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The influence of the fluid properties on the wave dispersion in the hollow cylinder with inhomogeneous initial stresses containing this fluid

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Abstract. *The paper studies the influence of the fluid properties on the dispersion of the axisymmetric waves propagating in the inhomogeneous pre-stressed hollow cylinder containing this fluid. The corresponding eigenvalue problem is formulated within the scope of the three-dimensional linearized theory of elastic waves in bodies with initial stresses and linearized Navier-Stokes equations for the inviscid compressible fluids. The discrete-analytical solution method is employed for the solution to the corresponding eigenvalue problem. Numerical results are presented for the cases where as fluids are selected Glycerin, Water, and Kerosene, however, as the material of the cylinder is selected as Steel. The dispersion curves are presented for each selected fluid case and these curves are compared with each other. Finally, according to this comparison, it is made concrete conclusions on the influence of the fluid properties on the velocity of the waves propagating in the inhomogeneous pre-stressed hollow cylinder contained these fluids. In particular, it is established that an increase in the values of the sound speed velocity of the fluid causes to increase in the propagation velocity of the studied axisymmetric waves.*
nonlinear, elliptic-parabolic, nonlinear boundary condition.

Keywords. wave dispersion · fluid properties · compressible inviscid fluid · inhomogeneous initial stresses · hollow cylinder + fluid system

Mathematics Subject Classification (2010): 74A60, 74E15

1 Introduction

The safety and control of the liquid transportations through the pipes (or hollow cylinders) which takes place in various branches of the modern industry requires fundamental theoretical investigations on the dynamics of the ”hollow cylinder + fluid” system. A review of these investigations is detailed in the paper [5] in which it is also investigated the influence of the

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inhomogeneous pre-stresses in the hollow cylinder on the dispersion of the axisymmetric waves propagating in the "hollow cylinder +fluid" system. What is more, the investigations carried out in the paper [5] are made within the scope of the three-dimensional linearized equations and relations of the theory of elastic waves in bodies with initial stresses [1, 7] and linearized Navier - Stokes equations for compressible inviscid fluids [8]. However, in the paper [5], concrete numerical results are obtained and analyzed for the case where the material of the cylinder is steel and the fluid is water.

In the present paper, we attempt to develop the investigations started in the paper [5] and to study how the fluid properties influence the velocity of the axisymmetric waves propagating in the inhomogeneous pre-stressed hollow cylinder contained this fluid. For this purpose, Glycerin, Water, and Kerosene are selected as fluids contained in the hollow cylinder and for each of these fluids, the dispersion curves are obtained and compared with each other. Numerical results are presented for the zeroth, first, and second modes for each selected fluid case. As in the paper [5], these results are obtained for various values of the fluid pressure which causes the inhomogeneous initial stresses in the cylinder.

2 Formulation of the problem

Consider a hydro-elastic system comprising the infinite hollow cylinder with inhomogeneous initial stresses and barotropic compressible inviscid fluid contained within this cylinder. We associate the cylindrical $Or\theta z$ and Cartesian $Ox_1x_2x_3$ ($x_3 = z$) (Fig. 1) systems of coordinates with the central axis of the cylinder. As usual, we distinguish two states, i.e. the initial and perturbed states in the hydro-elastic system under consideration and introduce the Lagrange and Euler coordinates for describing the motion of the cylinder and fluid respectively. In both initial and perturbed states, we will ignore the difference between these coordinates due to the insignificance of this difference in the case under consideration. Assume that in the initial state the fluid pressure p_0 acts on the interior of the cylinder and causes a static stress-strain state in that and it is called the "initial stress-strain state". Moreover, assume that in the initial state the fluid in the interior of the cylinder is rest.

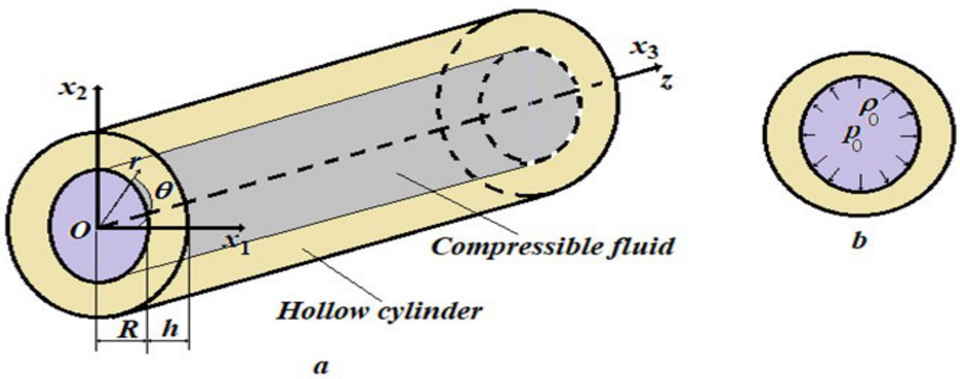

Fig. 1 The sketch of the hydro-elastic system under consideration

The mentioned initial stresses, according to the well-known Lame problem discussed almost in all textbooks related to the theory of elasticity (see, for instance, the book [10]), can be presented as follows.

$$\sigma_{rr}^{0} = \frac{p_0}{(1+h/R)^2 - 1}\left(1 - \frac{R^2}{r^2}\left(1 + \frac{h}{R}\right)^2\right),$$

$$\sigma^0_{\theta\theta} = \frac{p_0}{(1+h/R)^2-1}\left(1+\frac{R^2}{r^2}\left(1+\frac{h}{R}\right)^2\right),$$

$$\sigma^0_{zz}{=}\nu(\sigma^0_{rr}+\sigma^0_{\theta\theta}). \tag{2.1}$$

In (2.1) the conventional notation is used and the upper index "0" indicates the belonging of the related quantities to the initial stress state.

Thus, we assume that after appearing of the initial stress state in the cylinder, the hydro-elastic system gets a certain dynamical perturbation as a result of which the axisymmetric waves propagate in that. In the present paper, it is required to investigate how the properties of the fluid contained in the cylinder influence the dispersion of the mentioned waves in the presence of the initial inhomogeneous stresses in this cylinder. For carrying out this investigation we use the 3D linearized theory of elastic waves in initially stressed bodies for describing the motion of the cylinder and the linearized Navier-Stokes equations for describing the flow of the inviscid compressible barotropic fluid.

Now, according to the monograph [1, 8, 10], we write 3D linearized equations and corresponding relations describing the motion of the cylinder:

The equations of motion:

$$\frac{\partial t_{rr}}{\partial r}+\frac{\partial t_{zr}}{\partial z}+\frac{1}{r}(t_{rr}-t_{\theta\theta})=\rho\frac{\partial^2 u_r}{\partial t^2},\frac{\partial t_{rz}}{\partial r}+\frac{1}{r}t_{rz}+\frac{\partial t_{zz}}{\partial z}=\rho\frac{\partial^2 u_z}{\partial t^2}, \tag{2.2}$$

where

$$t_{rr}=\sigma_{rr}+\sigma^0_{rr}(r)\frac{\partial u_r}{\partial r},t_{rz}=\sigma_{rz}+\sigma^0_{rr}(r)\frac{\partial u_z}{\partial r},t_{\theta\theta}=\sigma_{\theta\theta}+\sigma^0_{\theta\theta}(r)\frac{u_r}{r},$$

$$t_{zr}=\sigma_{zr}+\sigma^0_{zz}(r)\frac{\partial u_r}{\partial z},t_{zz}=\sigma_{zz}+\sigma^0_{zz}(r)\frac{\partial u_z}{\partial z}. \tag{2.3}$$

The elasticity relations:

$$\sigma_{(jj)}=\lambda(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{zz})+2\mu\varepsilon_{(jj)},(jj)=rr;\,\theta\theta;\,zz,\sigma_{rz}=2\mu\varepsilon_{rz}. \tag{2.4}$$

The strain-displacement relations:

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\varepsilon_{\theta\theta}=\frac{u_r}{r},\varepsilon_{zz}=\frac{\partial u_z}{\partial z},\varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right). \tag{2.5}$$

In (2.2) and (2.3) the notation t_{rr}, t_{rz}, $t_{\theta\theta}$, $t_{zr}$and t_{zz} shows the components of the non-symmetric Kirchhoff stress tensor and the other notation used in (2.2) – (2.5) is conventional.

Thus, the motion in the cylinder with initial stresses described in (2.1) is described by the complete system of the linearized equations and relations in (2.2) – (2.5).

According to [8], the fluid flow into the cylinder is described by utilizing the following system of linearized Navier-Stokes equations for barotropic compressible inviscid fluids.

The continuity equation:

$$\frac{\partial\rho'}{\partial t}+\rho_0\left(\frac{\partial V_r}{\partial r}+\frac{V_r}{r}+\frac{\partial V_z}{\partial z}\right)=0; \tag{2.6}$$

Linearized equations of the fluid flow:

$$\frac{\partial V_r}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p'}{\partial r},\frac{\partial V_z}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p'}{\partial z}. \tag{2.7}$$

The state equation

$$a_0^2 = \frac{\partial p'}{\partial \rho'}, \tag{2.8}$$

where a_0 is the sound speed in the fluid and ρ_0 is the density of the fluid in the initial state.

The system of equation in equations (2.6) – (2.8) compose the complete system of equations within the scope of which the flow of the fluid in the perturbed state is described.

Note that the foregoing systems of equations are supplied with the following boundary and compatibility conditions.

The boundary conditions on the external surface of the cylinder:

$$t_{rr}|_{r=R+h} = 0,\, t_{rz}|_{r=R+h} = 0. \tag{2.9}$$

The compatibility conditions on the interface surface between the fluid and cylinder, i.e. on the internal surface of the cylinder:

$$t_{rr}|_{r=R} = -p',\, t_{rz}|_{r=R} = 0,\, \left.\frac{\partial u_r}{\partial t}\right|_{r=R} = V_r|_{r=R}\,. \tag{2.10}$$

The condition on boundedness of the quantities related to the fluid at the central axis of the cylinder:

$$\left\{|p'|\,, |\rho'|\,, |V_r|\,, |V_z|\right\}\big|_{r=0} < \infty. \tag{2.11}$$

This completes the mathematical formulation of the problem under consideration.

3 On the method of solution

For the solution to the system of equations (2.2) – (2.5) we employ the discrete- analytical method developed and employed in the references [2 - 5]. According to this method, the interval $[R, R+h]$ is divided into the N number sub-intervals which are determined through the expression $(R+(n-1)h/N) \leq r \leq (R+nh/N)$ where $1 \leq n \leq N$. It is assumed that the inhomogeneous initial stresses determined by expression (2.1) are homogeneous ones in each sub-intervals and the values of these stresses are determined as follows

$$\sigma_{rr}^0(r) \approx \sigma_{rr}^0(r_n), \sigma_{\theta\theta}^0(r) \approx \sigma_{\theta\theta}^0(r_n), \sigma_{zz}^0(r) \approx \sigma_{zz}^0(r_n),$$
$$r_n = R + (n-1)h/N + h/(2N). \tag{3.1}$$

Moreover, it is assumed that the following full contact conditions satisfy on the interfaces between the sub-intervals.

$$t_{rr}^1|_{r=R} = -p',\, t_{rz}^1|_{r=R} = 0,\, \left.\frac{\partial u_r^1}{\partial t}\right|_{r=R} = V_r|_{r=R}\,,$$
$$t_{rr}^1|_{r=R+h/N} = t_{rr}^2|_{r=R+h/N}\,,$$
$$t_{rz}^1|_{r=R+h/N} = t_{rz}^2|_{r=R+h/N}\,,\, u_r^1|_{r=R+h/N} = u_r^2|_{r=R+h/N}\,,$$
$$u_z^1|_{r=R+h/N} = u_z^2|_{r=R+h/N}\,, ...,$$
$$t_{rr}^{n-1}|_{r=R+(n-1)h/N} = t_{rr}^n|_{r=R+(n-1)h/N}\,,$$
$$t_{rz}^{n-1}|_{r=R+(n-1)h/N} = t_{rz}^n|_{r=R+(n-1)h/N}\,,$$
$$u_r^{n-1}|_{r=R+(n-1)h/N} = u_r^n|_{r=R+(n-1)h/N}\,,$$

$$times\ u_z^{n-1}\big|_{r=R+(n-1)h/N} = u_z^n\big|_{r=R+(n-1)h/N}, ...,$$

$$t_{rr}^N\big|_{r=R+h} = 0,\ t_{rz}^N\big|_{r=R+h} = 0. \tag{3.2}$$

By directed verification it is established that there are $4N+1$ conditions in (3.2) where the number N is determined from the convergence requirement of the numerical results which will be discussed below. Note that the upper indices in (3.2) and below indicate the number of the corresponding sub-interval.

Thus, using the assumptions in (3.1), it is obtained the following equations of motion from equations (2.2) and (2.3) which are satisfied in each n^{th} sub-interval separately.

$$\frac{\partial\sigma_{rr}^n}{\partial r} + \sigma_{rr}^0(r_n)\frac{\partial^2 u_r^n}{\partial r^2} + \frac{\partial\sigma_{zr}^n}{\partial z} + \sigma_{zz}^0(r_n)\frac{\partial^2 u_r^n}{\partial z^2} + \frac{1}{r}(\sigma_{rr}^n - \sigma_{\theta\theta}^n) +$$

$$+\sigma_{rr}^0(r_n)\frac{1}{r}\frac{\partial u_r^n}{\partial r} - \sigma_{\theta\theta}^0(r_n)\frac{u_r^n}{r^2} = \rho\frac{\partial^2 u_r^n}{\partial t^2},$$

$$\frac{\partial\sigma_{rz}^n}{\partial r} + \sigma_{rr}^0(r_n)\frac{\partial^2 u_z^n}{\partial r^2} + \frac{1}{r}\sigma_{rz}^n + \sigma_{rr}^0(r_n)\frac{1}{r}\frac{\partial u_z^n}{\partial r} +$$

$$+\frac{\partial\sigma_{zz}^n}{\partial z} + \sigma_{zz}^0(r_n)\frac{\partial^2 u_z^n}{\partial z^2} = \rho\frac{\partial^2 u_z^n}{\partial t^2}. \tag{3.3}$$

We add to these equations the elasticity relation (2.4) and the relation between deformations and displacements (2.5) rewritten for each subinterval separately and for the solution to the system of equations (3.3), (2.4), and (2.5) we employ the classical Lame decomposition (see, for instance, the monograph [6]), which for the axisymmetric problems can be written as follows.

$$u_r^n = \frac{\partial\Phi^n}{\partial r} + \frac{\partial^2\Psi^n}{\partial r\partial z},\ u_z^n = \frac{\partial\Phi^n}{\partial z} - \frac{\partial^2\Psi^n}{\partial r^2} - \frac{\partial\Psi^n}{r\partial r}. \tag{3.4}$$

Substituting the expressions in (3.4) into the system equations (2.5), (2.4), and (3.3), and doing corresponding cumbersome mathematical calculations we obtain the following equations for the potential functions Φ^n and Ψ^n in (3.4).

$$\left(1 + \frac{\sigma_{rr}^0(r_n)}{\lambda+2\mu}\right)\frac{\partial^2\Phi^n}{\partial r^2} + \left(1 + \frac{\sigma_{\theta\theta}^0(r_n)}{\lambda+2\mu}\right)\frac{\partial\Phi^n}{r\partial r} +$$

$$+\left(1 + \frac{\sigma_{zz}^0(r_n)}{\lambda+2\mu}\right)\frac{\partial^2\Phi^n}{\partial z^2} = \frac{1}{(c_1)^2}\frac{\partial^2\Phi^n}{\partial t^2},$$

$$\left(1 + \frac{\sigma_{rr}^0(r_n)}{\mu}\right)\frac{\partial^2\Psi^n}{\partial r^2} + \left(1 + \frac{\sigma_{\theta\theta}^0(r_n)}{\mu}\right)\frac{\partial\Psi^n}{r\partial r} +$$

$$+\left(1 + \frac{\sigma_{zz}^0(r_n)}{\mu}\right)\frac{\partial^2\Psi^n}{\partial z^2} = \frac{1}{(c_2)^2}\frac{\partial^2\Psi^n}{\partial t^2}. \tag{3.5}$$

In (3.5) the notation $c_1 = \sqrt{(\lambda+2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$ is used. Moreover, in the cases where $\sigma_{zz}^0(r_n) = 0$, $\sigma_{rr}^0(r_n) = 0$ and $\sigma_{\theta\theta}^0(r_n) = 0$, the equations in (3.5) coincide with the corresponding equations of classical elastodynamics [6].

Representing the functions Φ^n,u_r^n,σ_{rr}^n, $\sigma_{\theta\theta}^n$ and σ_{zz}^n with the multiplying $\sin(kz-\omega t)$and the functions Ψ^n, u_z^n and σ_{rz}^n with the multiplying $\cos(kz-\omega t)$, and denoting the amplitudes of the corresponding quantities with the same symbols, we obtain the following equations for the amplitudes of the potentials Φ^n and Ψ^n.

$$\frac{d^2\Phi^n}{d(r_2)^2} + \frac{\alpha_1(r_n)}{r_2}\frac{d\Phi^n}{dr_2} + \Phi^n = 0,\ \frac{d^2\Psi^n}{d(r_1)^2} + \frac{\alpha(r_n)}{r_1}\frac{d\Psi^n}{dr_1} + \Psi^n = 0, \tag{3.6}$$

where

$$\alpha(r_n) = \frac{1+\sigma^0_{\theta\theta}(r_n)/\mu}{1+\sigma^0_{rr}(r_n)/\mu},$$

$$\beta(r_n) = \frac{1+\sigma^0_{zz}(r_n)/\mu}{1+\sigma^0_{rr}(r_n)/\mu},$$

$$r_1^n = kr\sqrt{\frac{c^2}{(c_2)^2(1+\sigma^0_{rr}(r_n)/\mu)} - (\beta(r_n))^2},$$

$$c = \omega/\kappa, \alpha_1(r_n) = \frac{1+\sigma^0_{\theta\theta}(r_n)/(\lambda+2\mu)}{1+\sigma^0_{rr}(r_n)/(\lambda+2\mu)},$$

$$\beta_1(r_n) = \frac{1+\sigma^0_{zz}(r_n)/(\lambda+2\mu)}{1+\sigma^0_{rr}(r_n)/(\lambda+2\mu)},$$

$$r_2^n = kr\sqrt{\frac{c^2}{(c_1)^2(1+\sigma^0_{rr}(r_n)/(\lambda+2\mu))} - (\beta_1(r_n))^2}. \qquad (3.7)$$

According to [2, 11], the solution to the equations in (3.7) are found as follows.

$$\Phi^n = A_1^n(r_2)^{\gamma_1(r_n)}E_{\gamma_1(r_n)}(r_2^{n_i}) + A_2^n(r_2)^{\gamma_1(r_n)}F_{\gamma_1(r_n)}(r_2^n), \qquad (3.8)$$

$$\Psi^n = B_1^n(r_1)^{\gamma(r_n)}E_{\gamma(r_n)}(r_1^n) + B_2^n(r_1)^{\gamma(r_n)}F_{\gamma(r_n)}(r_1^n), \qquad (3.9)$$

where A_1^n, A_2^n, B_1^n and B_2^n are unknown constants and

$$\gamma_1(r_n) = (1-\alpha_1(r_n))/2, \gamma(r_n) = (1-\alpha(r_n))/2$$

$$E_{\gamma_1(r_n)}(r_2^n) = \begin{cases} J_{\gamma_1(r_n)}(r_2^n) \text{ if } (r_2^n)^2/r^2 > 0 \\ I_{\gamma_1(r_n)}(r_2^n) \text{ if } (r_2^n)^2/r^2 < 0 \end{cases},$$

$$F_{\gamma_1(r_n)}(r_2^n) = \begin{cases} Y_{\gamma_1(r_n)}(r_2^n) \text{ if } (r_2^n)^2/r^2 > 0 \\ K_{\gamma_1(r_n)}(r_2^n) \text{ if } (r_2^n)^2/r^2 < 0 \end{cases},$$

$$E_{\gamma(r_n)}(r_1^n) = \begin{cases} J_{\gamma(r_n)}(r_1^n) \text{ if } (r_1^n)^2/r^2 > 0 \\ I_{\gamma(r_n)}(r_1^n) \text{ if } (r_1^n)^2/r^2 < 0 \end{cases},$$

$$F_{\gamma(r_n)}(r_1^{n_i}) = \begin{cases} Y_{\gamma(r_{n_i})}(r_1^n) \text{ if } (r_1^n)^2/r^2 > 0 \\ K_{\gamma(r_n)}(r_1^n) \text{ if } (r_1^{n_i})^2/r^2 < 0 \end{cases}. \qquad (3.10)$$

In (3.10), $J_\delta(x)$ and $I_\delta(x)$ are the Bessel and modified Bessel functions of the first kind, however, $Y_\delta(x)$ and$K_\delta(x)$ are also the Bessel and Modified Bessel functions of the second kind.

Substituting these solutions into the presentations (3.4) we determine the expressions for displacements and then using the relations in (2.5) and 4) we obtain the expressions for the stresses within each sub-intervals. In this way we determine the analytic expressions related to motion of the cylinder within in each sub-intervals into which is divided the region $[R, R+h]$.

To reduce the volume of the paper the explicit forms of these expressions for displacements and stresses which enter the conditions in (3.2) are not presented here.

Consider also the determination of the quantities related to the flow of the fluid and for this purpose, according to [8], we use the following presentations for the general solution to equations in (2.6) – (2.8).

$$\rho' = a_0^{-2}\rho_0\left(-V_z^0\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)\Phi_f, p' = \rho_0\left(-V_z^0\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)\Phi_f,$$

$$V_r = \frac{\partial}{\partial r}\Phi_f, V_z = \frac{\partial}{\partial z}\Phi_f, \tag{3.11}$$

where

$$\left[\Delta - \frac{1}{a_0^2}\left(\frac{\partial}{\partial t}+V_z^0\frac{\partial}{\partial z}\right)^2\right]\Phi_f = 0, \Delta = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}. \tag{3.12}$$

Representing the functions V_z, p' and ρ' by multiplying $\sin(kz-\omega t)$, and the functions Φ_f and V_r by multiplying $\cos(kz-\omega t)$, we obtain the following equation from (3.12) for Φ_{f1} (where $\Phi = \Phi_{f1}(r)\cos(kz-\omega t)$).

$$\left(\frac{d^2}{dr_3^2}+\frac{1}{r_3}\frac{d}{dr_3}+1\right)\Phi_{f1}(r) = 0,$$

$$r_3 = kr\sqrt{\left(\frac{c}{a_0}\right)^2 + 2\frac{c}{a_0}\frac{V_z^0}{a_0}+\left(\frac{V_z^0}{a_0}\right)^2 - 1}. \tag{3.13}$$

According to the conditions in (2.11), the solution to equation (3.13) is found as follows.

$$\Phi_{f1}(r) = \begin{cases} FJ_0(r_3)\ if\ r_3^2 > 0 \\ FI_0(r_3)\ if\ r_3^2 < 0 \end{cases} \tag{3.14}$$

where $J_0(r_3)$ ($I_0(r_3)$) is the first kind Bessel (modified Bessel) function of the zeroth order and $F$is a unknown constant.

Using the expression (3.14) and substituting $\Phi = \Phi_{f1}(r)\cos(kz-\omega t)$into the equations in (3.12) we obtain the following expressions for the sought values related to the fluid.

$$p' = \rho_0\left(V_z^0 k+\omega\right)\sin(kz-\omega t)\begin{cases} FJ_0(r_3)\ if\ r_3^2 > 0 \\ FI_0(r_3)\ if\ r_3^2 < 0 \end{cases},$$

$$\rho' = a_0^{-2}\rho_0\left(V_z^0 k+\omega\right)\sin(kz-\omega t)\begin{cases} FJ_0(r_3)\ if\ r_3^2 > 0 \\ FI_0(r_3)\ if\ r_3^2 < 0 \end{cases},$$

$$V_r = k\frac{dr_3}{dr}\cos(kz-\omega t)\begin{cases} -FJ_1(r_3)\ if\ r_3^2 > 0 \\ FI_1(r_3)\ if\ r_3^2 < 0 \end{cases},$$

$$V_z = -k\sin(kz-\omega t)\begin{cases} FJ_0(r_3)\ if\ r_3^2 > 0 \\ FI_0(r_3)\ if\ r_3^2 < 0 \end{cases}. \tag{3.15}$$

After this determination, according to the foregoing discussions, we establish that the analytical expressions of the sought values contain $4N+1$ number unknown constants and these constants are A_1^n, A_2^n, B_1^n, B_2^n $(n = 1,2,...,N)$) and F. Using the $4N+1$ number of the conditions in (3.2) we obtain the system of homogeneous algebraic equations with respect to the mentioned unknown constants. According to the well-known procedure, equating to zero the determinant of the coefficient matrix of this system we obtain the dispersion equation. This equation can be formally presented as follows.

$$\det[a_{nm}(c/c_2, kR, p_0/\mu, \rho/\rho_0, h/R, a_0/c_2)] = 0, n\,; m = 1,2,...,4N+1. \tag{3.16}$$

The dispersion equation (3.16) is solved numerically by employing the "bi-section" method.

This completes the consideration of the solution procedure related to the problem under consideration.

4 Numerical results and discussions

As noted above, the aim of the present investigation is to determine how the fluid properties influences the dispersion curves of the axisymmetric waves propagating in the hydro-elastic system under consideration. For this purpose, as fluids, we select the water, kerosene and glycerin the mechanical properties for which, according to [8], are given in Table 1.

Table 1. The values of the mechanical constants of the fluids

Mechanical properties	The selected fluids		
	Kerosene	Water	Glycerin
Density ρ_0	$820 kg/m^3$	$1000 kg/m^3$	$1260 kg/m^3$
Sound speed a_0	$1324 m/s$	$1459.5 m/s$	$1927 m/s$

We assume that the material of the cylinder is steel with the Lame constants $\lambda = 1.075 \times 10^{11} Pa$, $\mu = 0.77 \times 10^{11} Pa$, with the material density $\rho = 7910 kg/m^3$and with the shear wave propagation velocity $c_2 = 3142 m/s$.

First, we consider the validation of the PC programs and solution method employed in the present paper and for this purpose, we use the numerical results obtained in the paper [9]. Note that in the paper [9] it is also considered the dispersion of the axisymmetric waves propagating in the hollow cylinder contained inviscid compressible fluid and under obtaining numerical results steel is taken as a material of the cylinder and water is taken as a fluid. Moreover, in the paper [9] it is assumed that $h/R = 0.2$ and the dispersion diagrams, i.e. the graphs of the dependencies between $\omega h/c_2$ and $kh$are constructed for some modes. It is evident that in the "steel + water" case under $h/R = 0.2$ and p_0/μ the corresponding results obtained by the present PC programs and algorithm must coincide with those obtained in the paper [9].

Taking this statement into consideration, by utilizing the present solution algorithm we construct the mentioned dispersion diagrams for the zeroth, first and second modes under various values of the ratio p_0/μ and these diagrams are presented in Fig. 2 and under obtaining the results for the cases where $p_0/\mu > 0$ it is assumed that $N = 50$ in the dispersion equation (3.16). Note that dispersion diagrams obtained in the case where $p_0/\mu = 0$and drawn by the dashed lines in Fig. 2 coincide completely with the corresponding diagrams obtained in the paper [9]. Consequently, this coinciding illustrates the trustiness of the used PC programs and solution method employed in the present investigation. At the same time, the dispersion diagrams presented in Fig. 2 show that after a certain value of kh the inhomogeneous initial stresses in the hollow cylinder cause to increase in the wave propagation velocity.

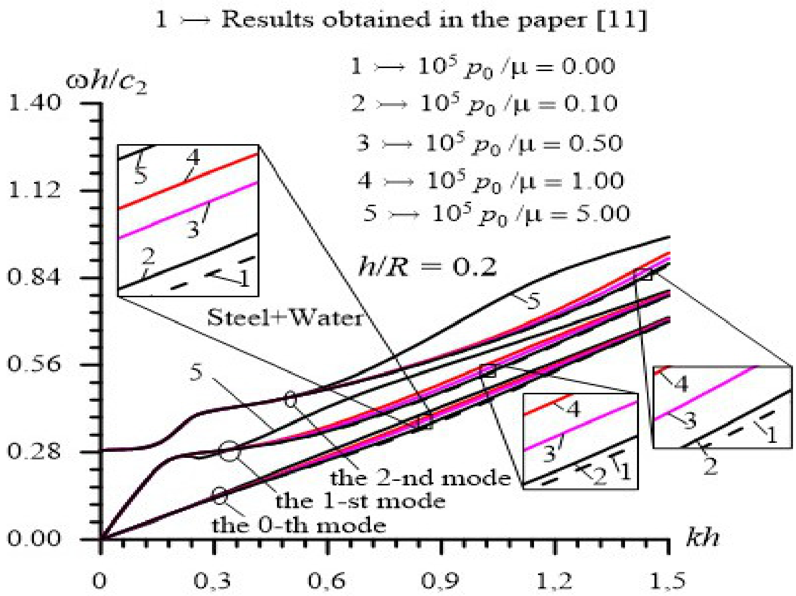

Fig. 2 Dispersion diagrams obtained for the "steel + water" system under various values of the ratio p_0/μ in the case where $h/R = 0.2$

Now we consider numerical results illustrated how the change of the fluid properties contained in the interior of the cylinder influence the wave propagation velocity under various values of the ratio p_0/μ. For this purpose consider dispersion curves, i.e. the graphs of the dependencies between the c/c_2 and kR for the fluid cases shown in Table 1. These dispersion curves for the zeroth, first and second modes are illustrated in Figs. 3, 4, and 5 respectively and these dispersion curves are constructed for various values of the ratio p_0/μ under $h/R = 0.2$.

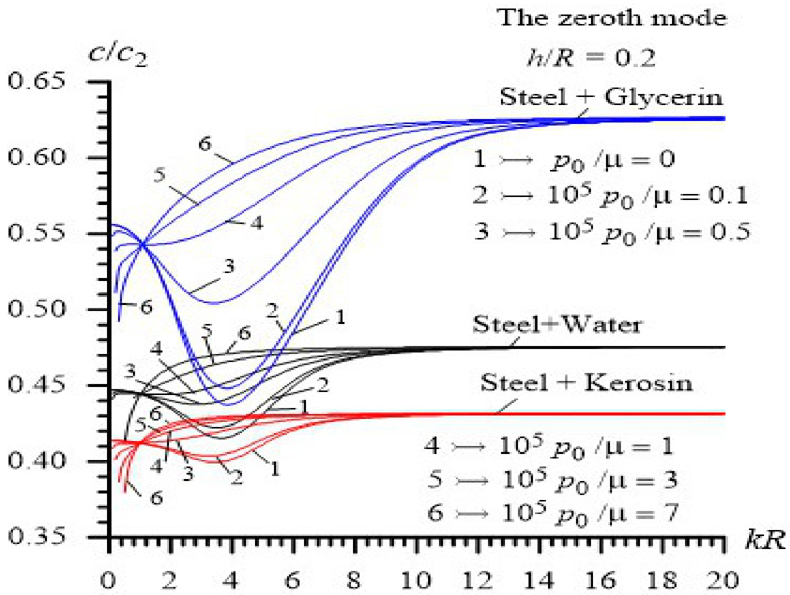

Fig. 3 Dispersion curves related to the zeroth mode for the "steel + glycerin", "steel +water", and "steel + kerosene" cases under various p_0/μ

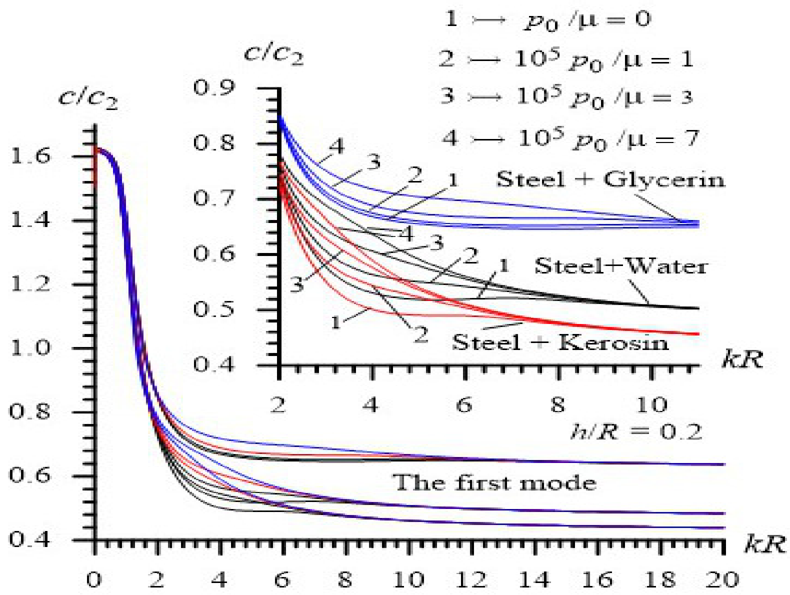

Fig. 4 Dispersion curves related to the zeroth mode for the "steel + glycerin", "steel +water", and "steel + kerosene" cases under various p_0/μ

We attempt to analyze the foregoing results and we begin this analysis with the results are given in Fig. 3 which relate to the zeroth mode and this mode appears namely as a result of the existence of the fluid in the interior of the cylinder. We recall that sometimes the zeroth mode is called the quasi-Scholte

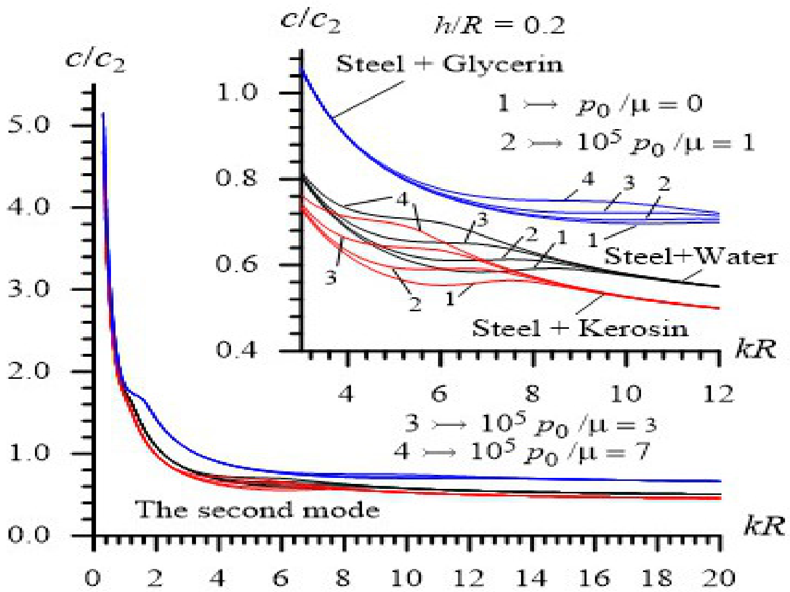

Fig. 5 Dispersion curves related to the zeroth mode for the "steel + glycerin", "steel +water", and "steel + kerosene" cases under various p_0/μ

waves mode the propagation velocity on which approach the velocity of the corresponding Scholte waves as $kR \to \infty$. Thus, it follows from Fig. 3 that for the zeroth mode there exist such value of the dimensionless wavenumber kR (denote it by $(kR)*$) after which, i.e. under $kR >(kR)*$ (before which, i.e. under $kR <(kR)*$) an increase of the fluid pressure in the initial state causes to increase (to decrease) of the wave propagation velocity of the quasi-Scholte waves. At the same time, these results show that in the case where $kR = (kR)*$ the initial inhomogeneous stresses in the cylinder do not influence the propagation velocity of the quasi-Scholte waves.

According to Table 1, we can write the following relations:

$$a_{0Gl}/c_{2St} = 1/1.5913 > a_{0Wt}/c_{2St} = 1/2.101 > a_{0Kr}/c_{2St} = 1/2.316,$$

$$\rho_{0Gl}/\rho_{St} = 0.15384 > \rho_{0Wt}/\rho_{St} = 0.1282 > \rho_{0Kr}/\rho_{St} = 0.10512, \qquad (4.1)$$

where $c_{2St}$is the shear wave propagation velocity in the steel, a_{0Gl}, a_{0Wt}, and a_{0Kr} are the sound speed in the Glycerin, Water and Kerosene respectively, ρ_{St} is the material density of the steel, ρ_{0Gl}, ρ_{0Wt}, and ρ_{0Kr} are the material density of the Glycerin, Water and Kerosene, respectively. Consequently, the sequence “steel + glycerin”, “steel + water”, and “steel + kerosene” means a decrease simultaneously in the values of the sound speed in the fluid and in the values of the fluid material density.

According to the well-known physico-mechanical considerations, an increase in the values of the sound speed in the fluid must cause to increase in the wave velocity propagating in the hydroelastic system for the concrete selected material for the cylinder. Moreover, the parametric study of the influence of the ratio ρ_0/ρ on the wave propagation velocity in the hydro-elastic system under consideration shows that under fixed values of ratio a_0/c_2 the character of this influence depends on the mode's number and on the values of the dimensionless wavenumber kR. For instance, in the modes, the appearance of which conditioned with the existence of the fluid (as in the zeroth mode) a decrease in the values of the fluid density causes to increase in the wave propagation velocity. As well as in the parts of the dispersion curves of the higher modes the appearance of which also conditioned with the existence of the fluid (for instance, in the first and second modes after a certain value of the wavenumber kR) a decrease in the values of the fluid density causes to increase in the wave propagation velocity. However, in the cases where the wave propagation velocity is determined mainly with the cylinder (as in the first and second modes before a certain value of the wavenumber kR), vice versa, a decrease in the values of the fluid density causes to increase insignificantly the wave propagation velocity. After mentioned wavenumber, the influence of the existence of the fluid on the wave propagation velocity becomes significant and again a decrease in the fluid density causes to increase in the wave propagation velocity. With all this, it should be noted that the magnitude of the influence of the ratio $a_0/c_2$is much bigger than that of the ratio ρ_0/ρ.

Namely, with the foregoing statements, it is explained the character of the results illustrated not only in Fig.3 but also the character of the results illustrated in Figs. 4 and 5. In other words, the character of the influence of the mechanical properties of the selected fluids on the dispersion curves in the modes under consideration is explained namely with the relations in (4.1).

We analyze the graphs illustrated in Figs. 4 and 5 from which follows that unlike the zeroth mode the influence of the difference of the fluid on the dispersion curves in the first and second modes appears after a certain value of the wavenumber kR (denote it by $(kR)'$). It follows from the numerical results that within the certain accuracy we can take $(kR)' = 1.5$ for the considered cases. Consequently, in the cases where $kR < (kR)'$ the dispersion curves obtained for the various fluid cases join with each other as $(kR)' > kR \to 0$. At the same time, under $kR > (kR)'$ the difference between the dispersion curves obtained for various fluid cases increase with kR and in the first mode approach the corresponding Scholte wave propagation velocity from "above" as $kR \to \infty$. In the second mode, the velocity of the waves also approaches certain asymptotes the values of which are greater than that of the corresponding Scholte waves.

According to the results illustrated in Figs. 4 and 5, it can be concluded that in the low wavenumber approach, i.e. in the cases where $kR < 1.5$the first and second modes of the dispersion curves similar to the corresponding dispersion curves related to the corresponding empty cylinder, however, in the high wavenumber approach, i.e. in the cases where $kR > 1.5$ the behavior of the first and second modes the dispersion curves approach the corresponding ones related to the waves propagating in a hypothetical liquid column.

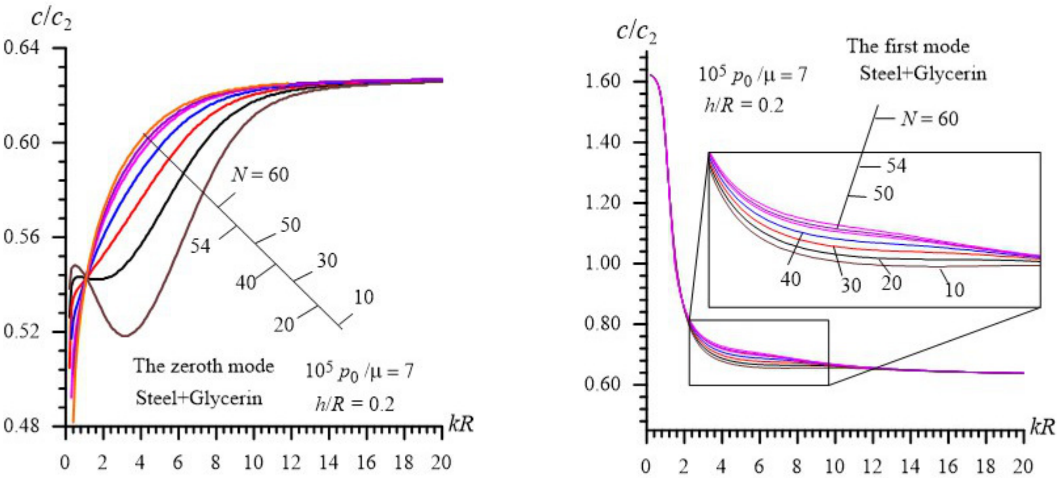

Fig. 6 Convergence of the numerical results with respect to the number N for the "Steel + Glycerin" for the zeoth (a) and first (b) modes

Namely, with the foregoing statements it is explained the character of the influence of the initial inhomogeneous stresses in the hollow cylinder on the wave dispersion curves obtained for the first and second modes shown in Figs. 4 and 5. According to these curves, the influence of the ratio p_0/μ on the mentioned curves becomes significant in the cases where $1.5 < kR < (kR)''$and under $kR > (kR)''$insignificant. Note the values of $(kR)''$depend on the number of the mode and of the selected pair of materials.

This completes the analysis of the numerical results which are obtained in the case where $N = 50$. We recall that N is the number of the sub-intervals into which the interval$[R, R + h]$ is divided. Now we consider numerical results illustrated the convergence of those with respect to the number N. These results are shown in Fig. 6 for the "Steel + Glycerin" case for the zeroth (Fig. 6a) and first (Fig. 6b) modes. It follows from Fig. 6 that the case where N=50 not only in the qualitative but also in the quantitative sense quite enough for obtaining numerical results with acceptable accuracy.

5 Conclusions

Thus, in present paper the influence of the fluid properties on the dispersion of the axisymmetric waves propagating in the inhomogeneous pre-stressed hollow cylinder containing this fluid is studied. The corresponding eigenvalue problem is formulated within the scope of the three-dimensional linearized theory of elastic waves in bodies with initial stresses and linearized Navier-Stokes equations for the inviscid compressible fluids. The discrete-analytical solution method is employed for the solution to the corresponding eigenvalue problem. Numerical results are presented for the cases where as fluids are selected Glycerin, Water, and Kerosene, however, as the material of the cylinder is selected as Steel. The dispersion curves are presented for each selected fluid case for the zeroth, first and second modes, and these curves are analyzed and compared with each other. According to this analysis, it can be made the following concrete conclusions:

-For selected fluids there exists the relations $c_{St+Gl} > c_{St+Wt} > c_{St+Kr}$ for each considered modes, for each value of the initial fluid pressure p_0/μ and for each selected value of the dimensionless wavenumber kR, where c_{St+Gl}, c_{St+Wt} and $c_{St+Kr}$are the wave propagation velocity in the "Steel + Glycerin", "Steel + Water", and "Steel + Kerosene" cases respectively;

- The character obtained results are explained with the relations in (4.1) and with the more significance of the influence of the sound speed in the fluid on the wave propagation velocity in the hydro-elastic system under consideration than the density of the fluid;

- For each selected pairs of materials in the quantitative sense the significant influence of the initial inhomogeneous stresses in the cylinder on the wave propagation velocity in the first and second modes appear within a certain finite change range of the dimensionless wavenumber kR, however, in the zeroth mode this influence is significant for all the considered change range of the kR.

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