\frac{\partial p}{\partial \rho}\right)_0, \tag{2.8}$$

where a_0 is the sound speed in the fluid and ρ_0 is the density of the fluid in the initial state.

We supply the foregoing system of equations with the following boundary and compatibility conditions.

The boundary conditions on the external surface of the cylinder:

$$t_{rr}|_{r=R+h} = 0, \, t_{rz}|_{r=R+h} = 0. \tag{2.9}$$

The compatibility conditions on the interface surface between the fluid and cylinder, i.e. on the internal surface of the cylinder:

$$t_{rr}|_{r=R} = -p', \, t_{rz}|_{r=R} = 0, \, \left.\frac{\partial u_r}{\partial t}\right|_{r=R} = V_r|_{r=R}\,. \tag{2.10}$$

The condition on boundedness of the quantities related to the fluid at the central axis of the cylinder:

$$\left\{|p'|\,, |\rho'|\,, |V_r|\,, |V_z|\right\}\big|_{r=0} < \infty. \tag{2.11}$$

This completes the mathematical formulation of the problem under consideration.

3 On the method of solution

To solve the system of equations (2.2) – (2.5) we employ the discrete-analytical method developed and employed in the references [1, 8 – 10], according to this which, the interval $[R, R+h]$ is divided into the Nnumber sub-intervals. These sub-intervals are determined through the expression $(R+(n-1)h/N) \leq r \leq (R+nh/N)$ where $1 \leq n \leq N$ and it is assumed that the inhomogeneous initial stresses determined by expression (2.1) are homogeneous ones in each sub-intervals and the values of these stresses are determined as follows:

$$\sigma_{rr}^0(r) \approx \sigma_{rr}^0(r_n), \sigma_{\theta\theta}^0(r) \approx \sigma_{\theta\theta}^0(r_n), \sigma_{zz}^0(r) \approx \sigma_{zz}^0(r_n),$$
$$r_n = R + (n-1)h/N + h/(2N). \tag{3.1}$$

At the same time, we satisfy the following full contact conditions on the interfaces between the mentioned sub-intervals.

$$t_{rr}^1\big|_{r=R} = -p', \, t_{rz}^1\big|_{r=R} = 0, \, \left.\frac{\partial u_r^1}{\partial t}\right|_{r=R} = V_r|_{r=R}\,, \, t_{rr}^1\big|_{r=R+h/N} = t_{rr}^2\big|_{r=R+h/N}\,,$$

$$t_{rz}^1\big|_{r=R+h/N} = t_{rz}^2\big|_{r=R+h/N}\,, \, u_r^1\big|_{r=R+h/N} = u_r^2\big|_{r=R+h/N}\,,$$

$$u_z^1\big|_{r=R+h/N} = u_z^2\big|_{r=R+h/N}\,, \ldots,$$

$$t_{rr}^{n-1}\big|_{r=R+(n-1)h/N} = t_{rr}^n|_{r=R+(n-1)h/N}\,, \, t_{rz}^{n-1}\big|_{r=R+(n-1)h/N} = t_{rz}^n|_{r=R+(n-1)h/N}\,,$$

$$u_r^{n-1}\big|_{r=R+(n-1)h/N} = u_r^n|'_{r=R+(n-1)h/N} \, u_z^{n-1}\big|_{r=R+(n-1)h/N} = u_z^n|_{r=R+(n-1)h/N}\,, \ldots,$$

$$t_{rr}^{N}\big|_{r=R+h}=0,\; t_{rz}^{N}\big|_{r=R+h}=0. \qquad (3.2)$$

Note that in (3.2) there are $4N+1$ conditions where the number N is determined from the convergence requirement of the numerical results which will be discussed below. Note that the upper indices in (3.2) and below indicate the number of the corresponding sub-interval.

Thus, taking into consideration the assumptions in (3.1), we obtain the following equations of motion from equations (2.2) and (2.3) which are satisfied in each n^{th} sub-interval separately.

$$\frac{\partial\sigma_{rr}^{n}}{\partial r}+\sigma_{rr}^{0}(r_n)\frac{\partial^2 u_r^n}{\partial r^2}+\frac{\partial\sigma_{zr}^{n}}{\partial z}+\sigma_{zz}^{0}(r_n)\frac{\partial^2 u_r^n}{\partial z^2}+\frac{1}{r}(\sigma_{rr}^{n}-\sigma_{\theta\theta}^{n})+$$

$$\sigma_{rr}^{0}(r_n)\frac{1}{r}\frac{\partial u_r^n}{\partial r}-\sigma_{\theta\theta}^{0}(r_n)\frac{u_r^n}{r^2}=\rho\frac{\partial^2 u_r^n}{\partial t^2},$$

$$\frac{\partial\sigma_{rz}^{n}}{\partial r}+\sigma_{rr}^{0}(r_n)\frac{\partial^2 u_z^n}{\partial r^2}+\frac{1}{r}\sigma_{rz}^{n}+\sigma_{rr}^{0}(r_n)\frac{1}{r}\frac{\partial u_z^n}{\partial r}+$$

$$\frac{\partial\sigma_{zz}^{n}}{\partial z}+\sigma_{zz}^{0}(r_n)\frac{\partial^2 u_z^n}{\partial z^2}=\rho\frac{\partial^2 u_z^n}{\partial t^2}. \qquad (3.3)$$

We add to these equations the elasticity relation (2.4) and the relation between deformations and displacements (2.5) rewritten for each subinterval separately and for the solution to the system of equations (3.3), (2.4), and (2.5) we employ the classical Lame decomposition (see, for instance, the monograph [7]), which for the axisymmetric problems can be presented as follows.

$$u_r^n=\frac{\partial\Phi^n}{\partial r}+\frac{\partial^2\Psi^n}{\partial r\partial z},\; u_z^n=\frac{\partial\Phi^n}{\partial z}-\frac{\partial^2\Psi^n}{\partial r^2}-\frac{\partial\Psi^n}{r\partial r}. \qquad (3.4)$$

Substituting the expressions in (3.4) into the system equations (2.5), (2.4), and (3.3), and doing corresponding cumbersome mathematical calculations we obtain the following equations for the potential functions Φ^n and Ψ^n in (3.4).

$$(1+\frac{\sigma_{rr}^{0}(r_n)}{\lambda+2\mu})\frac{\partial^2\Phi^n}{\partial r^2}+(1+\frac{\sigma_{\theta\theta}^{0}(r_n)}{\lambda+2\mu})\frac{\partial\Phi^n}{r\partial r}+(1+\frac{\sigma_{zz}^{0}(r_n)}{\lambda+2\mu})\frac{\partial^2\Phi^n}{\partial z^2}=\frac{1}{(c_1)^2}\frac{\partial^2\Phi^n}{\partial t^2},$$

$$(1+\frac{\sigma_{rr}^{0}(r_n)}{\mu})\frac{\partial^2\Psi^n}{\partial r^2}+(1+\frac{\sigma_{\theta\theta}^{0}(r_n)}{\mu})\frac{\partial\Psi^n}{r\partial r}+(1+\frac{\sigma_{zz}^{0}(r_n)}{\mu})\frac{\partial^2\Psi^n}{\partial z^2}=\frac{1}{(c_2)^2}\frac{\partial^2\Psi^n}{\partial t^2}. \qquad (3.5)$$

In (3.5) the notation $c_1=\sqrt{(\lambda+2\mu)/\rho}$ and $c_2=\sqrt{\mu/\rho}$ is used. Moreover, in the cases where $\sigma_{zz}^{0}(r_n)=0$, $\sigma_{rr}^{0}(r_n)=0$ and $\sigma_{\theta\theta}^{0}(r_n)=0$, the equations in (3.5) coincide with the corresponding equations of classical elastodynamics [7].

Representing the functions Φ^n,u_r^n,σ_{rr}^n, $\sigma_{\theta\theta}^n$ and σ_{zz}^n with the multiplying $\sin(kz-\omega t)$and the functions Ψ^n, u_z^n and σ_{rz}^n with the multiplying $\cos(kz-\omega t)$, and denoting the amplitudes of the corresponding quantities with the same symbols, we obtain the following equations for the amplitudes of the potentials Φ^n and Ψ^n.

$$\frac{d^2\Phi^n}{d(r_2)^2}+\frac{\alpha_1(r_n)}{r_2}\frac{d\Phi^n}{dr_2}+\Phi^n=0,\frac{d^2\Psi^n}{d(r_1)^2}+\frac{\alpha(r_n)}{r_1}\frac{d\Psi^n}{dr_1}+\Psi^n=0, \qquad (3.6)$$

where

$$\alpha(r_n)=\frac{1+\sigma_{\theta\theta}^{0}(r_n)/\mu}{1+\sigma_{rr}^{0}(r_n)/\mu},\beta(r_n)=\frac{1+\sigma_{zz}^{0}(r_n)/\mu}{1+\sigma_{rr}^{0}(r_n)/\mu},$$

$$r_1^n = kr\sqrt{\frac{c^2}{(c_2)^2(1+\sigma_{rr}^0(r_n)/\mu)} - (\beta(r_n))^2}, c = \omega/\kappa, \alpha_1(r_n) = \frac{1+\sigma_{\theta\theta}^0(r_n)/(\lambda+2\mu)}{1+\sigma_{rr}^0(r_n)/(\lambda+2\mu)},$$

$$\beta_1(r_n) = \frac{1+\sigma_{zz}^0(r_n)/(\lambda+2\mu)}{1+\sigma_{rr}^0(r_n)/(\lambda+2\mu)}, r_2^n = kr\sqrt{\frac{c^2}{(c_1)^2(1+\sigma_{rr}^0(r_n)/(\lambda+2\mu))} - (\beta_1(r_n))^2}. \quad (3.7)$$

According to [8, 11], the solution to the equations in (3.7) are found as follows.

$$\Phi^n = A_1^n(r_2)^{\gamma_1(r_n)}E_{\gamma_1(r_n)}(r_2^{n_i}) + A_2^n(r_2)^{\gamma_1(r_n)}F_{\gamma_1(r_n)}(r_2^n), \quad (3.8)$$

$$\Psi^n = B_1^n(r_1)^{\gamma(r_n)}E_{\gamma(r_n)}(r_1^n) + B_2^n(r_1)^{\gamma(r_n)}F_{\gamma(r_n)}(r_1^n), \quad (3.9)$$

where A_1^n, A_2^n, B_1^n and B_2^n are unknown constants and

$$\gamma_1(r_n) = (1-\alpha_1(r_n))/2, \gamma(r_n) = (1-\alpha(r_n))/2$$

$$E_{\gamma_1(r_n)}(r_2^n) = \begin{cases} J_{\gamma_1(r_n)}(r_2^n) \text{ if } (r_2^n)^2/r^2 > 0 \\ I_{\gamma_1(r_n)}(r_2^n) \text{ if } (r_2^n)^2/r^2 < 0 \end{cases},$$

$$F_{\gamma_1(r_n)}(r_2^n) = \begin{cases} Y_{\gamma_1(r_n)}(r_2^n) \text{ if } (r_2^n)^2/r^2 > 0 \\ K_{\gamma_1(r_n)}(r_2^n) \text{ if } (r_2^n)^2/r^2 < 0 \end{cases},$$

$$E_{\gamma(r_n)}(r_1^n) = \begin{cases} J_{\gamma(r_n)}(r_1^n) \text{ if } (r_1^n)^2/r^2 > 0 \\ I_{\gamma(r_n)}(r_1^n) \text{ if } (r_1^n)^2/r^2 < 0 \end{cases},$$

$$F_{\gamma(r_n)}(r_1^{n_i}) = \begin{cases} Y_{\gamma(r_{n_i})}(r_1^n) \text{ if } (r_1^n)^2/r^2 > 0 \\ K_{\gamma(r_n)}(r_1^n) \text{ if } (r_1^{n_i})^2/r^2 < 0 \end{cases}. \quad (3.10)$$

In (3.10), $J_\delta(x)$ and $I_\delta(x)$ are the Bessel and modified Bessel functions of the first kind, however, $Y_\delta(x)$ and $K_\delta(x)$ are also the Bessel and Modified Bessel functions of the second kind.

By usual procedure, we determine the expressions for displacements from the presentations (3.4) and then using the relations in (2.5) and 4) we obtain the expressions for the stresses within each sub-intervals. In this way we determine the analytic expressions related to motion of the cylinder within in each sub-intervals. To reduce the volume of the paper the explicit forms of these expressions for displacements and stresses which enter the conditions in (3.2) are not presented here.

Now we consider the determination of the quantities related to the flow of the fluid and for this purpose, according to [4], we use the following presentations for the general solution to equations in (2.6) - (2.8).

$$\rho' = a_0^{-2}\rho_0\left(-V_z^0\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\Phi_f, p' = \rho_0\left(-V_z^0\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\Phi_f,$$

$$V_r = \frac{\partial}{\partial r}\Phi_f, V_z = \frac{\partial}{\partial z}\Phi_f, \quad (3.11)$$

where

$$\left[\Delta - \frac{1}{a_0^2}\left(\frac{\partial}{\partial t} + V_z^0\frac{\partial}{\partial z}\right)^2\right]\Phi_f = 0, \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \quad (3.12)$$

Representing the functions V_z, p' and ρ' by multiplying $\sin(kz-\omega t)$, and the functions Φ_f and V_r by multiplying $\cos(kz-\omega t)$, we obtain the following equation from (3.12) for Φ_{f1} (where $\Phi = \Phi_{f1}(r)\cos(kz-\omega t)$).

$$\left(\frac{d^2}{dr_3^2}+\frac{1}{r_3}\frac{d}{dr_3}+1\right)\Phi_{f1}(r)=0, r_3=kr\sqrt{\left(\frac{c}{a_0}\right)^2+2\frac{c}{a_0}\frac{V_z^0}{a_0}+\left(\frac{V_z^0}{a_0}\right)^2-1}. \quad (3.13)$$

According to the conditions in (2.11), the solution to equation (3.13) is found as follows.

$$\Phi_{f1}(r)=\begin{cases} FJ_0(r_3)\ if\ r_3^2>0 \\ FI_0(r_3)\ if\ r_3^2<0 \end{cases} \quad (3.14)$$

where $J_0(r_3)$ ($I_0(r_3)$) is the first kind Bessel (modified Bessel) function of the zeroth order and $F$is a unknown constant.

Using the expression (3.14) and substituting $\Phi = \Phi_{f1}(r)\cos(kz-\omega t)$into the equations in (3.12) we obtain the following expressions for the sought values related to the fluid.

$$p'=\rho_0\left(V_z^0k+\omega\right)\sin(kz-\omega t)\begin{cases} FJ_0(r_3)\ if\ r_3^2>0 \\ FI_0(r_3)\ if\ r_3^2<0 \end{cases},$$

$$\rho'=a_0^{-2}\rho_0\left(V_z^0k+\omega\right)\sin(kz-\omega t)\begin{cases} FJ_0(r_3)\ if\ r_3^2>0 \\ FI_0(r_3)\ if\ r_3^2<0 \end{cases},$$

$$V_r=k\frac{dr_3}{dr}\cos(kz-\omega t)\begin{cases} -FJ_1(r_3)\ if\ r_3^2>0 \\ FI_1(r_3)\ if\ r_3^2<0 \end{cases},$$

$$V_z=-k\sin(kz-\omega t)\begin{cases} FJ_0(r_3)\ if\ r_3^2>0 \\ FI_0(r_3)\ if\ r_3^2<0 \end{cases}. \quad (3.15)$$

In this way, we obtain the analytical expressions of the sought values which contain $4N+1$ number unknown constants and these constants are $A_1^n, A_2^n, B_1^n, B_2^n$ $(n=1,2,...,N))$ and F. Using the $4N+1$ number of the conditions in (3.2) we obtain the system of homogeneous algebraic equations with respect to the mentioned unknown constants. According to the well-known procedure, equating to zero the determinant of the coefficient matrix of this system we derive the dispersion equation. This equation can be formally presented as follows.

$$\det[a_{nm}(c/c_2, kR, p_0/\mu, \rho/\rho_0, h/R, a_0/c_2)]=0, n\,; m=1,2,...,4N+1. \quad (3.16)$$

The dispersion equation (3.16) is solved numerically by employing the “bi-section” method.

This completes the consideration of the solution procedure related to the problem under consideration.

4 Numerical results and discussions

As noted above, the aim of the present investigation is to determine how the material properties of cylinder influences the dispersion curves of the longitudinal axisymmetric waves propagating in the hydro-elastic system under consideration. For this purpose, as the cylinder materials, we select the Steel, Aluminum, Lucite, and Soft-rubber the mechanical properties for which, according to [2, 3, 4, 12], are given in Table 1.

Table 1. The values of the mechanical constants of the cylinder material

Mechanical properties	The selected materials			
	Steel	Aluminum	Lucite	Soft-rubber
Lame constants μ (upper number) and λ (lower number)	$\frac{79\times10^9 Pa}{94.4\times10^9 Pa}$	$\frac{28\times10^9 Pa}{43.1\times10^9 Pa}$	$\frac{1.86\times10^9 Pa}{3.96\times10^9 Pa}$	$\frac{1.2\times10^6 Pa}{57.6\times10^6 Pa}$
Material density ρ	$7790 kg/m^3$	$2770 kg/m^3$	$1160 kg/m^3$	$1200 kg/m^3$
Shear wave propagation velocity $c_2=\sqrt{\mu/\rho}$	$3184 m/s$	$3179 m/s$	$1266 m/s$	$31.6 m/s$

In the present paper, as the fluid is taken water with the density $\rho_0 = 1000kg/m^3$ and with the sound speed $a_0 = 1459.5m/s$ [4, 12].

First, we consider the validation of the PC programs and algorithms which are used under obtaining present results. For this purpose, we compare those obtained in the case where the initial stresses are absent in the cylinder with corresponding ones obtained in the paper [13]. Note that in the paper [13] it is also studied the dispersion of the axisymmetric waves propagating in the hollow cylinder contained inviscid compressible fluid and under obtaining numerical results steel is taken as a material of the cylinder and water is taken as a fluid. Moreover, in the paper [13] it is assumed that$h/R = 0.2$and the dispersion diagrams, i.e. the graphs of the dependencies between $\omega h/c_2$ and kh are constructed for some modes and also for the first and second modes. It is evident that in the "steel + water" case under $h/R = 0.2$ and $p_0/\mu= 0.0$ the results obtained by the present PC programs and algorithm must coincide with corresponding ones obtained in the paper [13].

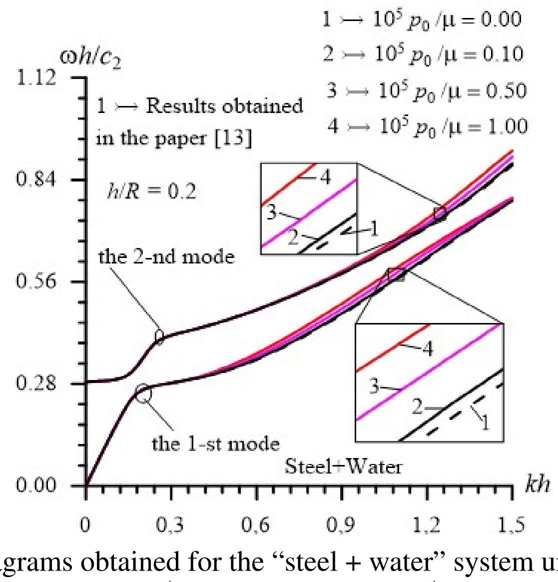

Fig. 2 Dispersion diagrams obtained for the "steel + water" system under various values of the ratiop_0/μ in the case where $h/R = 0.2$

Fig. 2 shows the mentioned dispersion diagrams of the first and second modes constructed for various values of the ratio p_0/μ through which the magnitude of the initial stresses in the cylinder is characterized. Note that in Fig. 2 the dispersion diagram obtained in the case where $p_0/\mu= 0.0$is drawn by the dashed line and this line coincides with the corresponding line constructed in the paper [13]. At the same time, it follows from the graphs illustrated in Fig. 2 and constructed under $N = 50$ (where N is the number of the sub-intervals into which is divided the interval $[R, R + h]$) for various values of the ratio p_0/μ shows that an increase in the magnitude of the internal pressure of the fluid causes to increase the wave propagation velocity in the first and second modes. The mentioned increase of the wave propagation velocity agrees with the well-known physicomechanical considerations. According, namely, to these statements, we can conclude that the solution method and PC programs used under obtaining the presented results are true.

Now we consider the dispersion curves of the first and second modes related to each selected cylinder material the values of the mechanical constants of which are given in Table 1. These curves are presented in Figs. 3, 4, 5, and 6 for the Steel + Water, Aluminum + Water, Lucite + Water, and Soft-rubber + Water systems respectively. All these results are obtained for various values of the ratio p_0/μ in the case where $h/R = 0.2$ and $N = 50$. Note that in these figures the dispersion curves drawn by the dashed lines relate to the case where $p_0/\mu = 0$.

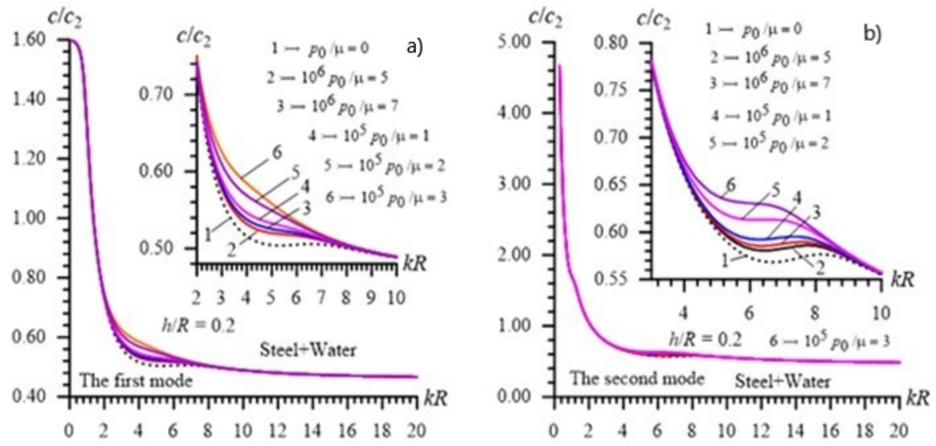

Fig. 3 Dispersion curves of the first (a) and second (b) modes related to the Steel + Water system

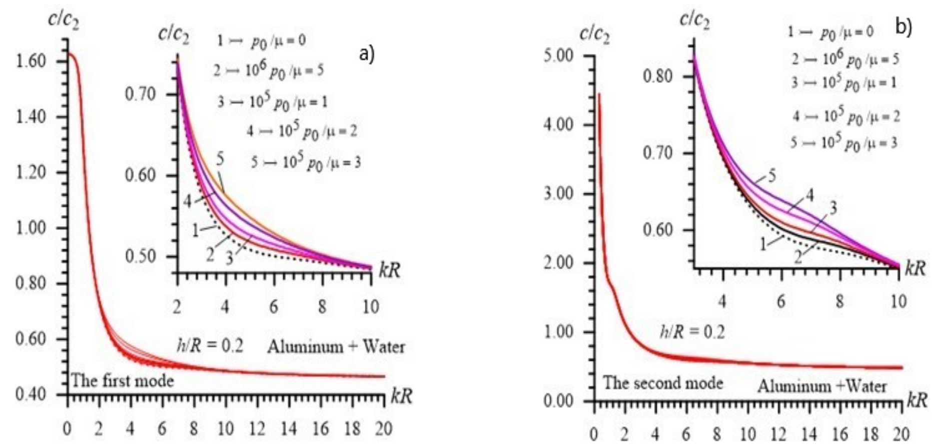

Fig. 4 Dispersion curves of the first (a) and second (b) modes related to the Aluminum + Water system

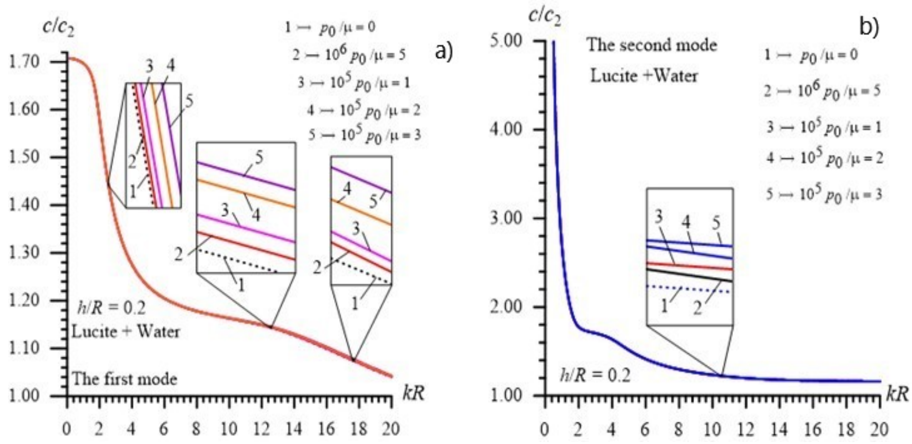

Fig. 5 Dispersion curves of the first (a) and second (b) modes related to the Lucite +Water system

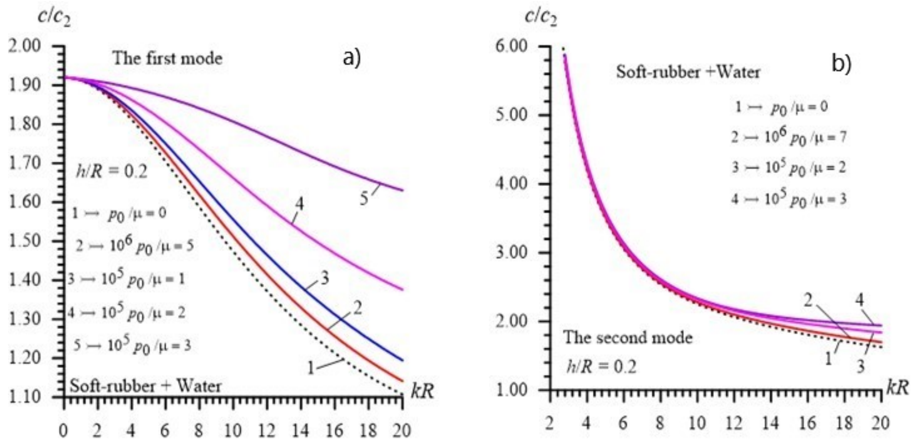

Fig. 6 Dispersion curves of the first (a) and second (b) modes related to the Soft-rubber + Water system

It follows from the analyses of the results given in Figs 3 and 4 that for the stiffer metal materials such as stel and aluminum the significant influence of the initial inhomogeneous stresses on the wave propagation velocity appear before a certain value of the dimensionless wavenumber kR (denote it by $(kR)'$). After this value of the kR, i.e in the cases $kR > (kR)'$ the magnitude of the mentioned influence decrease with kR. However, for the non-metal materials, the results for which are presented in Figs. 5 and 6, the magnitude of the influence of the inhomogeneous initial stresses on the wave propagation velocity increase with kR. This increasing is more considerable for the soft-rubber material.

Analyses of the numerical results shows that in the cases where the material of the cylinder is non-metal the influence of the initial stresses on the wave propagation velocity in the first mode is more considerable than that in the second mode. This rule is also valid for the cases where the material of the cylinder is metal, although in these cases the magnitude of this difference is not so much as in the non-metal case.

If we compare the shear wave propagation velocity c_2 of the selected materials with the sound speed a_0 of the fluid then it can be established that in the cases where the difference beteween c_2 and a_0 is greter (such as between water and selected metals and between water and selected soft-rubber type non-metals) the influence of the initial stresses on the wave propagation velocity becomes significantly, however in the cases where this difference is not so much (as for the Lucity and water) the magnitude of the mentioned influence becomes insignificantly.

Note that all the foregoing results are obtained in the case where the number N (i.e. the number of the sub-intervals) is selected as $N = 50$. Now we consider numerical results illustrated the convergence of the emloyed method with respect to this number. These results for the Aluminum +Water pair are given in Fig. 7 for the dispersion curves obtained for the first mode. Comparison of the results obtained for various values of the number N shows that an increase in the values of this number influences on the dispersion curves quantitatively only. This comparison also shows that the value$N = 50$ is enough for obtaining the results with accuracy 10^{-4}.

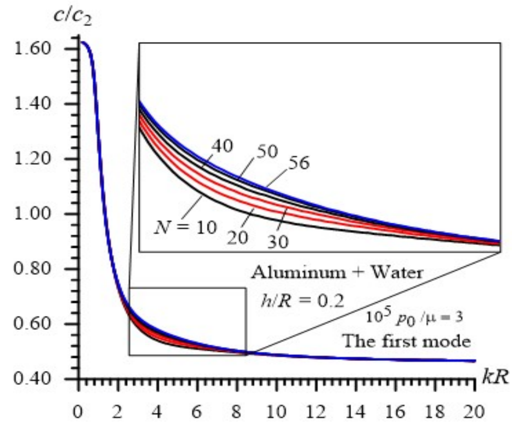

Fig. 7 Convergence of the numerical results with respect to the number for the Aluminum + Water system system

This completes the analyses of the obtained numerical results.

5 Conclusions

Thus, in the present paper, the influence of the material properties of the inhomogeneously pre-stressed hollow cylinder containing a compressible inviscid fluid on the dispersion of the first and second modes of axisymmetric waves propagating in this cylinder is studied. The corresponding eigenvalue problem is formulated within the scope of the three-dimensional linearized theory of elastic waves in bodies with initial stresses and linearized Navier-Stokes equations for the inviscid compressible fluids. The discrete-analytical solution method is employed for the solution to the corresponding eigenvalue problem. Numerical results are presented for the cases where Steel, Aluminum, Lucite, and Soft-rubber are selected as a cylinder material and Water is selected as a fluid contained in the cylinder. Numerical results are presented and discussed. According to these discussions it is established the following concrete results:

1 For all selected cylinder materials initial inhomogeneous stresses in the cylinder caused by the internal pressure of the fluid acting on the inner surface of the cylinder causes to increase the wave propagation velocity and magnitude of this increase grows with the mentioned pressure;
2 In the cases where the difference between c_2 (shear wave propagation velocity in the cylinder material) and a_0(sound speed in the fluid) is greater (such as between water and selected metals and between water and selected soft-rubber type non-metals) the influence of the initial stresses on the wave propagation velocity becomes significantly, however in the cases where this difference is not so much (as for the Lucite and water) the magnitude of the mentioned influence becomes insignificantly;

3 For the selected cylinder materials the influence of the initial stresses on the wave propagation velocity in the first mode is more considerable than that in the second mode;
4 For the metal type materials the influence of the initial stresses on the wave propagation velocity is more considerable in the low wavelength cases, however, for the non-metal type materials in the high wavelength cases.

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