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To solve the problem of propagation of unsteady waves in limited areas

Mubariz B. Rasulov and Gulnar R. Mirzayeva

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Abstract. *The paper presents a method based on the application of a new type of functionally invariant solution, which is simultaneously a two-fold Laplace-Fourier transformant and ensures the reduction of elastic dynamics equations to algebraic ones.*

Keywords. wave equation · permutation of parameters · algebraic expression.

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1 Introduction

Currently, the problems of propagation of unsteady waves in an elastic half-space have been studied in sufficient detail. At the same time, there are practically no analytical solutions to the corresponding problems for finite domains. Recently, in [3], a new type of functionally invariant solutions of the wave equation was presented, which has a unique property - preserving its appearance after applying two-fold Laplace-Fourier transformations to them, it changes the dimensionless parameter.

2 Solution method

This paper is devoted to identifying some properties of these equations in order to apply them to solving problems for finite domains, as well as reducing the equations of plane problems of elastic dynamics to algebraic ones.

The result published in [3] and obtained using Efros' theorem is presented in the form:

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{ikx}\int\limits_{0}^{\infty}\frac{F(\theta)}{\sqrt{t^2c^2-x^2-y^2}}e^{-st}dxdt = e^{y\sqrt{s^2\overline{c}^2+k^2}}\frac{F\left(\overline{\theta}\right)}{\sqrt{s^2\overline{c}^2+k^2}}, \qquad (2.1)$$

Mubariz B. Rasulov
Institute of Mathematics and Mechanics of NAS Azerbaijan
Baku, Azerbaijan

E-mail: mubariz1950@mail.ru

where

$$\theta(x,y,t)=\frac{tc\sqrt{t^2c^2-x^2-y^2}+ixy}{t^2c^2-y^2},\qquad t^2c^2\geq x^2+y^2,\ y\leq 0 \tag{2.2}$$

$$\overline{\theta}(s,k)=\sqrt{\frac{s^2\overline{c}^2}{k^2}+1}$$

S, *k* parameters of Laplace and Fourier transforms. Solving [2] with respect to t we get:

$$tc=\frac{x}{\sqrt{1-\theta^2}}-iy\frac{\theta}{\sqrt{1-\theta^2}} \tag{2.3}$$

Representation of equation (2.3) in integral (Laplace) (2.1), performing the replacement of variable integration sequentially according to the scheme $t\to\theta(x,y,t)\to\overline{\theta}(s,\tau)\to\tau$ and taking into account the obvious relations

$$\frac{dt}{\sqrt{t^2c^2-x^2-y^2}}=\frac{d\theta}{1-\theta^2};\qquad \frac{d\bar{\theta}}{1-\theta^2}=\frac{d\tau}{\sqrt{s^2c^2+\tau^2}};$$

and also, by reducing it to the complex form of the Fourier integral [2], we obtain another proof of the validity of the result obtained in [3].
To study the behavior of these functions when differentiating (integrating) in *x*, *y* and taking into account that the right part (2.1) is multiplied by, respectively

$$ik=\frac{s}{c}\left(1-\theta^2\right)^{-\frac{1}{2}}=\frac{s}{c}\mathrm{ch}\overline{p} \tag{2.4}$$

$$\sqrt{s^2\overline{c}^2+k^2}=i\frac{s}{c}\left(1-\theta^2\right)^{-\frac{1}{2}}\overline{\theta}=i\frac{s}{c}\mathrm{sh}\overline{p},\ \ \overline{\theta}=tn\overline{p}$$

Given the zero initial data and the properties of the Laplace and Fourier transforms in differentiation

$$\varphi=\frac{F(\theta)}{\sqrt{t^2c^2-x^2-y^2}}=0\ \text{ for } t=0$$

from (2.1), taking into account (2.4) and (2.1), it can be argued that:

$$\begin{aligned}\frac{\partial^n\varphi}{\partial x^n}&=\frac{1}{c^n}\frac{\partial^n}{\partial t^n}\left(\varphi(\text{-ch}p^{\pm})^n\right)\\ \frac{\partial^n\varphi}{\partial y^n}&=\frac{1}{c^n}\frac{\partial^n}{\partial t^n}\left((\pm i\text{sh}p^{\pm})^n\right)\end{aligned} \tag{2.5}$$

Here the $\pm$ sign takes into account the waves propagating in positive and negative directions.
Here, to simplify calculations, it is accepted:

$$\theta(x,\pm y,t)=thp^{\pm};$$

$$\left(1-\theta^2\right)^{-\frac{1}{2}}=chp^{\pm}=\frac{1}{r^2}\left(tcx\pm iy\sqrt{t^2c^2-x^2-y^2}\right)$$

$$\theta\left(1-\theta^2\right)^{-\frac{1}{2}}=shp^{\pm}=\frac{1}{r^2}\left(x\sqrt{t^2c^2-x^2-y^2}\pm iytc\right),\ \ r^2=x^2+y^2$$

A general satisfying solution for displacements in terms of ψ and ω.

$$U_i=a^2\psi,i+2b^2\left(\omega_{ij,j}+\omega_{ik,k}\right)-\text{not summation.} \tag{2.6}$$

We consider the potential of expansion and displacement, which are associated with horizontal and vertical movements using [4]:

$\psi = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} -$ dilatation;

$\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) -$ rotation;

$a^2 = \frac{\lambda + 2\mu}{\rho}$ - the speed of a dilatation wave in a solid;

$b^2 = \frac{\mu}{\rho} -$the velocity of a rotational wave in a solid;

$\lambda, \mu -$Lame coefficient, $\rho -$density.

Substituting (2.6) into the expression for the stresses

$\sigma_{ij} = \lambda \psi \delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right),\ \delta_{ij}$ - Kronecker symbol,

$$\begin{aligned} \ddot{\sigma}_{ii} &= \lambda \ddot{\psi} + 2\mu a^2 \psi_{,ii} + 4\mu b^2\left(\omega_{ij,ij} + \omega_{ik,ik}\right) \\ \ddot{\sigma}_{i,j} &= 2\mu a^2 \psi_{,ij} + 2\mu \ddot{\omega}_{ij} + 4\mu b^2\left(\omega_{ij,ij} + \omega_{ik,ik}\right) \end{aligned} \tag{2.7}$$

It is taken into account here: $\frac{\partial \omega_1}{\partial x} + \frac{\partial \omega_2}{\partial y} + \frac{\partial \omega_3}{\partial z} = 0$

$$\begin{aligned} \omega_1 &= \omega_{32} = -\omega_{23} \\ \omega_2 &= \omega_{13} = -\omega_{13} \\ \omega_3 &= \omega_{13} = -\omega_{12} \end{aligned}$$

ψ_i and ω_i satisfy the wave equations

$$\Delta\psi - \frac{1}{a^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

$$\Delta\omega_i - \frac{1}{b^2}\frac{\partial^2\omega_i}{\partial t^2} = 0$$

For flat problems, expressions have the following form:

$$\ddot{u}_1 = a_1^2 \frac{\partial\psi}{\partial x} - b^2 \frac{\partial\omega_3}{\partial y}$$

$$\ddot{u}_2 = a_1^2 \frac{\partial\psi}{\partial y} + b^2 \frac{\partial\omega_3}{\partial x}$$

$$\ddot{\sigma}_{yy} = \lambda\ddot{\psi} + 2\mu a^2 \frac{\partial^2\psi}{\partial y^2} - 4\mu b^2 \frac{\partial^2\omega_3}{\partial x \partial y}$$

$$\ddot{\sigma}_{xy} = 2\mu a^2 \frac{\partial^2\psi}{\partial x \partial y} - 2\mu\ddot{\omega}_3 + 4\mu b^2 \frac{\partial^2\omega}{\partial x^2}$$

The solution of two-dimensional wave equations is represented in the form:

$$\begin{aligned} \psi(x,y,t) &= \frac{\psi^+\left(thp^+\right)}{\sqrt{t^2a^2 - x^2 - y^2}} + \frac{\psi^-\left(thp^-\right)}{\sqrt{t^2a^2 - x^2 - y^2}} \\ \omega_3(x,y,t) &= \frac{\omega_3{}^+\left(thq^+\right)}{\sqrt{t^2b^2 - x^2 - y^2}} + \frac{\omega_3{}^-\left(thq^-\right)}{\sqrt{t^2b^2 - x^2 - y^2}} \end{aligned},$$

where

$$thp^{\pm} = \frac{ta\sqrt{t^2a^2 - x^2 - y^2} + ixy}{t^2a^2 - y^2}$$

$$thq^{\pm} = \frac{tb\sqrt{t^2b^2 - x^2 - y^2} + ixy}{t^2b^2 - y^2}$$

Using (2.5) in (2.6) and applying integration over t to both parts, we obtain expressions for displacements and stresses in matrix form $S = T\phi$, where:

$$S(y) = [\dot{u}(y), \dot{v}(y), \sigma_{yy}(y), \sigma_{yx}(y)]^T$$

$$\phi(y) = [\psi^+(y), \omega^+(y), \psi^-(y), \omega^-(y)]^T$$

$$T(y) = \begin{vmatrix} -achp^+ \; 2ibshq^+ \; -achp^- \; 2ibshq^- \\ -ashp^+ \; -2bchq^+ \; -iashp^- \; -2bchq^- \\ \lambda - \mu sh^2p^+ \; -2i\mu sh2q^+ \; \lambda - 2\mu sh^2p^+ \; 2\mu ish2q^- \\ i\mu sh2p^+ \; 2\mu ch2q^+ \; i\mu sh2p^- \; 2\mu ch2q^- \end{vmatrix}$$

$$p^{\pm} = iaratg\left(\pm\frac{y}{x}\right) + \ln\left(\frac{ta}{r} + \sqrt{\left(\frac{ta}{r}\right)^2 - 1}\right) \text{ as } t > \frac{r}{a}$$
$$q^{\pm} = iaratg\left(\pm\frac{y}{x}\right) + \ln\left(\frac{tb}{r} + \sqrt{\left(\frac{tb}{r}\right)^2 - 1}\right) \text{ as } t > \frac{r}{b}$$
$$q^{\pm} = iaratg\left(\pm\frac{y}{x}\right) + \ln\left(\frac{tb}{r} + i\sqrt{1 - \left(\frac{tb}{r}\right)^2}\right) \text{ as } \frac{r}{a} < t < \frac{r}{b}$$

For the first time, the matrix method for stationary waves was proposed by William T. Thomson [4], and W. Dunkin [1] obtained a similar result for non-stationary waves in Laplace and Fourier images.

3 Conclusion

Using new types of functionally invariant solutions for wave equations, algebraic expressions for displacements and stresses are obtained, which allow solving non-stationary problems of elastic dynamics.

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