Tran. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mechanics, **41** (7), 53–56 (2021).

A nonlinear elliptic-parabolic problem with nonlinear boundary conditions

Gulnara M. Zulfaliyeva

Received: 28.05.2021 / Revised:17.06.2021 / Accepted: 12.08.2021

Abstract. *We consider a general nonlinear degenerate elliptic-parabolic problem with nonlinear boundary conditions. We prove existence and uniqueness of weak solutions, comparison principle. These include classical model in filtration theory.*

Keywords. nonlinear · elliptic-parabolic · nonlinear boundary condition.

Mathematics Subject Classification (2010): 35J25

1 Introduction

The nonlinear dynamical boundary conditions, although not too widely considered in the mathematical literature, are very natural in many mathematical models including heat transfer in a solid in contact with a moving fluid, thermoelasticity, diffusion phenomena, problems in fluid dynamics, etc. (see [1], [2]). These nonlinear boundary conditions also appear in the study of the Stefan problem when the boundary material has large thermal conductivity and sufficiently small thickness. Hence, the boundary material is regraded as the boundary of the domain. For instance, this is the case if one considers an iron ball in which water and ice coexist, in the study of the Hale-Shaw problem. Notice that general nonlinear diffusion operators of Leray-Lions type, different from the Laplacian, appear when one deals with non-Newtonian fluids (see, [3], [4]) and the references therein for the case of the Hale-Shaw problem with non-Newtoninan fluids). Another application in mind concerns the filtration equation with dynamical nonlinear boundary conditions, which appears in the study of rainfall infiltration through soil, when accumulation of water on the ground surfaces caused by saturation of the surface layer is taken into account.

In contrast to the Dirichlet boundary condition, for the nonhomogeneous Neumann and dynamical boundary conditions, the problem is non-coercive, and moreover, the conservation of mass exhibits a necessary condition for the existence of a solution related to the ranges of the nonlinearities γ. See also papers [5,6,7,8].

Gulnara M. Zulfaliyeva
Sumgayit State University
Baku, Azerbaijan

E-mail: gulnaraz@mail.ru

Let $\Omega \subset R^n$ be a smooth bounded domain and $T > 0$. Let us denote $Q_T = \Omega \times [0,T]$. The purpose of paper is to establish the existence and uniqueness of a weak solution for a nonlinear degenerate elliptic-parabolic problem with nonlinear dynamical boundary condition

$$\begin{cases} \frac{\partial \gamma(u)}{\partial t} - diva(x,Du) = f & \text{in} \quad Q_T \\ u_t + a(x,Du)\eta = g & \text{on} \quad S_T = \partial\Omega \times (0,T) \\ \gamma\left(u\left(0\right)\right) = u_0 & \text{in} \quad \Omega \end{cases} \tag{1.1}$$

where $u_0 \in L_1(\Omega),\ f \in L_1\left(0,T; L_1\left(\Omega\right)\right),\ g \in L_1\left(0,T; L_1\left(\partial\Omega\right)\right)$ and η is the unit outward normal on $\partial\Omega$. Here $a : \Omega \times R^N \to R^n$ is a Caratheodory function. For the function $\gamma : R \to R$ we assume that: 1. γ is increasing and Lipshitz; 2. $\gamma(s) = 0$, when $s = 0$; 3 $\gamma \in C(R) \cap C^1\left([0,\infty]\right)$.

The Caratheodory function $a : \Omega \times R^N \to R^n$ satisfies: a) there exists $\lambda > 0$ such that $a(x,\xi)\xi \geq \lambda\left|\xi\right|^p$ for a.e. $x \in \Omega$ and for all $\xi \in R^N,\ p > 1$; b) there exist $c > 0$ such that $|a(x,\xi)| \leq c\left(1 + |\xi|^{p-1}\right)$ for a.e. $x \in \Omega$ and for all $\xi \in R^N$; c) $\left(a\left(x,\xi\right) - a\left(x,\eta\right)\right)\left(\xi - \varphi\right) > 0$ for a.e. $x \in \Omega$ and for all $\xi, \eta \in R^N, \xi \neq \eta$.

The hypothesis a)-c) are classical in the study of nonlinear operators in divergence form. The model example of a function a satisfying these conditions is $a(x,\xi) = |\xi|^{p-2}\xi$. The corresponding operator is the p-Laplacian operator $\triangle_p(u) = div\left(|Du|^{p-2} Du\right)$. We denote by $|E|$ the Lebesgue measure of a set $E \subset R^N$ or its $(N-1)$ Hansdorff measure. For $1 \leq p < \infty,\ L_p\left(\Omega\right)$ and $W_p^1\left(\Omega\right)$ denote respectively the Lebesgue and Sobolev spaces, and $\overset{\circ}{W}{}_p^1\left(\Omega\right)$ is the closure $C^\infty\left(\Omega\right)$ functions which vanishing in $W_p^1(\Omega)$.

The weak solution is the following. Given $f \in L_1\left(0,T;\ L_1(\Omega)\right),\ g \in L_1\left(0,T;\ L_1\left(\partial\Omega\right)\right),\ u_0 \in L_1\left(\Omega\right)$, a weak solution of (11.) in $[0,T]$ such that $\gamma(u) \in C\left([0,T];\ L_1\left(\Omega\right)\right),\ \gamma\left(u(0)\right) = u_0$ and there exists $u \in L_p\left(0,T; W_p^1\left(\Omega\right)\right)$. Such that

$$\frac{d}{dt}\int\limits_{\Omega} \gamma(t)\xi dx + \frac{d}{dt}\int\limits_{\partial\Omega} u(t)\xi ds + \int\limits_{\partial\Omega} a(x,Du(t))D\xi dx =$$

$$= \int\limits_{\Omega} f(x)\xi dx + \int\limits_{\partial\Omega} g(s)\xi ds \tag{1.2}$$

for any $\xi \in C^1\left(\overline{\Omega}\right)$.

Remark 1.1 If we taking $\xi = 1$ in the above definition, we get

$$\int\limits_{\Omega} \gamma(t)dx + \int\limits_{\partial\Omega} u(s)ds = \int\limits_{\Omega} u_0 dx + \int\limits_0^t \left(\int\limits_{\Omega} f dx + \int\limits_{\partial\Omega} g ds\right) dt, \tag{1.3}$$

for $\forall t \in [0,T]$. Then existence and uniqueness of weak solutions for this problem is known to hold only if

$$\int\limits_{\Omega} u_0 dx + \int\limits_0^t \left(\int\limits_{\Omega} f dx + \int\limits_{\partial\Omega} g ds\right) \in (0, |\Omega|)$$

for any $t \in [0,T]$. For the monotone γ and u, we set

$$R^+_{\gamma,u} = \gamma_+\left|\Omega\right| + u_+\left|\partial\Omega\right|,\ R^-_{\gamma,u} = \gamma_-\left|\Omega\right| + u_-\left|\partial\Omega\right|$$

where $v_- = \inf v, \;\; v_+ = \sup v$.

We suppose $R^-_{\gamma,u} < R^+_{\gamma,u}$ and we write $R_{\gamma,u} = \left(R^-_{\gamma,u}, R^+_{\gamma,u}\right)$.

2 Main results.

The main results of this paper are the following contraction principle and the following existence and uniqueness theorem.

Theorem 2.1 *Let $u(x,t)$ be a weak solution in $[0,T]$ of problem (1.1) and $T > 0$. For $i = 1,2$, let $f_i \in L_1(0,T;L_1(\Omega)),\ g_i \in L_1(0,T;L_1(\partial\Omega)),\ u_{i0} \in L_1(\Omega)$. Then*

$$\int\limits_{\Omega} (\gamma_1(u) - \gamma_2(u))^+ \, dx + \int\limits_{\partial\Omega} (u_1(s) - u_2(s))^+ \, ds \leq$$

$$\int\limits_{\Omega} (\gamma_1 (u_1(0)) - \gamma_2 (u_2(0)))^+ \, dx + \int\limits_{\partial\Omega} (u_{10} - u_{20})^+ \, ds +$$

$$+ \int\limits_0^t \int\limits_{\Omega} (f_1(\tau) - f_2(\tau))^+ \, d\tau + \int\limits_0^t \int\limits_{\partial\Omega} (g_1(\tau) - g_2(\tau))^+ \, d\tau \tag{2.1}$$

for almost every $t \in (0T)$.

The following relation for $u, v \in L_1(\Omega)$ is defined: $u \leq v$ if

$$\int\limits_{\Omega} (u-k)^+ dx \leq \int\limits_{\Omega} (v-k)^+ dx \quad \text{and} \quad \int\limits_{\Omega} (u+k)^- dx \leq \int\limits_{\Omega} (v+k)^- dx$$

for any $k > 0$.

Theorem 2.2 *Assume $R^-_{\gamma,u} < R^+_{\gamma,u}$ and $T > 0$. Let $f \in L_{p'}(0,T;L_{p'}(0,T;L_{p'}(\partial\Omega))$, $g \in L_{p'}\left(0,T;L_{p'}(\partial\Omega)\right), u_0 \in L_{p'}(\Omega)$ be such that*

$$\gamma_- \leq \gamma(u_0) \leq \gamma_+ \tag{2.2}$$

and

$$\int\limits_{\Omega} \gamma(u_0) dx + \int\limits_{\partial\Omega} u_0 dx + \int\limits_0^t \left(\int\limits_{\Omega} f dx + \int\limits_{\partial\Omega} g ds \right) \in R_{\gamma,u}. \forall t \in [0,T].$$

Then there exists unique weak solution of problem (1.1).

Proof. The uniqueness part of Theorem 2.2 follows from Theorem 2.1. To prove Theorem 2.1, and the existence part of Theorem 2.2 we shall use the theory of nonlinear semi groups (see [3]). We will show the existence of a mild solution and we will prove that it is a weak solution of problem (1.1). To prove the contraction principle we will show that weak solutions are integral solutions. To do this we need to rewrite (1.1) as an abstract Cauchy problem and to use the results obtained in [6] for the associated elliptic problem.

References

1. Astarita G., Marrucci G.: *Principles of Non-Newtomain Fluid Mechanics.* New York, (1974).
2. Colli P., Rodrigues J.: *Diffusion through thin layers with high specific heat*. Asymptotic Anal. **(3)**, 249-263 (1990).
3. Escher J.: *Qusilinear parabolic systems with dynamical boundary conditions*. Commun. Partial Differ. Eq. **(18)**, 1309-1364 (1993).
4. Gadjiev T., Rustamov J., Yagnaliyeva A.: *The behavior of solutions to degenerate double nonlinear parabolic equations*. Proc. Fourt. Inter. Conf. on Manag. Science, 2020, ICMSEM, p. 1-15 (2020).
5. Gadjiev T., C. Aliev, M.Kerimova.: Proceeding of Institute of Applied Mathematics, 8**(1)**, 14-23 (2019).
6. Gadjiev T., Kerimova M.: *Solvability of a boundary-value problem for degenerate equations*. Ukranian Mathematical journal, V. 72, Issue 4, 495-514 (2020).
7. Gadjiev T., Kerimova M.: *Coercive estimate for degenerate elliptic-parabolic equations*. Proceeding of the Institute of Mathem. and Mechanics, 41**(1)**, 123-134 (2015).
8. Kondic I., Shelley M.: Non-Newtonian Hele-Shauflow. Phys. Rev. Lett. 80, 1433-1437 (1998).