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The estimates for solution boundary value problem for biharmonic equations

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Abstract. *The estimates for the solutions boundary value problem for biharmonic equations are obtained. A problem in generalized Morrey spaces is considered. The solvability of problem in bounded smooth domain is proved. Also $L_{p,\lambda} \to L_{q,\lambda}$ reqularity estimates are obtained.*

Keywords. biharmonic equation · generalized Morrey spaces · mixed boundary value problem · regularity.

Mathematics Subject Classification (2010): 35125, 35B40

1 Introduction

The estimates for the solutions mixed boundary value problem for the biharmonic equations in generalized Morrey spaces are obtained. The better inclusion between the Morrey and Holder spaces permits to obtain regularity of the solution to boundary problems.

The properties of the classical Morrey spaces we refer to [6]. For study generalized Morrey spaces we refer to [7], [5]. These are new functional spaces and having many applications in the differential equations theory see also [1], [3], [2], [4].

The domain $\Omega \subset R^n$, $n \geq 2$ is supposed to be bounded with $\partial\Omega = \Gamma_1 \cup \Gamma_2$.

Definition 1.1 *Let $\varphi : \Omega \times R_+ \to R_+$ be a measurable function and $1 \leq p \leq \infty$. For any domain Ω, the generalized Morrey space $M_{p,\varphi}(\Omega)$ consist of all $f \in L_{p,loc}(\Omega)$*

$$\|f\|_{M_{p,\varphi}(\Omega)} = \sup_{x\in\Omega, 0<r<d} \varphi^{-1}(x,r)\, r^{-\frac{n}{p}} \|f\|_{L_p(\Omega(x,r))} < \infty,$$

where $d = \sup\limits_{x,y\in\Omega} |x-y|$, $B(x,r) = \{y \in R^n : |x-y| < r\}$ *and* $\Omega(x,r) = \Omega \cap B(x,r)$.

In the case of $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$, the generalized Morrey space $M_{p,\varphi}$ is a classical Morrey space $L_{p,\lambda}$.

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Definition 1.2 *The generalized Sobolev-Morrey space $W_{m,p,\varphi}(\Omega)$ consists of all Sobolev functions $u \in W_{m,p}(\Omega)$ with distubutional derivatives $D^s u \in M_{p,\varphi}(\Omega)$, endowed with the norm*

$$\|u\|_{W_{m,p,\varphi}(\Omega)} = \sum_{0 \le s \le m} \|D^s u\|_{M_{p,\varphi}(\Omega)}.$$

The space $W_{m,p,\varphi}(\Omega) \cap W_{1,p,0}(\Omega)$ consist of all functions $u \in W_{m,p}(\Omega) \cap W_{1,p,0}(\Omega)$ with $D^s u \in M_{p,\varphi}(\Omega)$ and endowed with the same norm. $W_{1,p,0}(\Omega)$ in the closure functions of $C^{\infty}(\Omega)$ vanising on Γ_1, with respect to the norm in $W_p^1(\Omega)$.

We consider mixed boundary value problem for biharmonic equation

$$\Delta^2 u = f \qquad in \qquad \Omega, \tag{1.1}$$

$$u|_{\Gamma_1} = \frac{\partial u}{\partial n}|_{\Gamma_1} = 0 \qquad and \qquad \frac{\partial^2 u}{\partial n^2}|_{\Gamma_2} = g. \tag{1.2}$$

Now we give estimates for the Green function and the Poisson kernels for the solution of the problem (1.1)-(1.2). Later we will obtain a priore estimates for the solution and the solvability of problem (1.1)-(1.2) in generalized Morrey spaces.

Let $G_2(x,y)$ be the Green function and $K_j(x,y)$, $j = 0,1$ be the Poisson kernels of problem (1.1)-(1.2). Then the solution of this problem can be written as

$$u(x) = \int_{\Omega} G_2(x,y) f(y)\, dy + \sum_{j=0}^{1} \int_{\partial\Omega} K_j(x,y) g(y)\, d\sigma_y.$$

It is known that there is a constant C such that

$$|G_2(x,y)| \le C d(x) d(y) \min\left\{1, \frac{d(x) d(y)}{|x-y|^2}\right\}, \tag{1.3}$$

where d is the distance between x and the boundary $\partial\Omega$

$$d(x) = \inf_{\tilde{x} \in \partial\Omega} |x - \tilde{x}|. \tag{1.4}$$

It follows from (1.3)that the solution of problem (1.1)-(1.2) satisfies for appropriate of at $g = 0$ following estimate

$$\left\|u d^{-2}\right\|_{L_\infty(\Omega)} \le C \|f\|_{L_1(\Omega)},$$

$$\|u\|_{L_\infty(\Omega)} \le C \left\|f d^2\right\|_{L_1(\Omega)}.$$

2 Main results.

Now we give main result.

Theorem 2.1 *Let $\Omega \subset R^n,\ n \ge 2$ be a bounded domain with $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Let $1 < p < \infty$ and the pair of functions (φ_1, φ_2) satisfy the condition*

$$\int_r^d \frac{ess \inf_{t<s<\infty} \varphi_1(x,s)\, s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C \varphi_2(x,r), \tag{2.1}$$

where $d = \sup_{x,y\in\Omega} |x-y|$ *and constant* C *is independent of* $x \in \Omega$ *and* $r > 0$*. We assume that* $f \in M_{p,\varphi}$ *and* $u(x)$ *is a solution of problem (1.1)-(1.2). Then there exist a constant* C *which depends only on* n, φ_1, φ_2 *and* Ω *such that*

$$\|u\|_{W_{2,p,\varphi_2,0}(\Omega)} \le C \|f\|_{M_{p,\varphi_1}(\Omega)}. \tag{2.2}$$

Proof. The proof follows from estimates from the Green function and the Poisson kernels of the problem (1.1)-(1.2). Also we used inequalities

$$|u(x)| + |D_{x_i} u(x)| \le CMf(x), \tag{2.3}$$

$$\left|D_{x_i x_j} u(x)\right| \le C\left(Kf(x) + Mf(x) + |f(x)|\right), \tag{2.4}$$

hear $Mf(x)$-maximal operator, $Kf(x)$ is a singular Calderon-Zygmund operator.

Theorem 2.2 *Let* $G_2(x,y)$ *be the Green function of the problem (1.1)-(1.2). Then for every* $x, y \in \Omega$ *the following estimates hold:*

1. if $4-n>0$*, then*

$$|G_2(x,y)| \le d^{2-\frac{1}{2}n}(x) \min\left(1, \frac{d(x)\,d(y)}{|x-y|^2}\right)^{\frac{n}{2}},$$

2. if $4-n=0$*, then*

$$|G_2(x,y)| \le \log\left(1 + \frac{d(x)\,d(y)}{|x-y|^2}\right)^2,$$

3. if $4-n<0$*, then*

$$|G_2(x,y)| \le |x-y|^{4-n} \min\left(1, \frac{d(x)\,d(y)}{|x-y|^2}\right)^2.$$

Also we have following for the Posson kernels

Theorem 2.3 *Let* $K_j(x,y),\ j=0,1$ *be the Poisson kernels of the problem (1.1)-(1.2). Then for every* $x \in \Omega, y \in \partial\Omega$ *the following estimates holds:*

$$|K_j(x,y)| \le \frac{d^2(x)}{|x-y|^{n-j+1}}. \tag{2.5}$$

We prove estimates for $G_2(x,y)$ and $K_j(x,y)$ depending on the distance to the boundary. We will do so by estimating the j-th derivative through an integration of the $(j+1)$-th derivative along a path to the boundary. Distance to the boundary $d(x)$ will depend on the proportionality between the arch which joins internal point with the boundary. The arch will be constructed explicitly.

For the proof we used integral representation of solution of problem (1.1)-(1.2)

$$D_x^2 D_y^2 G_2(x,y) = D_x^2 D_y^2 G_2(\tilde{x},y) + \int_{\gamma_x^y} \nabla_z D_z^2 D_y^2 G_2(x,y)\,dz, \tag{2.6}$$

where $\tilde{x} \in \partial\Omega$, γ_x^y arch which above defined.

3 Estimates for the solution.

We define $K_\gamma(x) = |x|^{-\gamma}$ and

$$(K_\gamma * f)(x) = \int_\Omega |x-y|^{-\gamma} f(y)\, dy.$$

The following lemma holds.

Lemma 3.1 *Let $\Omega \subset R^n$ be bounded domain, $\gamma < n$ and $1 < p < \frac{n-\lambda}{p}$. Then condition $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n-\lambda}$ is necessary and sufficient for the boimdednese of $K_\gamma * f$ from $L_{p,\lambda}(\Omega)$ to $L_{q,\lambda}(\Omega)$ and there exist is constant C such that for all $f \in L_p$*

$$\|K_\gamma * f\|_{L_{q,\lambda}(\Omega)} \le C \|f\|_{L_{p,\lambda}(\Omega)}. \tag{3.1}$$

As a consequence of the pointwize estimates and using lemma 3.1, we next state the optimal $L_{p,\lambda} \to L_{q,\lambda}$ regularity results mentioned before.

Theorem 3.1 *Let $u \in C^4(\overline{\Omega})$ and $f \in C(\overline{\Omega})$ satisfy of the problem (1.1)-(1.2). 1. If $4 > n$, then there exists $C > 0$ such that for all $\theta \in [0,1]$*

$$\left\|d^{-m+n\theta}(\cdot)\, u\right\|_{L_\infty(\Omega)} \le C \left\|d^{m-(1-\theta)n}(\cdot)\, f\right\|_{L_1(\Omega)}. \tag{3.2}$$

2. Let $1 < p < \frac{n-\lambda}{\gamma}$. If $\left(\frac{1}{p} - \frac{1}{q}\right) = \frac{\gamma}{n-\lambda} < \min\left\{\frac{4}{n,1}\right\}$, $\alpha \in \left\{\left(\frac{1}{p} - \frac{1}{q}\right) = \frac{\gamma}{n-\lambda}, \min\left\{1, \frac{4}{n}\right\}\right\}$.
Then there exists $C > 0$ such that for all $\theta \in [0,1]$

$$\left\|d^{-m+\theta n\alpha}(\cdot)\, u\right\|_{L_{q,\lambda}} \le C \left\|d^{m-(1-\theta)n\alpha}(\cdot)\, f\right\|_{L_{p,\lambda}}. \tag{3.3}$$

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