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Regularity of solutions of degenerate parabolic nonlinear obstacle problems

Aybeniz H. Yagnaliyeva

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Abstract. *In this paper we study regularity of solutions of degenerate parabolic nonlinear obstacle problems. We prove optimal regularity results for obstacle problems involving nonlinear parabolic operators.* nonlinear, elliptic-parabolic, nonlinear boundary condition.

Keywords. degenerate · nonlinear · parabolic · obstacle problem · regularity.

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1 Introduction

Let we to the study in cylindrical domains $Q = \Omega \times (0, T)$,where $\Omega \subset R^n,\ n \geq 2$ is a bounded Lipschitz domain, $T > 0$, of regularity of solutions of obstacle problems involving degenerate nonlinear parabolic operators Lu of the type

$$-Lu = u_t - div - \left(\omega(x)\,|Du|^{p-2}\,Du\right) \tag{1.1}$$

Let $\Gamma(Q_T) = \left(\overline{\Omega} \times \{0\} \cup (\partial\Omega \times [0, T])\right)$ the parabolic boundary of $Q_T{\cdot}\omega(x)$-Mackenxoupt weight function (see [1]). Let $h : Q_T \to R$ continuous boundary datum and $\psi : Q_T \to R$ a continuous obstacle, such that $h \geq \psi$ on $\Gamma(Q_T)$. We consider the problem

$$\begin{cases} \max\{Lu, \psi - u\} = 0 & \text{in} \quad Q_T \\ \quad u = h & \text{on} \quad \Gamma(Q_T). \end{cases} \tag{1.2}$$

We are interested in the optima regularity of the solution u conditioned on the regularity of h and ψ.

The goal of the paper is to prove that solutions to (2) have the same degree of regularity as the data ψ and we emphasize that a key point of this paper is that we assume no differentiability of the obstacle ψ with respect to time.

Aybeniz H. Yagnaliyeva
Sumgayit State University
Baku, Azerbaijan

E-mail: aybaniz69@mail.ru

Also this problem have many applying in mexanics.

In the case of linear and uniformly parabolic equations we refer to [4]. Optimal regularity of the solution to the initial boundary problem for the heat equation was first proved in [6,7] and the technique is based on the Harnack inequality for harmonic functions. This is the basic idea of DiBenedetto's intrinsic geometry and for this reason the cylinders considered are referred to as intrinsic cylinders. One is led to consider cylinders of the type

$$B(x,r)\times\left(t-\lambda^{2-p}r^{p},\ t+\lambda^{2-p}r^{p}\right)\quad\text{or}\quad B(x,r)\times\left(t-\lambda^{2-p}r^{2},\ t+\lambda^{2-p}r^{2}\right),$$

where $\lambda>0$ is a parameter related to the size of the solution on the cylinder. Here $B(x,r)$ is the standard Euclidean ball in R^n, centered at x and with radius $r>0$. Note that when R^n both of the above cylinders reduce to the standard parabolic cylinders used in the context of the heat equation. Later we show in more detail describe the way intrinsic geometries are used to obtain regularity results.

Let $C^{\alpha}_{\omega(x)}(Q_T)$ weighted space, where norm following:

$$\|f\|_{C^{\alpha}_{\omega(x)}(Q_T)}=\sup_{z_1,z_2\in Q_T}\frac{|f(z_1)\omega(x_1)-f(z_2)\omega(x_2)|}{\|z_1-z_2\|^{\alpha}}<\infty$$

where the parabolic metric is defined as

$$\|(x_1,t_1)-(x_2,t_2)\|_{\alpha}=\max\left\{|x_1-x_2|\,,\ |t_1-t_2|^{\frac{1}{|p-\alpha(p-2)|}}\right\},\ \alpha\in(0,1]\,.$$

The metric is depending on the degree of regularity considered. When $p=2$, these spaces coincide with the spaces of functions which are Holder continuous of order α with respect to the standard parabolic metric.

Generalization of this result to the nonlinear setting of operators of p-Laplace type first came with the work of [4], while more recent work under assumptions of Lipschitz and Holder continuity of the solution can be found in [3], [4]. Similarly results the fundamental work of [10], under assumption Holder continuity of the solution can be found in [5-10].

The paper organized as follows. We will give in section 1 some information about previous results and some definitions. In section 2 we to the study of regularity of solution to initial boundary problem for degenerate parabolic nonlinear obstacle problems. In section 3 a removability theorem for weak solution are proved. Also we define a Hausdorff measures, see, for instance, [10]. In sense Hausdorff measures we give a removability theorem for weak solutions. Out results is the optimal parabolic analog of a series of results known in the elliptic case and we recall that Carleson [9] was the first to prove that a sufficient condition for a set $E\subset R^n$ to be removable with respect to a Lipchitz harmonic function.

2 Main results

We are now ready to state our result which concerns regularity for solutions to the problem (12). This is optimal interior regularity in the obstacle problem assuming that the obstacle is in the space $C^{\alpha}_{\omega(x)}(Q_T)$.

Theorem 2.1 *(interior regularity). Let's consider problem (1.1), (1.2) and $u(x,t)$ solve this problem. Also $\psi\in C^{\alpha}_{\omega(x)}(Q_T)$. Let $Q'_T\subset Q_T$ be a bounded space-time cylinder such that $Q'_T\cap\Gamma(Q_T)=\oslash$. Then $u\in C^{\alpha}_{\omega(x)}(Q'_T)$ and*

$$\|u(x,t)\|_{C^{\alpha}_{\omega(x)}(Q'_T)}\le C\left(n,p,\omega(x),Q'_T,Q_T,\underset{Q_T}{osch},|\psi|_{C^{\alpha}_{\omega(x)}(Q_T)}\right). \tag{2.1}$$

Theorem 2.1 concerns optimal interior regularity. We also establish optimal regularity up to the initial state. In particular, in this case we prove $C^{\alpha}_{\omega(x)}$ estimates on $Q'_T = \Omega' \times (0,T)$ for every $\Omega' \subset \Omega$. We doing remark that in this case Q'_T is not a compact subset of Q_T. Therefore main result is the following.

Theorem 2.2 *(Initial time regularity). Let's consider problem (1.1), (1.2), $u(x,t)$ solve problem (1.2) and $h(x) \in C^{\alpha}_{\omega(x)}(\Omega)$, $\psi \in C^{\alpha}_{\omega(x)}(Q_T)$. Let $\Omega' \subset \Omega$ and $Q'_T = \Omega' \times (0,T)$. Then $u \in C^{\alpha}_{\omega(x)}(Q'_T)$ and*

$$\|u(x,t)\|_{C^{\alpha}_{\omega(x)}(Q'_T)} \leq C\left(n, p, \omega(x), Q_T, Q'_T, \underset{Q_T}{osch}, |\psi|_{C^{\alpha}_{\omega(x)}(Q_T)}\right). \tag{2.2}$$

Corollary 2.1. Let $u(x,t)$ be a solution to (1.2) with $D\psi,\ h \in L^{\infty}(Q_T)$. Then $Du \in L^{\infty}_{loc}(Q_T)$.

The obstacle problem in Sobolev spaces we refer to [3] for details. Optimal regularity of the solution to the obstacle problem for the Laplas equation was first proved in [3,4]. The parabolic obstacle problems have been treated in [2-4], see also [5-8].

Theorem 2.3 *(removable singularities). Let $Q_T \subset R^{n+1}$ be a cylindrical domain, $E \subset Q_T$ be a closed set. Assume that $u(x,t)$ is a weak solution to $Lu = 0$ in $Q_T \backslash E$ and that $u \in C^{\alpha}_{\omega(x),loc}(Q_T)$. Also let $H^{\omega(x)}(E) = 0$. Then the set E is removable, i.e. $u(x,t)$ can be extended to be a weak solution in Q_T.*

We note $H^{\omega(x)}(E)$-Hausdorff measures (se, [36]). Let L be as in (1.1). We say that a function $u(x,t)$ is a weak supersolution (subsolution) to (1.1) in an open set $K \subset R^{n+1}$ if, whenever $K' = U \times (t_1,t_2) \in K$ with $U \subset R^n$ and $t_1 < t_2$, then $u \in L^P\left(t_1,t_2; W'_p(U)\right)$ and

$$\underset{K'}{S}\left(\omega(x)\,|Du|^{p-2}\,Du - u\varphi_t\right) dxdt \geq (\leq)\, 0 \tag{2.3}$$

for all nonnegative $\varphi \in C_0^{\infty}(K')$. A weak solution is a distributional solution satisfying (2.3) with equality and without sign restrictions for the test functions.

We now give the definition of solutions to the obstacle problem. In the following we assume that the obstacle ψ and boundary value function h are continuous on Q_T and that $h \geq \psi$ on the parabolic boundary of $Q_T = \Omega \times (0,T)$.

A function $u(x,t)$ is a solution (1.2) if it satisfies the following properties:

1. u is continuous on $Q_T, u \geq \psi$ in Q_T and $u = h$ on $\Gamma(Q_T)$,

2. u is a weak supersolution in Q_T,

3. u is a weak solution in $Q_T \cap \{u > \psi\}$.

4. As for the property 3 we recall that $u(x,t)$ is a weak solution in $Q_T \cap \{u > \psi\}$ means that u is a standard distributional solution in the sense (2.3).

Proof of Theorem 2.1. Let $\overline{u}$ be the unique solution to (1.2). By the uniqueness $\overline{u} = u$ in $\Omega \times [0,T]$ and hence $\overline{u}$ is an extension of u. Let $R = \max\left\{1, diam\ \Omega,\ T^{1/2}\right\}$. As clearly

$$T \leq (\psi(R))^{2-p} R^p \leq R^2,$$

whenever $R \geq 1$. By maximum principle implies that

$$\underset{Q_T}{oscu} \leq \underset{Q_T}{osch} \leq \overline{C}\left(\Omega, T, \underset{\Gamma(Q_T)}{osch}\right). \tag{2.4}$$

We may assume that $Q'_{T,\tau} = \Omega' \times (\tau, T)$, where $\Omega' \subset \Omega$ and $\tau > 0$. Let R be a number subject to restrictions

$$R \leq dist\left(\Omega', \partial\Omega\right), \quad \tau \geq R^p \max\left\{\underset{Q_T}{osch}, SR\right\}^{2-p}.$$

As so $\psi(1) = 1$, we see that these conditions are satisfied if we take

$$R = \min\left\{dist\left(\Omega', \partial\Omega\right), \max\left\{\tau^{1/p}\overline{c}\left(\Omega, \underset{\Gamma(Q_T)}{osch}, T\right)^{\frac{p-2}{2}}, \tau^{\frac{1}{p}}, S^{\frac{p-2}{R}}\right\}\right\}.$$

Taking corresponding λ it follows that $Q_R^{\lambda\psi(R)}(z) \subset Q_T$, whenever $z \in Q'_{T,\tau}$.

Now we prove that the following holds whenever $z_0 \in Q'_{T,\tau}$

$$\underset{Q_{T,r}^{\lambda\psi(R)} \cap Q_T}{oscu} \leq \underset{Q_T}{oscu} = \frac{\underset{Q_T}{oscu}}{\psi(r)}\psi(r) \leq \frac{\underset{Q_T}{oscu}}{\psi(R/2)}\psi(r) \leq 2\lambda\psi(r).$$

This completes the proof of Theorem 2.1.

Proof of Theorem 2.2. We extend $u(x,t)$ and choose $R = dist\,(\Omega', \partial\Omega)$. We define $\lambda =$
$= \max\left\{\overline{c}/\psi(R),\ |h|_{C^\alpha_{\omega(x)}}, \frac{SR}{\psi(R)}\right\}$, where $\overline{c} = \overline{c}\left(\Omega, T, \underset{Q_T}{osch}\right)$. Let $Z = \overline{\Omega}' \times \{0\}$. Then $\underset{Q_\tau^{\lambda\psi(r)}(z)\cap Q'_T}{osch} \leq 2\lambda\psi(r)$ for every $r \in (0,R)$, whenever $z \in Z$. We consider $z_1 \in F = \overline{Q}'_T \cap \{t > 0\}$ and define

$$\overline{r} = \overline{r}(z_1) \pm \sup\left\{r \leq R; Q_r^{\lambda\psi(r)}(z_1) \cap Z = \oslash \notin\right\}$$

If $r > R/2$, then

$$\underset{Q_r^{2\lambda\psi(r)}(z_1)}{oscu} \leq 2\lambda\psi(r) \quad \text{for every } \ z \in (0,R).$$

In the final

$$\overline{\lambda} = \max\left\{\varphi\lambda\psi(\overline{r}),\ s\overline{r}/\psi(\overline{r})\right\} \leq \varphi\max\left\{\lambda, SR/\psi(R)\right\} = c\lambda$$

implies that

$$\underset{Q_r^{c\lambda\psi(r)}(z_1)}{oscu} \leq c\lambda\psi(r) \quad \text{for every } \ z \in [0,\overline{r}],$$

whenever $z_1 \in \left\{Q'_T \cap \left(\overline{\Omega}' \times \{0\}\right)\right\}$.

This completes the proof of Theorem 2.2.

Proof of Theorem 2.3. Let $u(x,t)$ weakly solution of problem (1.1), (1.2) in $Q_T\backslash E$ and assume that $u(x,t) \in C^\alpha_{\omega(x),loc}(Q_T)$ and $H^{\omega(x)}(E) = 0$. By the assumption $u \in C^\alpha_{\omega(x),loc}(Q_T)$. There exist $M > 0$ such that

$$\underset{Q_r^{\psi(r)}\cap Q_T}{oscu}\ (x,t) \leq M\psi(r).$$

Later we doing some calculations get of proof theorem.

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