Tran. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mechanics, **41** (8), 21–29 (2021).

Analysis of an axially-symmetric problem of elasticity theory for a radially inhomogeneous sphere under mixed boundary conditions on lateral surfaces

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Received: 18.05.2021 / Revised:17.07.2021 / Accepted: 14.10.2021

Abstract. *An axially -symmetric problem of elasticity theory for a radially-inhomogeneous isotropic open sphere containing none of the poles 0 and π, is considered. It is assumed that the elasticity module are linear functions of the radius of the sphere. It is supposed that mixed homogeneous boundary conditions are given on the lateral surfaces of the sphere, arbitrary stresses keeping the sphere in equilibrium are given on the conical sections. The solutions are structured and after fulfilling homogeneous boundary conditions given on lateral surfaces of the sphere, a characteristic equation for a spectral parameter is obtained. The classification of the roots of the characteristic equation with respect to a small parameter characterizing the thickness of the sphere, is performed. Asymptotic solutions dependent on the roots of the characteristic equation are constructed. The behavior of the constructed solutions is studied both in the inside of the sphere and near conical sections. Asymptotic expansions of solutions allowing to calculate the stress – stain state are obtained. It is shown that the constructed solutions are of boundary lager character localized in conical sections.*

Keywords. equilibrium equation · boundary layer · edge effect Legendre function · characteristic equation

Mathematics Subject Classification (2010): 74B10

1 Introduction

Inhomogeneous materials are widely used in various fields of engineering. Various materials whose characteristics, in particular elasticity module, can continuously vary along some directions, are developed and created [4]. These materials have unique advantages compared to traditional materials. The study of inhomogeneous shells occupies a special place in theory of shells [17].

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There is a number of studies on three-dimensional problems of elasticity theory. An elasticity theory problem for a sphere was studied by Saint-Venant [9]. In [7] based on elasticity theory equations for a sphere, a general solution satisfying boundary conditions on the contour in Saint-Venant's sense was obtained, and the stress –strain state of the sphere was analyzed. In [10], homogeneous solutions dependent on the roots of a transcendental equation were constructed for a thick isotropic sphere based on the elasticity theory equations. In [16], the exactness of the existing applied theories was studied for a small thickness sphere on the base of solutions of there-dimensional elasticity theory problems and a method for constructing refined applied theories was given. In [12] three-dimensional asymptotic theory of a transversally isotropic sphere of small thickness was stated. A system of homogeneous solutions of a not axially-symmetric problem of elasticity theory for a thick isotropic and transversally isotropic sphere was constructed in [13]. In [5], a three dimensional stress-strain state of a three-layer sphere with a soft filler was analyzed. A torsional [1] problem for a radially-layered sphere with arbitrary number of alternating soft and hard layers was studied. The existence of weakly damping boundary layer solutions and possible violation of the Saint-Venant principle in its classical formulation was shown.

In [8], by means of the finite elements and spline collocation methods a problem of elasticity theory was studied for a radially-inhomogeneous hollow sphere. The results obtained by means of the finite elements and spline-collocation methods were compared. In [2] an axially-symmetric problem of elasticity theory for a radially-inhomogeneous transversally-isotropic sphere of small thickness was studied by the method of asymptotic integration of elasticity theory equations. An axially-symmetric problem of elasticity theory was studied for a small thickness sphere with variable elasticity module in [3] by the method of homogeneous solutions. Asymptotic formulae for displacements and stresses were obtained.

In [14], an axially-symmetric thermoelastic problem for a radially-inhomogeneous spherical shell was considered. Considering the Poisson ratio constant, in the case of exponential and power law of change of the Young modules, an analytic solution was obtained, the stress-stain state of the sphere was studied.

2 Setting boundary value problems for a radially-inhomogeneous sphere and constructing solutions

We consider an axially-symmetric problem of elasticity theory for a small thickness radially-inhomogeneous isotropic sphere. Suppose that the sphere contain none of the poles 0 and π (Fig.1). In the spherical system of coordinates, we denote the domain occupied by the sphere by $\Gamma = \{r \in [r_1; r_2], \theta \in [\theta_1; \theta_2], \varphi \in [0; 2\pi]\}$.

Assume that change in the modulus of elasticity along the radius occurs according to the linear law

$$G(r) = G_* r, \lambda(r) = \lambda_* r, \tag{2.1}$$

where G_* and λ_* are some constant quantities.

The system of equilibrium equations in the absence of mass forces in spherical system of coordinates r, θ, φ has the form [12]:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{2\sigma_{rr} - \sigma_{\varphi\varphi} - \sigma_{\theta\theta} + \sigma_{r\theta} ctg\theta}{r} = 0, \tag{2.2}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{3\sigma_{r\theta} + (\sigma_{\theta\theta} - \sigma_{\varphi\varphi}) ctg\theta}{r} = 0, \tag{2.3}$$

where σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\varphi\varphi}$, $\sigma_{\theta\theta}$ –are stress tensor components expressed by the displament vectors $\upsilon_r = \upsilon_r(r, \theta)$, $\upsilon_\theta = \upsilon_\theta(r, \theta))$ as followrs [12]:

$$\sigma_{rr} = (2G + \lambda)\frac{\partial \upsilon_r}{\partial r} + \frac{\lambda}{r}\left(2\upsilon_r + \upsilon_{\theta} ctg\theta + \frac{\partial \upsilon_\theta}{\partial \theta}\right), \tag{2.4}$$

$$\sigma_{\varphi\varphi} = (2G+\lambda)\frac{v_\theta}{r}ctg\theta + \lambda\left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\bullet\frac{\partial v_\theta}{\partial \theta}\right) + 2(G+\lambda)\frac{v_r}{r}, \quad (2.5)$$

$$\sigma_{\theta\theta} = (2G+\lambda)\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \lambda\left(\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}ctg\theta\right) + 2(G+\lambda)\frac{v_r}{r}, \quad (2.6)$$

$$\sigma_{r\theta} = G\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\right) \quad (2.7)$$

Substituting (2.4)-(2.7) in (2.2),(2.3), allowing for (2.1) we get an equilibrium equation in displacements

$$(2G_0+\lambda_0)\frac{\partial^2 u_\rho}{\partial \rho^2} + 2\varepsilon(2G_0+\lambda_0)\frac{\partial u_\rho}{\partial \rho} - 4G_0\varepsilon^2 u_\rho - 3G_0\varepsilon^2\left(\frac{\partial u_\theta}{\partial \theta} + u_\theta ctg\theta\right) +$$

$$+\varepsilon(G_0+\lambda_0)\left(\frac{\partial u_\theta}{\partial \rho}ctg\theta + \frac{\partial^2 u_\theta}{\partial\theta\partial\rho}\right) + \varepsilon^2 G_0\left(\frac{\partial^2 u_\rho}{\partial \theta^2} + \frac{\partial u_\rho}{\partial \theta}ctg\theta\right) = 0, \quad (2.8)$$

$$G_0\frac{\partial^2 u_\theta}{\partial \rho^2} + 2\varepsilon G_0\frac{\partial u_\theta}{\partial \rho} + (5G_0+2\lambda_0)\varepsilon^2\frac{\partial u_\rho}{\partial \theta} + \varepsilon(G_0+\lambda_0)\frac{\partial^2 u_\rho}{\partial\rho\partial\theta} +$$

$$+\varepsilon^2(2G_0+\lambda_0)\left(\frac{\partial u_\theta}{\partial \theta}ctg\theta + \frac{\partial^2 u_\theta}{\partial \theta^2} - u_\theta ctg^2\theta\right) - \varepsilon^2(\lambda_0+3G_0)u_\theta = 0, \quad (2.9)$$

Here $u_\rho = \frac{v_r}{r_0}$, $u_\theta = \frac{v_\theta}{r_0}$ are displacement vector components; $G_0 = \frac{r_0 G_*}{t}$, $\lambda_0 = \frac{\lambda_* r_0}{t}$–are pure variables; $t-$ is some characteristic parameter with the dimension of shear modulus; $\rho = \frac{1}{\varepsilon}\cdot\ln\left(\frac{r}{r_0}\right)$ – is a new radial variable; $\varepsilon = \frac{1}{2}\ln\left(\frac{r_2}{r_1}\right)$ – is a small parameter characterizing the thickness of the sphere; $r_0 = \sqrt{r_1 r_2}$; $\rho \in [-1;1]$.

Assume that on the lateral part of the boundary of the sphere we are given the boundary conditions

$$u_\rho = 0 \text{ for } \rho = \pm 1 \quad (2.10)$$

$$\sigma_{\rho\theta} = 0 \text{ for } \rho = \pm 1 \quad (2.11)$$

Assume that the following stresses are given on the ends of the sphere (on conical sections)

$$\sigma_{\theta\theta}|_{\theta=\theta_n} = f_{1n}(\rho), \sigma_{\rho\theta}|_{\theta=\theta_n} = f_{2n}(\rho). \quad (2.12)$$

Here $f_{1n}(\rho)$. $f_{2n}(\rho)$ $(n = 1;2)-$ are rather smooth functions satisfying the equilibrant conditions.

We will look for the solution of (2.8)-(2.11) in the form

$$u_\rho(\rho,\theta) = a(\rho)m(\theta), \;\; u_\theta(\rho,\theta) = d(\rho)m'(\theta), \quad (2.13)$$

where the function $m(\theta)$ satisfies the Legendre equation [12]:

$$m''(\theta) + ctg\theta\cdot m'(\theta) + \left(z^2 - \frac{1}{4}\right)m(\theta) = 0. \quad (2.14)$$

Substituting (2.13) in (2.8)-(2.11), allowing for (2.14), we get the following boundary value problems with a spectral parameter z:

$$(2G_0+\lambda_0)(a''(\rho) + 2\varepsilon a'(\rho)) - \varepsilon^2 G_0\left(z^2 + \frac{15}{4}\right)a(\rho) +$$

$$+\varepsilon\left(z^2-\frac{1}{4}\right)\left(3\varepsilon G_0 d(\rho)-(G_0+\lambda_0)d'(\rho)\right)=0, \quad (2.15)$$

$$G_0(d''(\rho)+2\varepsilon d'(\rho))-\varepsilon^2\left(\left(z^2-\frac{1}{4}\right)(2G_0+\lambda_0)+G_0\right)d(\rho)+$$

$$+\varepsilon^2(5G_0+2\lambda_0)a(\rho)+\varepsilon(G_0+\lambda_0)a'(\rho)=0, \quad (2.16)$$

$$a\,(\rho)=0 \; as \; \rho=\pm 1 \quad (2.17)$$

$$G_0\left(d'\,(\rho)+\varepsilon\,(a\,(\rho)-d\,(\rho))\right)=0 \text{ as } \rho=\pm 1 \quad (2.18)$$

The solution of the system (2.15),(2.16) is of the form:

$$\begin{array}{l} a(\rho)=e^{-\varepsilon\rho}\left[p_1e^{\varepsilon s_1\rho}A_1+p_1e^{-\varepsilon s_1\rho}A_2+p_2e^{\varepsilon s_2\rho}A_3+p_2e^{-\varepsilon s_2\rho}A_4\right]\,, \\ d(\rho)=e^{-\varepsilon\rho}\left[t_1e^{\varepsilon s_1\rho}A_1+q_1e^{-\varepsilon s_1\rho}A_2+t_2e^{\varepsilon s_2\rho}A_3+q_2e^{-\varepsilon s_2\rho}A_4\right]\,, \end{array} \quad (2.19)$$

where $A_n(n=\overline{1,4})$ are arbitrary constants; $q_{\,k}=(G_0+\lambda_0)s_k-(4G_0+\lambda_0)$; $p_{\,k}=G_0s_k^2-\left(z^2-\frac{1}{4}\right)(2G_0+\lambda_0)-2G_0$; $t_{\,k}=-(G_0+\lambda_0)s_k-(4G_0+\lambda_0)$; s_k are the roots of the equation

$$\begin{array}{l}(2G_0+\lambda_0)G_0s^4-\left[2G_0(\lambda_0+2G_0)\left(z^2-\frac{1}{4}\right)+10G_0^2+3G_0\lambda_0\right]s^2+ \\ +G_0(2G_0+\lambda_0)\left(z^2-\frac{1}{4}\right)^2-2G_0^2\left(z^2-\frac{1}{4}\right)+2G_0(6G_0+\lambda_0)=0.\end{array}$$

Satisfying boundary conditions (2.17), (2.18) with respect to $A_n(n=\overline{1;4})$, we get the following homogeneous system of linear algebraic equations

$$\begin{cases} p_1e^{-\varepsilon s_1}A_1+p_1e^{\varepsilon s_1}A_2+p_2e^{-\varepsilon s_2}A_3+p_2e^{\varepsilon s_2}A_4=0; \\ D_{11}e^{-\varepsilon s_1}A_1+D_{21}e^{\varepsilon s_1}A_2+D_{12}e^{-\varepsilon s_2}A_3+D_{22}e^{\varepsilon s_2}A_4=0; \\ p_1e^{\varepsilon s_1}A_1+p_1e^{-\varepsilon s_1}A_2+p_2e^{\varepsilon s_2}A_3+p_2e^{-\varepsilon s_2}A_4=0; \\ D_{11}e^{\varepsilon s_1}A_1+D_{21}e^{-\varepsilon s_1}A_2+D_{12}e^{\varepsilon s_2}A_3+D_{22}e^{-\varepsilon s_2}A_4=0; \end{cases} \quad (2.20)$$

From the condition of existence of non-trivial solutions of this system, we have a characteristic equation for determining z :

$$\varDelta=(p_1D_{22}-p_2D_{21})(p_2D_{11}-p_1D_{12})\cdot sh^2(\varepsilon(s_1+s_2))+$$

$$+(p_1D_{12}-p_2D_{21})(p_1D_{22}-p_2D_{11})\cdot sh^2(\varepsilon(s_2-s_1))=0 \quad (2.21)$$

where

$$D_{ik}=-\lambda_0s_k^2+(-1)^i(2G_0-\lambda_0)s_k-\left(z^2-\frac{1}{4}\right)(2G_0+\lambda_0)+(6G_0+2\lambda_0);$$

The transcendental equation (2.21) determines a countable set of roots z_k, and the appropriate constants A_1, A_2, A_3, A_4 are proportional to algebraic complement of the elements of any line of the determinant of the system (2.20). Choosing algebraic complement of the elements of the first line of the determinant of the system (2.20), we have

$$A_{1n}=A_n\varDelta_{11};A_{2n}=-A_n\varDelta_{12};A_{3n}=A_n\varDelta_{13};A_{4n}=-A_n\varDelta_{14}. \quad (2.22)$$

where

$$
\begin{aligned}
&\Delta_{11} = p_2 D_{21}(D_{22} - D_{12})e^{\varepsilon s_1} - D_{12}(p_1 D_{22} - p_2 D_{21})e^{-\varepsilon(s_1+2s_2)}+\\
&+ D_{22}(p_1 D_{12} - p_2 D_{21})e^{\varepsilon(2s_2-s_1)};\\
&\Delta_{12} = p_2 D_{11}(D_{22} - D_{12})e^{-\varepsilon s_1} - D_{12}(p_1 D_{22} - p_2 D_{11})e^{\varepsilon(s_1-2s_2)}+\\
&+D_{22}(p_1 D_{12} - p_2 D_{11})e^{\varepsilon(s_1+2s_2)};\\
&\Delta_{13} = D_{11}(p_1 D_{22} - p_2 D_{21})e^{\varepsilon(2s_1+s_2)} - D_{21}(p_1 D_{22} - p_2 D_{11})e^{\varepsilon(2s_1-s_2)}+\\
&+ D_{22} p_1 (D_{21} - D_{11})e^{\varepsilon s_2};\\
&\Delta_{14} = D_{11}(p_1 D_{12} - p_2 D_{21})e^{\varepsilon(s_2-2s_1)} - D_{21}(p_1 D_{12} - p_2 D_{11})e^{\varepsilon(2s_1+s_2)}+\\
&+ D_{12} p_1 (D_{21} - D_{11})e^{-\varepsilon s_2}
\end{aligned}
\tag{2.23}
$$

Substituting (2.22) in (2.19) and summing over all the roots of the characteristic equation (2.21), allowing for (2.13) we get solutions of the following form :

$$u_\rho = \sum_{k=1}^{\infty} A_k a_k(\rho) m_k(\theta), \tag{2.24}$$

$$u_\theta = \sum_{k=1}^{\infty} A_k d_k(\rho) m_k^{'}(\theta), \tag{2.25}$$

where

$$a_k(\rho) = e^{-\varepsilon\rho}\left[p_1 e^{\varepsilon s_1 \rho}\Delta_{11} - p_1 e^{-\varepsilon s_1 \rho}\Delta_{12} + p_2 e^{\varepsilon s_2 \rho}\Delta_{13} - p_2 e^{-\varepsilon s_2 \rho}\Delta_{14}\right],$$

$$d_k(\rho) = e^{-\varepsilon\rho}\left[t_1 e^{\varepsilon s_1 \rho}\Delta_{11} - q_1 e^{-\varepsilon s_1 \rho}\Delta_{12} + t_2 e^{\varepsilon s_2 \rho}\Delta_{13} - q_2 e^{-\varepsilon s_2 \rho}\Delta_{14}\right].$$

For stresses we have:

$$\sigma_{\rho\rho} = \varepsilon^{-1}\sum_{k=1}^{\infty} A_k\left[(2G_0+\lambda_0)a_k^{'}(\rho) + \varepsilon\lambda_0\left(2a_k(\rho) - \left(z_k^2 - \frac{1}{4}\right)d_k(\rho)\right)\right] m_k(\theta),$$

$$\sigma_{\rho\theta} = \varepsilon^{-1}\sum_{k=1}^{\infty} A_k G_0\left[d_k^{'}(\rho) + \varepsilon(a_k(\rho) - d_k(\rho))\right] m_k(\theta),$$

$$\sigma_{\theta\theta} = \varepsilon^{-1}\sum_{k=1}^{\infty} A_k\left\{\left[\lambda_0 a_k^{'}(\rho) + 2\varepsilon(G_0+\lambda_0)a_k(\rho) - \right.\right.$$

$$\left.\left.-\varepsilon\left(z_k^2 - \frac{1}{4}\right)(2G_0+\lambda_0)d_k(\rho)\right] m_k(\theta) - 2\varepsilon G_0 d_k(\rho) ctg\theta m_k^{'}(\theta)\right\},$$

$$\sigma_{\varphi\varphi} = \varepsilon^{-1}\sum_{k=1}^{\infty} A_k\left\{\left[\lambda_0 a_k^{'}(\rho) + 2\varepsilon(G_0+\lambda_0)a_k(\rho) - \right.\right.$$

$$\left.\left.-\varepsilon\left(z_k^2 - \frac{1}{4}\right)\lambda_0 d_k(\rho)\right] m_k(\theta) + 2\varepsilon G_0 d_k(\rho) ctg\theta m_k^{'}(\theta)\right\}$$

3 Constructing asymptotic formulas for displacements and stresses

Equation (2.21) has a countable set of roots with a condensation point at infinity and $\Delta(z,\varepsilon)$ even function of its own arguments. As in [12] we can show that all the zeros of the function $\Delta(z,\varepsilon)$ tend to ∞ as $\varepsilon \to 0$ and we can divide then into the following groups depending on their behavior as $\varepsilon \to 0$:
1^0) $\varepsilon z_k \to 0$ as $\varepsilon \to 0$.
2^0) $\varepsilon z_k \to const$ as $\varepsilon \to 0$.
3^0) $\varepsilon z_k \to \infty$ as $\varepsilon \to 0$.
Here only the case 2^0) $\varepsilon z_k \to const$ as $\varepsilon \to 0$.
We look for z_k in the form:

$$z_k = \frac{\delta_k}{\varepsilon} + O(\varepsilon). \tag{3.1}$$

After substituting (3.1) in characteristic equation (2.21), for δ_k we have

$$\delta^4 sh^2\delta ch^2\delta = 0 \tag{3.2}$$

We give asymptotic construction of solutions corresponding to the roots of the characteristic equation (2.21).

$$a) z_k = \varepsilon^{-1}(\delta_k + \varepsilon\delta_{1k} + ...). \tag{3.3}$$

$$u_\rho^{(3)}(\rho;\theta) = \sum_{k=1}^{\infty} D_k\delta_k^3 \left[(G_0+\lambda_0)ch\delta_k \cdot sh(\delta_k\rho) + O(\varepsilon)\right] m_k(\theta); \tag{3.4}$$

$$u_\theta^{(3)}(\rho;\theta) = \varepsilon\sum_{k=1}^{\infty} D_k\delta_k^2 \left[(G_0+\lambda_0)ch\delta_k \cdot ch(\delta_k\rho) + O(\varepsilon)\right] m_k'(\theta); \tag{3.5}$$

$$\sigma_{\rho\theta}^{(3)} = 2G_0(G_0+\lambda_0)\sum_{k=1}^{\infty} D_k\delta_k^3 \left[ch\delta_k \cdot sh(\delta_k\rho) + O(\varepsilon)\right] m_k'(\theta); \tag{3.6}$$

$$\sigma_{\rho\rho}^{(3)} = 2G_0(G_0+\lambda_0)\sum_{k=1}^{\infty} \frac{D_k\delta_k^4}{\varepsilon} \left[ch\delta_k \cdot ch(\delta_k\rho) + O(\varepsilon)\right] m_k(\theta); \tag{3.7}$$

$$\sigma_{\theta\theta} = 2G_0(G_0+\lambda_0)\sum_{k=1}^{\infty} \frac{D_k\delta_k^4}{\varepsilon} \left[-ch\delta_k \cdot ch(\delta_k\rho) + O(\varepsilon)\right] m_k(\theta); \tag{3.8}$$

$$\sigma_{\varphi\varphi} = O(1) \tag{3.9}$$

where δ_k are the solutions of the equation $\delta^2 sh^2\delta = 0$.

$$b) z_k = \varepsilon^{-1}(\delta_k + \varepsilon\delta_{1k} + ...). \tag{3.10}$$

$$u_\rho^{(3)}(\rho;\theta) = \sum_{k=1}^{\infty} F_k\delta_k^3 \left[(G_0+\lambda_0)sh\delta_k \cdot ch(\delta_k\rho) + O(\varepsilon)\right] m_k(\theta); \tag{3.11}$$

$$u_\theta^{(3)}(\rho;\theta) = \varepsilon\sum_{k=1}^{\infty} F_k\delta_k^2 \left[(G_0+\lambda_0)sh\delta_k \cdot sh(\delta_k\rho) + O(\varepsilon)\right] m_k'(\theta) \tag{3.12}$$

$$\sigma_{\rho\theta}^{(3)} = 2G_0(G_0+\lambda_0)\sum_{k=1}^{\infty} F_k\delta_k^3 \left[sh\delta_k \cdot ch(\delta_k\rho) + O(\varepsilon)\right] m_k'(\theta); \tag{3.13}$$

$$\sigma_{\rho\rho}^{(3)} = 2G_0(G_0+\lambda_0)\sum_{k=1}^{\infty}\frac{F_k\delta_k^4}{\varepsilon}\left[sh\delta_k\cdot sh(\delta_k\rho)+O(\varepsilon)\right]m_k(\theta); \quad (3.14)$$

$$\sigma_{\theta\theta}^{(3)} = 2G_0(G_0+\lambda_0)\sum_{k=1}^{\infty}\frac{F_k\delta_k^4}{\varepsilon}\left[-sh\delta_k\cdot sh(\delta_k\rho)+O(\varepsilon)\right]m_k(\theta); \quad (3.15)$$

$$\sigma_{\varphi\varphi}^{(3)} = O(1). \quad (3.16)$$

where δ_k are the solutions of the equation $\delta^2 ch^2\delta = 0$.

If $q_0(\theta) \neq 0 (\theta_1 < \theta < \theta_2)$; $Re\left(\lambda^{-1}\sqrt{q_0(\theta_0)}\right) \geq 0$ $(\lambda \in (0;\lambda_0))$ and as $\lambda \to 0$ the asymptotic expansion $q(\theta;\lambda) \sim \sum\limits_{k=0}^{\infty} q_k(\theta)\lambda^k$, is valid, then the principle term of the asymptotic solutions of the equation

$$\lambda^2 y''(\theta) - q(\theta;\lambda)y(\theta) = 0 \quad (3.17)$$

as $\lambda \to 0$ is of the form [6]:

$$y_{1,2}(\theta;\lambda) = q_0^{-\frac{1}{4}}(\theta)\exp\left[\pm\lambda^{-1}\int_{\theta_0}^{\theta}\sqrt{q_0(t)}dt + \frac{1}{2}\int_{\theta_0}^{\theta}\frac{q_1(t)}{\sqrt{q_0(t)}}dt\right](1+O(\varepsilon)). \quad (3.18)$$

The substitution

$$m(\theta) = \frac{y(\theta)}{\sqrt{\sin\theta}} \quad (3.19)$$

reduces the equation (2.14) to the form:

$$y''(\theta) + \left(z^2 + \frac{1}{4\sin^2\theta}\right)y(\theta) = 0. \quad (3.20)$$

Substituting (3.3), (3.10) in (3.20), based on (3.18) allowing for (3.19), for the third group of zeros the principle term of the asymptotic solution of the equation (2.14) as $\varepsilon \to 0$ takes the form:

$$m_k(\theta) =$$

$$= \begin{cases} \frac{1}{\sqrt{\delta_k\sin\theta}}\exp\left[-\varepsilon^{-1}\sqrt{\delta_k^2}(\theta-\theta_1)\right](1+O(\varepsilon)); \text{ in the vicinity of } \theta=\theta_1, \\ \frac{1}{\sqrt{\delta_k\sin\theta}}\exp\left[\varepsilon^{-1}\sqrt{\delta_k^2}(\theta-\theta_2)\right](1+O(\varepsilon)); \text{ in the vicinity of } \theta=\theta_2. \end{cases} \quad (3.21)$$

(3.4)-(3.9),(3.11)-(3.16) have a boundary layer character and is absent in Kirchhoff – Liav theory. The first terms of its asymptotic expansion are equivalent to Saint-Venant's edge effect of an inhomogeneous isotropic plate [15]. From (3.21) we obtain that when moving away from conical sections $\theta = \theta_j (j = 1, 2)$ the solutions decrease exponentially.

4 Satisfying the boundary conditions at the ends of the sphere

To determine the constants D_k ,F_k we use the Lagrange variation principle [11]. Since the solutions satisfy the equilibrium condition and boundary conditions on the lateral surface, the variation principle takes the following form [12]:

$$\sum_{j=1}^{2}\int_{-1}^{1}\left[(\sigma_{\theta\theta}-f_{1j}(\rho))\delta u_\theta + (\sigma_{\rho\theta}-f_{2j}(\rho))\delta u_\rho)\right]\Bigg|_{\theta=\theta_j} e^{2\varepsilon\rho}d\rho = 0. \quad (4.1)$$

Substituting (3.4)-(3.6),(3.8),(3.11)-(3.13),(3.15) in (4.1) and considering δD_k ,δF_k independent variations, from (4.1) we get an infinite system of linear algebraic equations:

a)

$$\sum_{k=1}^{\infty} {M'}_{kj} D_{k0} = \tau_j^{'} \tag{4.2}$$

$$M'_{kj} = \frac{2G_0(G_0+\lambda_0)\delta_j^2\delta_k^3}{\sqrt[4]{\delta_k^2\delta_j^2}}\left(\frac{1}{\sin\theta_2} - \frac{1}{\sin\theta_1}\right) ch\delta_k ch\delta_j \cdot$$

$$\cdot \left[\sqrt{-\delta_k^2}\delta_j \cdot \int\limits_{-1}^{1} sh(\delta_k\rho)sh(\delta_j\rho)d\rho - \sqrt{-\delta_j^2}\delta_k \cdot \int\limits_{-1}^{1} ch(\delta_k\rho)ch(\delta_j\rho)d\rho\right];$$

$$\tau_j^{'} = \sum_{s=1}^{2} \frac{\delta_j^2 ch\delta_j}{\sqrt[4]{-\delta_j^2}\cdot\sqrt{\sin\theta_s}} \cdot \left[\delta_j \int\limits_{-1}^{1} f_{2s}(\rho)sh(\delta_j\rho)d\rho + (-1)^s\cdot\sqrt{-\delta_j^2}\int\limits_{-1}^{1} f_{1s}(\rho)ch(\delta_j\rho)d\rho\right];$$

$$D_k = D_{k0} + \varepsilon D_{k1} + ...$$

b)

$$\sum_{k=1}^{\infty} M''_{kj} F_{k0} = \tau_j^{''} \tag{4.3}$$

$$M''_{kj} = \frac{2G_0(G_0+\lambda_0)\delta_j^2\delta_k^3}{\sqrt[4]{\delta_k^2\delta_j^2}}\left(\frac{1}{\sin\theta_2} - \frac{1}{\sin\theta_1}\right) sh\delta_k sh\delta_j \times$$

$$\times \left[\sqrt{-\delta_k^2}\delta_j \cdot \int_{-1}^{1} ch(\delta_k\rho)ch(\delta_j\rho)d\rho - \sqrt{-\delta_j^2}\delta_k \cdot \int_{-1}^{1} sh(\delta_k\rho)sh(\delta_j\rho)d\rho\right];$$

$$\tau_j^{''} = \sum_{s=1}^{2} \frac{\delta_j^2 sh\delta_j}{\sqrt[4]{-\delta_j^2}\cdot\sqrt{\sin\theta_s}} \cdot \left[\delta_j \int_{-1}^{1} f_{2s}(\rho)ch(\delta_j\rho)d\rho + (-1)^s\cdot\sqrt{-\delta_j^2}\int_{-1}^{1} f_{1s}(\rho)sh(\delta_j\rho)d\rho\right];$$

$$F_k = F_{k0} + \varepsilon F_{k1} + ...$$

The definition $D_{kn}, F_{kn}(n = 1, 2, ...$invariably is reduced to the systems whose matrices coincide with the matrices of the system (4.2),(4.3).

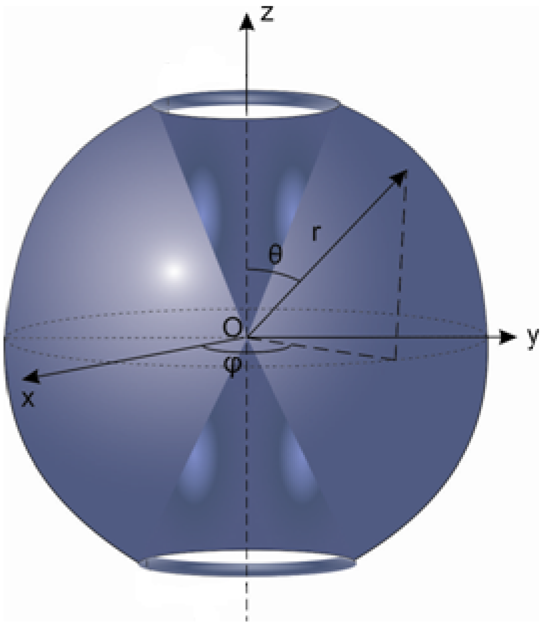

Fig. 1.

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