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Existence and uniqueness of an inverse boundary value problem for Benny-Luke equation with not self-adjoint boundary conditions

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Abstract. *We study an inverse problem for Benny-Luke linearized equation with boundary conditions. As first the original problem is reduced to the equivalent problem (in the certain sense) for which the existence and uniqueness theorem is proved. Further, based on this fact, we prove the existence and uniqueness of the classic solution of the original problem.*
nonlinear, elliptic-parabolic, nonlinear boundary condition.

Keywords. inverse problem · Benney-Luke equation · existence · uniqueness · classic solution.

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1 Introduction

Many problems of mathematical physics, continuum mechanics are boundary value problem reduced to the integration of a differential equation or the system of partial equations under the given boundary and initial conditions. A number of problems of gas dynamics, elasticity theory, theory of shells and plates are reduced to the consideration of higher order partial differential equations [1]. Fourth order differential equations are of great interest from the point of view of applications (see e.i. [2, 12]). Benney-Luke partial differential equations have applications in mathematical physics (see [2]).

The problems in which together with the solution of their or other differential equation it is required to determine also the coefficient (coefficient) of the equation itself or the right-hand side of the equation in mathematics and in mathematical simulation, are called inverse problems. Theory of inverse problems for differential equations is a dynamically developing section of contemporary science. Recently, inverse problems arise in very different fields of human activity such as seismology, mineral exploration, biology, medicine, quality control of industrial product, etc. that puts them among the topical problems of modern mathematics. Different inverse problem for individual types of partial differential equations were

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studied in many works. First of all, we note the works of note works of A.N. Tikhonov [13], M.M. Lavrentev [7, 8], V.K. Ivanov [5] and their followers. More detailed information can be found in the monograph of A.M. Denisov [4].

Theory of inverse boundary value problems for fourth order equations still remains poorly studied. The works [3, 11, 15, 16] were devoted to inverse boundary value problems for the Benney-Luke equation.

The goal of the paper is to prove the existence and uniqueness of the solution of an inverse boundary value problem for the Benny-Luke equation with not self-adjoint boundary conditions.

2 Statement of the problem and its reduction to the equivalent problem

Let $D_T = \{(x,t) : \ 0 \le x \le 1, \ \ 0 \le t \le T\}$. In the rectangle D_T we consider the following inverse boundary value problem: to find the pair $\{u(x,t)\,,\ a(t)\}$ functions $u(x,t)$, $a(t)$ satisfying the equation [2]

$$u_{tt}(x,t) - u_{xx}(x,t) + \alpha u_{xxxx}(x,t) - \beta u_{xxtt}(x,t) =$$
$$= a(t)u(x,t) + f(x,t) \ \ (x,t) \in D_T, \tag{2.1}$$

with nonlocal initial conditions

$$u(x,0) = \varphi(x) + \int_0^T p_1(t)u(x,t)\,dt\,,\ u_t(x,0) =$$
$$= \psi(x) + \int_0^T p_2(t)u(x,t)\,dt \ (0 \le x \le 1), \tag{2.2}$$

not self-adjoint boundary conditions

$$u(1,t) = 0, u_x(0,t) = u_x(1,t), \ \ u_{xx}(1,t) \ = 0,$$
$$u_{xxx}(0,t) = u_{xxx}(1,t) \ \ (0 \le t \le T) \tag{2.3}$$

and with the additional condition

$$u(0,t) = h(t) \quad \ (0 \le t \le T), \tag{2.4}$$

where $\alpha > 0$, $\beta > 0$ - are fixed numbers, $f(x,t)$, $\varphi(x)$, $\psi(x)$, $p_1(t)$, $p_2(t)$, $h(t)$ - are the given functions.

Denote

$$\tilde{C}^{4,2}(D_T) = \{u(x,t) : \ u(x,t) \in C^2(D_T)\ , u_{ttx}(x,t),$$
$$u_{ttxx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t) \in C(D_T)\}\,.$$

Definition 2.1 *Under the classical solution of the inverse boundary value problem (2.1) - (2.4) we understand the pair* $\{u(x,t)\,,\ a(t)\}$ *of functions* $u(x,t) \ \in \ \tilde{C}^{4,2}(D_T)$, $a(t) \in C[0,T]$ *satisfying equation (2.1) and conditions (2.2) - (2.4) in the usual sense.*

Along with the inverse boundary value problem (2.1) - (2.4) we consider the following auxiliary inverse boundary value problem it is required to determine the pair $\{u(x,t), a(t)\}$ of functions $u(x,t) \ \in \tilde{C}^{4,2}(D_T)$, $a(t) \ \in C[0,T]$, from the relations (2.1) - (2.3),

$$h''(t) - u_{xx}(0,t) + \alpha u_{xxxx}(0,t) - \beta u_{xxtt}(0,t) = \ a(t)h(t) + f(0,t) \ \ (\, 0 \le t \le T). \tag{2.5}$$

Similar to [9], we prove the following theorem

Theorem 2.1 *Let $\varphi(x)$, $\psi(x) \in C[0,1]$, $h(t) \in C^2[0,T]$, $h(t) \neq 0 \quad (0 \le t \le T)$, $f(x,t) \in C(D_T)$ and the terms of argument*

$$\varphi(0) = h(0) \, - \int_0^T p_1(t)h(t)\,dt, \quad \psi(0) = h'0) - \int_0^T p_2(t)h(t)\,dt.$$

be fulfilled.

Then, the following statements are valid:

A. Each classic solution $\{u(x,t), a(t)\}$ of problem (2.1) - (2.4) is also solution of the problem (2.1) - (2.3), (2.5);

B. Each solution $\{u(x,t), a(t)\}$ of the problem (2.1) - (2.3), (2.5), is such that

$$\left(\|p_1(t)\|_{C[0,T]} + T\,\|p_2(t)\|_{C[0,T]} + \|a(t)\|_{C[0,T]} \right) T^2 < 1 \tag{2.6}$$

is a classic solution of (2.1) - (2.4).

3 Solvability of the inverse boundary value problem

It is known [14], that the sequences of the function

$$X_0(x) = 2(1-x), \;\; X_{2k-1}(x) = 4(1-x)\cos\lambda_k x, \;\; X_{2k}(x) = 4\sin\lambda_k x \;\, (k = 1, 2, ...), \tag{3.1}$$

$$Y_0(x) = 1, \;\; Y_{2k-1}(x) = \cos\lambda_k x, \;\; Y_{2k}(x) = x\sin\lambda_k x \;\, (k = 1, 2, ...) \tag{3.2}$$

From a biorthogonal system, and the system (3.7) forms a Rises basic in $L_2(0,\ 1)$, where $\lambda_k = 2k\pi\,(k = 1, 2, ...)$. Then the arbitrary function expands in biorthogonal series: $\vartheta(x) \in L_2(0,1)$

$$\vartheta(x) = \vartheta_0\,X_0(x) + \sum_{k=1}^{\infty} \vartheta_{2k-1}\;X_{2k-1}(x) \; + \; \sum_{k=1}^{\infty} \vartheta_{2k}\;X_{2k}(x)\,,$$

where

$$\vartheta_0 = \int_0^1 \vartheta_0\,Y_0\,(x)dx\,, \;\; \vartheta_{2k-1} \,=\int_0^1 \vartheta_{2k-1}Y_{2k-1}\,(x)dx\,, \;\; \vartheta_{2k-1} \,=\int_0^1 \vartheta_{2k-1}Y_{2k-1}\,(x)dx\,.$$

It is known [10], that if

$$\vartheta(x) \in C^{2i-1}[0,1], \quad \vartheta^{(2i)}(x) \in L_2(0,1)\,,$$

$$\vartheta^{(2s)}(1) = 0, \;\; \vartheta^{(2s+1)}(0) = \vartheta^{(2s)}(1) \;\, (s = \overline{0, i-1} \;\;),$$

then

$$\sum_{k=1}^{\infty} \left(\lambda_k^{2i}\vartheta_{2k-1}\,\right)^2 \le \frac{1}{2}\left\|\vartheta^{(2i)}(x)\right\|_{L_2(0,1)}^2,$$

$$\sum_{k=1}^{\infty} \left(\lambda_k^{2i}\vartheta_{2k}\,\right)^2 \le \frac{1}{2}\left\|\vartheta^{(2i)}(x)x + 2i\vartheta^{(2i-1)}(x)\right\|_{L_2(0,1)}^2. \tag{3.3}$$

Under the assumptions

$$\vartheta(x) \in C^{2i}[0,1], \quad \vartheta^{(2i+1)}(x) \in L_2(0,1)\,,$$

$$\vartheta^{(2s)}(1) = 0,\ \vartheta^{(2s-1)}(0) = \vartheta^{(2s-1)}(1)\ \ (i \geq 1,\ s = \overline{0, i}\ ),$$

the validity of the following estimations [10] is established:

$$\sum_{k=1}^{\infty} \left(\lambda_k^{2i+1} \vartheta_{2k-1}\ \right)^2 \leq \frac{1}{2} \left\| \vartheta^{(2i+1)}(x) \right\|^2_{L_2(0,1)},$$

$$\sum_{k=1}^{\infty} \left(\lambda_k^{2i+1} \vartheta_{2k}\ \right)^2 \leq \frac{1}{2} \left\| \vartheta^{(2i+1)}(x)x + (2i+1)\vartheta^{(2i)}(x) \right\|^2_{L_2(0,1)}. \tag{3.4}$$

To study the problem (2.1) - (2.3), (2.6) we consider the following space.

By $B_{2,T}^5$ [10] we denote the totality of all the functions $u\,(x, t)$ of the form

$$u\,(x, t) = \sum_{k=0}^{\infty} u_k\,(t)\,X_k\,(x)\,,$$

considered on D_T,and for which all the functions $u_k\,(t) \in C[0, T]$ and

$$J_T(u) \equiv \|u_0(t)\|_{C[0,T]} +$$

$$+ \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 \|u_{2k-1}(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 \|u_{2k}(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}},$$

where the functions $X_k\,(x)\ \ (k = 0, 1, 2, ...)$ are determined by the relations (2.1).

The norm in this set is determined as: $\|u(x, t)\|_{B_{2,T}^5} = J_{"}(u)$.

Denote by E_T^5 such a space of vector-functions $\{u(x, t), a(t)\}$ that $u\,(x, t) \in B_{2,T}^5$, $a\,(t) \in C[0, T]$. Supply this space with the norm

$$\|z\|_{E_T^5} = \|u(x, t)\|_{B_{2,T}^5} + \|a(t)\|_{C[0,T]}\,.$$

Obviously, $B_{2,T}^5$ and E_T^5 are Banach spaces.

Since the system (2.1) forum the Riesz basis $L_2\,(0, 1)$ and the system (2.1) and (2.2) forms a system of functions biorthogonal in $L_2\,(0, 1)$ then we will book for the first component $u\,(x, t)$ of the solution $\{u(x, t), a(t)\}$ of the problem (2.1) - (2.3), (2.6) in the form

$$u\,(x, t) = u_0\,(t)\,X_0\,(x) + \sum_{k=1}^{\infty} u_{2k-1}\,(t)\,X_{2k-1}\,(x) + \sum_{k=1}^{\infty} u_{2k}\,(t)\,X_{2k}\,(x)\,, \tag{3.5}$$

where

$$u_0(t) = \int_0^1 u(x, t)Y_0(x)dx,$$

$$u_{2k-1}(t) = \int_0^1 u(x, t)Y_{2k-1}(x)dx,\ u_{2k}(t) = \int_0^1 u(x, t)Y_{2k}(x)dx\ (k = 1, 2, ...)\,, \tag{3.6}$$

is the solution of the following problem:

$$u_0''\,(t) = F_0\,(t; u, a) \quad (0 \leq t \leq T), \tag{3.7}$$

$$u_{2k-1}''(t) + \beta_k^2 u_{2k-1}(t) = \frac{1}{1 + \beta\lambda_k^2} F_{2k-1}\,(t; u, a) \quad (0 \leq t \leq T,\ k = 1, 2, ...), \tag{3.8}$$

$$u''_{2k}(t)+\beta_k^2 u_{2k}(t)=\frac{1}{1+\beta\lambda_k^2}F_{2k}(t;u,a)+$$

$$+\frac{2\lambda_k(1+2\alpha\lambda_k^2)}{1+\beta\lambda_k^2}u_{2k-1}(t)+\frac{2\beta\lambda_k}{1+\beta\lambda_k^2}\,u''_{2k-1}(t)\ \ (0\le t\le T,\ k=1,2,...), \tag{3.9}$$

$$u_k(0)=\varphi_k+\int\limits_0^T p_1(t)u_k(t)\,dt,\quad u'_k(0)=\psi_k+\int\limits_0^T p_2(t)u_k(t)\,dt\quad (k=0,1,2,...)\,, \tag{3.10}$$

moreover

$$\beta_k^2=\frac{\lambda_k^2(1+\alpha\lambda_k^2)}{1+\beta\lambda_k^2},F_k(t;u,a)=a(t)u_k(t)+f_k(t),\quad f_k(t)=\int\limits_0^1 f(x,t)Y_k(x)dx,$$

$$\varphi_k=\int\limits_0^1\varphi(x)Y_k(x)dx,\quad \psi_k=\int\limits_0^1\psi(x)Y_k(x)dx\ \ (k=0,1,...).$$

Solving the problem (3.7) - (3.10) we find:

$$u_0(t)=\varphi_0+\int\limits_0^T p_1(t)u_0(t)\,dt+$$

$$+\left(\psi_0+\int\limits_0^T p_2(t)u_0(t)\,dt\right)t+\int\limits_0^t(t-\tau)F_0(\tau;u,a)d\tau, \tag{3.11}$$

$$u_{2k-1}(t)\ =\ \left(\varphi_{2k-1}+\int_0^T p_1(t)u_{2k-1}(t)\,dt\right)\cos\beta_k t+$$

$$+\frac{1}{\beta_k}\left(\psi_{2k-1}+\int_0^T p_2(t)u_{2k-1}(t)\,dt\right)\sin\beta_k t+$$

$$+\ \frac{1}{\beta_k(1\,+\,\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,a)\sin\beta_k(t-\tau)d\tau, \tag{3.12}$$

$$u_{2k}(t)\ =\ \left(\varphi_{2k}+\int_0^T p_1(t)u_{2k}(t)\,dt\right)\cos\beta_k t+\frac{1}{\beta_k}\left(\psi_{2k}+\int_0^T p_2(t)u_{2k}(t)\,dt\right)\sin\beta_k t+$$

$$+\frac{\lambda_k(1+2\alpha\lambda_k^2+\alpha\beta\lambda_k^4)}{(1+\beta\lambda_k^2)^3}\left[t\left(\varphi_{2k-1}+\int_0^T p_1(t)u_{2k-1}(t)\,dt\right)\sin\beta_k t\,+\right.$$

$$+\left(\frac{1}{\beta_k}\sin\beta_k t-t\cos\beta_k t\right)\frac{1}{\beta_k}\left(\psi_{2k-1}+\int_0^T p_2(t)u_{2k-1}(t)\,dt\right)+$$

$$+\ \frac{1}{\beta_k(1\,+\,\beta\lambda_k^2)}\int_0^t F_{2k}(\tau;u,a)\sin\beta_k(t\,-\,\tau)d\tau\,+$$

$$\left.+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t\left(\int_0^\tau F_{2k-1}(\xi;u,a)\sin\beta_k(t-\xi)\,d\xi\right)\sin\beta_k(t-\tau)\,d\tau\right]+$$

$$+\frac{2\beta\lambda_k}{\beta_k(1+\beta\lambda_k^2)^2}\int_0^t F_{2k-1}(\tau;u,a)\sin\lambda_k(t-\tau)d\tau. \quad (3.13)$$

After substituting the expression $u_k(t)$ $(k = 0, 1, ...)$ in (2.5), for determining the components $u(x,t)$ of the solution of the problem (2.1) - (2.3), (2.6) we get:

$$u(x,t) = \left(\varphi_0 + \int_0^T p_1(t)u_0(t)\,dt + \left(\psi_0 + \int_0^T p_2(t)u_0(t)\,dt\right)t+\right.$$

$$\left.+\int_0^t (t-\tau)F_0(\tau;u,a)d\tau\right)X_0(x) + \sum_{k=1}^{\infty}\left\{\left(\varphi_{2k-1} + \int_0^T p_1(t)u_{2k-1}(t)\,dt\right)\cos\beta_k t+\right.$$

$$\frac{1}{\beta_k}\left(\psi_{2k-1} + \int_0^T p_2(t)u_{2k-1}(t)\,dt\right)\sin\beta_k t+$$

$$\left.+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,a)\sin\beta_k(t-\tau)d\tau\right\}X_{2k-1}(x)+$$

$$+\sum_{k=1}^{\infty}\left\{\left(\varphi_{2k} + \int_0^T p_1(t)u_{2k}(t)\,dt\right)\cos\beta_k t + \frac{1}{\beta_k}\left(\psi_{2k} + \int_0^T p_2(t)u_{2k}(t)\,dt\right)\sin\beta_k t+\right.$$

$$+\frac{\lambda_k(1+2\alpha\lambda_k^2+\alpha\beta\lambda_k^4)}{(1+\beta\lambda_k^2)^3}\left[t\left(\varphi_{2k-1} + \int_0^T p_1(t)u_{2k-1}(t)\,dt\right)\sin\beta_k t+\right.$$

$$+\left(\frac{1}{\beta_k}\sin\beta_k t - t\cos\beta_k t\right)\frac{1}{\beta_k}\left(\psi_{2k-1} + \int_0^T p_2(t)u_{2k-1}(t)\,dt\right)+$$

$$+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k}(\tau;u,a)\sin\beta_k(t-\tau)d\tau+$$

$$\left.+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t\left(\int_0^\tau F_{2k-1}(\xi;u,a)\sin\beta_k(t-\xi)\,d\xi\right)\sin\beta_k(t-\tau)\,d\tau\right]+$$

$$\left.+\frac{2\beta\lambda_k}{\beta_k(1+\beta\lambda_k^2)^2}\int_0^t F_{2k-1}(\tau;u,a)\sin\lambda_k(t-\tau)d\tau\right\}X_{2k}(x). \quad (3.14)$$

Now, from (2.6), allowing for (3.5), we have:

$$a(t) = [h(t)]^{-1}\left\{h''(t) - f(0,t) + 4\sum_{k=1}^{\infty}((\lambda_k^2+\alpha\lambda_k^4)u_{2k-1}(t) + \beta\lambda_k^2 u''_{2k-1}(t))\right\}. \quad (3.15)$$

Further, from (3.7), allowing for (3.12), we obtain:

$$(\lambda_k^2+\alpha\lambda_k^4)u_{2k-1}(t) + \beta\lambda_k^2 u''_{2k-1}(t) = F_{2k-1}(t;u,a) - u''_{2k-1}(t) =$$

$$= \frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}(t;u,a) - \beta_k^2 u_{2k-1}(t) =$$

$$= \frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}(t;u,a) - \beta_k^2\left(\left(\varphi_{2k-1} + \int_0^T p_1(t)u_{2k-1}(t)\,dt\right)\cos\beta_k t +\right.$$

$$+\frac{1}{\beta_k}\left(\psi_{2k-1}+\int_0^T p_2(t)u_{2k-1}(t)\,dt\right)\sin\beta_k$$

$$+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,a)\sin\beta_k(t-\tau)d\tau\Bigg). \tag{3.16}$$

In other to obtain an equation for the second component $a(t)$ of the solution $\{u(x,t),a(t)\}$ of the problem (2.1) – (2.3), (2.6), we substitute the expression (3.16) in (3.15):

$$a(t)=[h(t)]^{-1}\Bigg\{h''(t)-f(0,t)+4\sum_{k=1}^{\infty}\Bigg[\frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}(t;u,a)-$$

$$-\beta_k^2\Bigg(\Bigg(\varphi_{2k-1}+\int_0^T p_1(t)u_{2k-1}(t)\,dt\Bigg)\cos\beta_k t+$$

$$+\frac{1}{\beta_k}\left(\psi_{2k-1}+\int_0^T p_2(t)u_{2k-1}(t)\,dt\right)\sin\beta_k t+$$

$$+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,a)\sin\beta_k(t-\tau)d\tau\Bigg)\Bigg)\Bigg]\Bigg\}. \tag{3.17}$$

Thus, the solution of the problem (2.1) - (2.3), (2.6) is reduced to the solution of the system (3.14), (3.17) with respect to the unknown function $u(x,t)$ and $a(t)$.

The following lemma is very important for studying the uniqueness of the problem (2.1) -(2.3), (2.6).

Lemma 3.1 *If $\{u(x,t),a(t)\}$ is any solution of problem (2.1) - (2.3), (2.6), then the functions $u_k(t)$ $(k=0,1,2,\ldots)$, determined by the relation (2.6), satisfy on $[0,T]$ the denumerable system (2.11), (2.12) and (2.13).*

Obviously, if $u_k(t)=\int_0^1 u(x,t)Y_k(x)dx$ $(k=0,1,...)$ is the solution of the system $u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x)$ and $a(t)$ the pair $\{u(x,t),a(t)\}$ of the function (3.11), (3.12) and (3.13), is the solutions of the system (3.14), (3.17).

Lemma 3.1 implies the following corollary

Corollary 3.1 *Let the system (2.14), (2.17) have a unique solution. Then the problem (2.1) - (2.3), (2.6) can have at most solution, i.e. if the problem (2.1) - (2.3), (2.6) has a solution then this solution is unique.*

Now in the space E_T^5 we consider the operator

$$\Phi(u,a)=\{\Phi_1(u,a),\Phi_2(u,a)\}\ ,$$

where

$$\Phi_1(u,a)=\tilde{u}(x,t)=\sum_{k=0}^{\infty}\tilde{u}_k(t)X_k(x),\Phi_2(u,a)=\tilde{a}(t),$$

while $\tilde{u}_0(t),\ \tilde{u}_{2k-1}(t),\ \tilde{u}_{2k}(t)$ and $\tilde{a}(t)$ are equal to the right hand sides of (3.11), (3.12), (3.13) and (3.17) respectively.

It is easy to see that

$$1+\beta\lambda_k^2>\beta\lambda_k^2,\quad \frac{1}{1+\beta\lambda_k^2}<\frac{1}{\beta\lambda_k^2},$$

$$\sqrt{\frac{\alpha}{1+\beta}}\,\lambda_k \le \beta_k \le \sqrt{\frac{1+\alpha}{\beta}}\,\lambda_k,\ \ \sqrt{\frac{\beta}{1+\alpha}}\,\frac{1}{\lambda_k} \le \frac{1}{\beta_k} \le \sqrt{\frac{1+\beta}{\alpha}}\,\frac{1}{\lambda_k},$$

Considering this relation, we find:

$$\|\tilde{u}_0\,(t)\|_{C[0,T]} \le |\varphi_0| + T\,|\psi_0| + T\sqrt{T}\left(\int_0^T |f_0\,(\tau)|^2\,d\tau\right)^{\frac{1}{2}} +$$
$$+\left(\|p_1(t)\|_{C[0,T]} + T\,\|p_2(t)\|_{C[0,T]}\right)T\,\|u_0(t)\|_{C[0,T]} +$$
$$+T^2\,\|a(t)\|_{C[0,T]}\,\|u_0(t)\|_{C[0,T]}\,, \tag{3.18}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le$$
$$\le 3\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}} + 3\sqrt{\frac{1+\beta}{\alpha}}\,\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_{2k-1}\,|)^2\right) +$$
$$+\left(\|p_1(t)\|_{C[0,T]} + \|p_2(t)\|_{C[0,T]}\right)T\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} +$$
$$+\frac{3}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\ |)^2 d\tau\right)^{\frac{1}{2}} +\right.$$
$$\left. + T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right], \tag{3.19}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le$$
$$\le 4\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k}|)^2\right)^{\frac{1}{2}} + 4\sqrt{\frac{1+\beta}{\alpha}}\,\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_{2k}\,|)^2\right)^{\frac{1}{2}} +$$
$$+\left(\|p_1(t)\|_{C[0,T]} + \|p_2(t)\|_{C[0,T]}\right)T\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} +$$
$$+\frac{4}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k}\,(\tau)\ |)^2 d\tau\right)^{\frac{1}{2}} +\right.$$
$$\left. + T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right] +$$

$$+\frac{4(1+2\alpha+\alpha\beta)}{\beta^3}\left[T\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\varphi_{2k-1}\right|)^2\right)^{\frac{1}{2}}+\right.$$

$$+\left(\sqrt{\frac{1+\beta}{\alpha}}+T\right)\sqrt{\frac{1+\beta}{\alpha}}\left(\sum_{k=1}^{\infty}(\lambda_k^4\left|\psi_{2k-1}\right|)^2\right)^{\frac{1}{2}}+$$

$$+\frac{4}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left(T\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\;|)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.+T^2\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right)\right]+$$

$$+4\left(\left\|p_1(t)\right\|_{C[0,T]}+\left\|p_2(t)\right\|_{C[0,T]}\right)T\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\frac{6}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\;|)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.+\,T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right], \qquad (3.20)$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]}+\right.$$

$$+4\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\{\frac{1+\alpha}{\beta}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\varphi_{2k-1}\right|)^2\right)^{\frac{1}{2}}+\sqrt{\frac{1+\beta}{\alpha}}\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_{2k-1}\,|)^2\right)+\right.\right.$$

$$+\left(\left\|p_1(t)\right\|_{C[0,T]}+\left\|p_2(t)\right\|_{C[0,T]}\right)T\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\frac{1}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\;|)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.+\,T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right]+\left(\sum_{k=1}^{\infty}(\lambda_k^2\left\|f_{2k}\,(t)\right\|_{C[0,T]}\;|\right)^{\frac{1}{2}}$$

$$\left.\left.+\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right\}\right\}. \qquad (3.21)$$

Assume that the data of problem (2.1) - (2.3), (2.6) satisfy the following conditions:

$$1.\alpha>0,\beta>0,p_i(t)\in C[0,T](i=1,2),\;\; h(t)\in C^2[0,T],$$

$$h(t) \neq 0 \ (0 \leq t \leq T).$$

$$2.\varphi(x) \in C^4[0,1],\ \varphi^{(5)}(x) \in L_2(0,1),\ \ \varphi(1)=0, \varphi'(0)=\varphi'(1),$$
$$\varphi''(1)=0, \varphi'''(0)=\varphi'''(1),\ \ \varphi^{(4)}(1)=0.$$
$$3.\psi(x) \in C^3[0,1],\ \psi^{(4)}(x) \in L_2(0,1),\ \ \psi(1)=0,$$
$$\psi'(0)=\psi'(1),\ \ \psi''(1)=0, \psi'''(0)=\psi'''(1).$$
$$4.f(x,t), f_x(x,t) \in C(D_T),\ f_{xx}(x,t) \in L_2(D_T),$$
$$f(1,t)=0,\ \ f_x(0,t)=f_x(1,t)\ \ \ (0 \leq t \leq T).$$

Then from (3.18)- (3.21) we find:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}} =$$
$$= A_1(T) + B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}} + C_1(T)\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{3.22}$$
$$\|\tilde{a}(t)\|_{C[0,T]} =$$
$$= A_2(T) + B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}} + C_2(T)\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{3.23}$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\,\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)} +$$
$$+\sqrt{2}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}\sqrt{\frac{2(1+\beta)}{\alpha}}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)} + \sqrt{\frac{2(1+\beta)}{\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)} +$$
$$\frac{3}{\sqrt{2}}\left\|\varphi^{(5)}(x)+4\varphi^{(3)}(x)\right\|_{L_2(0,1)} + \frac{4}{\sqrt{2}}\sqrt{\frac{1+\beta}{\alpha}}\left\|\psi^{(4)}(x)+3\psi^{(3)}(x)\right\|_{L_2(0,1)} +$$
$$+\frac{4}{\beta}\sqrt{\frac{T(1+\beta)}{2\alpha}}\,\|f_{xx}(x,t)+2f_x(x,t)\|_{L_2(D_T)} +$$
$$+\frac{4(1+2\alpha+\alpha\beta)}{\beta^3}\left(\frac{T}{\sqrt{2}}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)} + \left(\sqrt{\frac{1+\beta}{\alpha}}+T\right)\sqrt{\frac{1+\beta}{2\alpha}}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)} +\right.$$
$$\left.+\frac{T}{\beta}\sqrt{\frac{T(1+\beta)}{2\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\right) + \frac{6}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\,,$$
$$B_1(T) = T^2 + \frac{11T}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left(1+\frac{3(1+2\alpha+\alpha\beta)}{\beta^3}T\right),$$
$$C_1(T) = 4\left(\|p_1(t)\|_{C[0,T]} + \|p_2(t)\|_{C[0,T]}\right)T$$
$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]} +\right.$$
$$+2\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\{\frac{1+\alpha}{\beta}\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)} + \sqrt{\frac{1+\beta}{\alpha}}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)} +\right.\right.$$

$$+\frac{1}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}\Bigg]+\left\|\left\|f_{xx}(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}\Bigg\},$$

$$B_2(T)=2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\frac{1+\alpha}{\beta^2}\sqrt{\frac{1+\beta}{\alpha}}T+1\right).$$

$$C_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left(\left\|p_1(t)\right\|_{C[0,T]}+\left\|p_2(t)\right\|_{C[0,T]}\right)T.$$

From inequalities (3.21), (3.22) we deduce:

$$\left\|\tilde{u}(x,t)\right\|_{B^5_{2,T}}+\left\|\tilde{a}(t)\right\|_{C[0,T]}$$

$$\leq A(T)+B(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^5_{2,T}}+C(T)\left\|u(x,t)\right\|_{B^5_{2,T}}, \tag{3.24}$$

where

$$A(T)=A_1(T)+A_2(T),\quad B(T)=B_1(T)+B_2(T), C(T)=C_1(T)+C_2(T).$$

Thus, we proved the following theorem

Theorem 3.1 *Let conditions 1-4 be fulfilled, and*

$$(B(T)(A(T)+2)+C(T))A(T)+2)<1. \tag{3.25}$$

Then problem (2.1)*,* (2.3)*, (2.5) has in the sphere* $K=K_R(\|\,z\,\|_{E^5_T}\leq R=A(T)+2)$ *from* E^5_T *a unique solution.*

Proof. In the space E^5_T we consider the equation

$$z=\Phi z, \tag{3.26}$$

where $z=\{u,a\}$, the components of the operator $\Phi_i(u,a)\,(i=1,2)$ are determined $\Phi(u,a)$ by the right hand sides of equations (3.14), (3.17), respectively.

Let us consider the operator $\Phi(u,a)$ in the sphere $K=K_R$ from E^5_T. Similar to (3.24) we get that for any $z,\ z_1,\ z_2\in K_R$ the following estimates are valid:

$$\left\|\Phi z\right\|_{E^5_T}\leq A(T)+B(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^5_{2,T}}+C(T)\left\|u(x,t)\right\|_{B^5_{2,T}}\leq$$

$$\leq A(T)+B(T)(A(T)+2)^2+C(T)(A(T)+2), \tag{3.27}$$

$$\left\|\Phi z_1-\Phi z_2\right\|_{E^5_T}\leq B(T)R\left(\left\|a_1(t)-a_2(t)\right\|_{C[0,T]}+\left\|u_1(x,t)-u_2(x,t)\right\|_{B^5_{2,T}}\right)+$$

$$+C(T)\left\|u_1(x,t)-u_2(x,t)\right\|_{B^5_{2,T}}. \tag{3.28}$$

Then, allowing for (3.25), it following from estimates (3.27), (3.28) that the operator Φ act in the sphere $K=K_R$ and is compressive. Therefore, in the sphere $K=K_R$ the operator Φ has a unique fixed point $\{u,a\}$, being the solution of the equation (2,26), i.e. it is a unique solution of the system in the sphere (3.15), (3.17) in the sphere $K=K_R$.

The function $u(x,t)$, as an element of the space $B^5_{2,T}$, has continuous derivatives $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $u_{xxx}(x,t)$, $u_{xxxx}(x,t)$ in D_T.

Similar to [10], we can show that $u_t(x,t)$, $u_{tt}(x,t)$, $u_{tt}(x,t)$, $u_{ttx}(x,t)$, $u_{ttxx}(x,t)$ are continuous in D_T.

It is easy to verify that equation (2.1) and conditions (2.1)-(2.3), and (2.5) are satisfied in the usual sense. So, $\{u(x,t),a(t)\}$ is the solution of the problem (2.1)-(2.3), (2.5), and by the corollary of lemma 3.1 it is unique. The theorem is proved. By means of Theorem 3.1 we prove the following Theorem.

Theorem 3.2 *Let all the conditions of theorem 3.2 and the argument terms*

$$\varphi(0) = h(0) - \int_0^T p_1(t)h(t)\,dt, \quad \psi(0) = h'0) - \int_0^T p_2(t)h(t)\,dt.$$

be fulfilled.

Then the problem (2.1) - (2.4) has in the sphere $K = K_R(||\,z\,||_{E_T^5} \leq R = A(T) + 2)$ *from* E_T^5 *a unique classic solution.*

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