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The influence of the inhomogeneous initial thermo-stresses in the hollow cylinder on the dispersion of the torsional waves propagated in this cylinder

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Abstract. *This paper deals with the study of the influence of the inhomogeneous initial thermo-stresses in the hollow cylinder on the dispersion of the torsional waves propagated in this cylinder. It is assumed that the mentioned initial thermo-stresses are caused by the inhomogeneous temperature field which appears as a result of the cooling or heating of the inner and outer surfaces of the cylinder. The initial inhomogeneous thermo-stresses are determined within the scope of the classical uncoupled linear theory of thermo-elasticity, however, the study of the torsional wave propagation in the hollow cylinder under consideration is studied within the scope of the second version of the three-dimensional linearized theory of elastic waves in initially stressed bodies. Numerical results are presented for the first lowest mode which under absent of the inhomogeneous initial stresses is non-dispersive one. It is established that this mode becomes dispersive as a result of the aforementioned inhomogeneous initial stresses. It is also studied the influence of the problem parameters on the dispersion curves of the first lowest mode.*

Keywords. inhomogeneous initial thermo-stresses · torsional waves · hollow cylinder · wave dispersion · cut-off frequencies · cut-off wavelengths

Mathematics Subject Classification (2010): 74H55

1 Introduction

Theoretical results on the influence of the initial stresses on the dispersion and propagation velocity of the waves in the elements of constructions have a great significance under the non-destructive determination of defects (NDT) and stresses in those. Therefore, the subject of the present paper which relates to the study of the influence of the non-homogeneous

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initial thermo-stresses in the hollow cylinder on the dispersion of the torsional waves in that has not only theoretical but also application significance.

Note that the application of the NDT are carried out in the exploitation conditions which are sources for appearing of the various reference particularities in the pipelines before the dynamic excitation of that. One of these particularities is the inhomogeneous initial stresses which can appear in the pipelines as a result of the temperature change of the surrounding airier or of the liquid contained in the pipelines or transported with these pipelines. The mentioned temperature change causes to appear non-homogeneous temperature field in the pipeline and that, in turn, causes to appear the non-homogeneous initial stresses in this pipeline. The theoretical investigation of how these inhomogeneous initial stresses influence on the dispersion of the torsional waves propagated in the single-layer hollow cylinder, is the subject of the present paper. This investigation will be made by utilizing the three-dimensional linearized theory of elastic waves in initially stressed bodies (TDLTEWISB).

If to say in the historical aspect, the investigations of the axisymmetric torsional wave propagations in the hollow and solid cylinders within the scope of the exact equations and relations of the classical linear elastodynamics were started more than hundred years ago. The detail analyses of these results can be found in the monographs [1, 2] and others listed therein. One of the main and well-known classical conclusions is the following: the first lowest mode of the torsional waves in hollow and solid cylinders is non-dispersive. However, later in the papers [3 – 6], it was established the aforementioned lowest non-dispersive mode becomes dispersive one for the bi-material (or bi-layered) solid and hollow cylinders and the wave propagation velocity c of this mode is limited with the shear wave propagation velocities of the constituents of the cylinder, i.e. $\min\{c_2^{(2.1)}; c_2^{(2.2)}\} < c < \max\{c_2^{(2.1)}; c_2^{(2.2)}\}$, where $c_2^{(2.1)}(c_2^{(2.2)})$ is the shear wave propagation velocity in the outer (inner) layer material of the cylinder.

The investigations carried out in the papers [7 – 10] showed that the foregoing conclusions remain also valid for the cases where the cylinders have homogeneous initial stresses caused by the uniaxial tension or compression of those along the cylinders axis.

Besides all this, in the paper [11] it was studied the influence of the inhomogeneous initial stresses in the hollow circular cylinder appearing as a result of the action of "dead" hydrostatic pressures on the outer and inner face surfaces on the dispersion of the torsional wave propagating in this cylinder. The problem considered in the paper [11] was considered again a further 35 years later, in the paper [12] and it was assumed that the hydrostatic pressures causing the initial stresses in the hollow cylinder are the "follower" forces. Note that in the papers [11, 12] it was assumed that the material of the cylinder is incompressible highly elastic one and concrete numerical results were examined for the Mooney-Rivlin materials. At the same time, in the paper [11] it was established that in the cases where the external forces (hydrostatic pressures) are "dead" forces the first lowest mode of the torsional wave become dispersive. However, in the paper [12] it was established that in the cases where these pressures are "follower" ones the first lowest mode of the torsional wave remain non-dispersive one.

It is evident that the inhomogeneous initial stresses can appear in the hollow cylinders as a result of the temperature change of the inner and outer surfaces of that. So that this change causes the inhomogeneous temperature distribution in the cylinder and this temperature field in turn causes the corresponding in-homogeneous thermo-stresses in the hollow cylinder. How this inhomogeneous initial thermo-stresses influences on the dispersion of the first lowest mode of the torsional waves propagated in the hollow cylinder made of the moderately rigid, homogeneous and isotropic material, is studied in the present paper.

2 Determination of the initial thermal-stresses and formulation of the wave ropagation problem

We consider the single-layer hollow cylinder and associate with the central axis of this cylinder the cylindrical $Or\theta z$ and Cartesian $Ox_1x_2x_3$ system of coordinates (Fig. 1a).

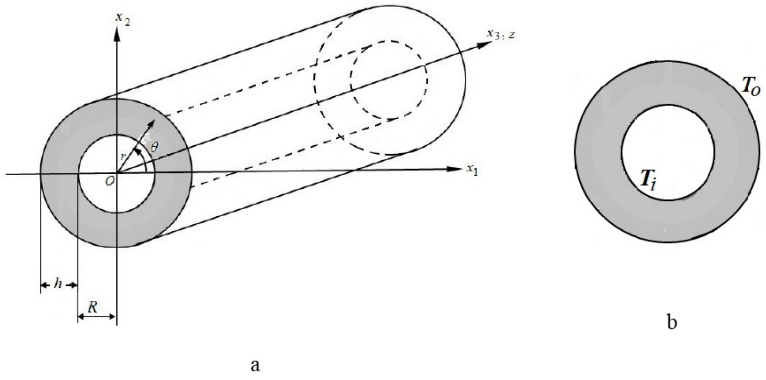


Fig. 1. The sketch of the single-layered cylinder (a) and the temperature on the inner and outer surfaces (b)

We suppose that in the initial state (i.e. before the dynamical excitation of the cylinder) as a result of the heating or cooling of the inner or outer surfaces of the cylinder it appears the non-homogeneous temperature field in the layers of the cylinder which in turn causes to appear the inhomogeneous thermo-stresses. The mentioned "heating or cooling" of the inner and outer surfaces of the cylinder is made through the change of the temperatures T_i and T_o (Fig. 1b) of these surfaces, respectively. At the same time, assume that the quantities of the T_i and T_o do not depend on the coordinates of the surface points and according to this assumption, we conclude that the temperature field in each layer depends on the radial coordinate r only and consequently, satisfies the following well-known equation [13].

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0, \tag{2.1}$$

for which, according to Fig. 1b, there exist the following boundary conditions.

$$T|_{r=R+h} = T_o, T|_{r=R} = T_i, \tag{2.2}$$

The solution of the equations in (2.1) within the scope of the conditions (2.2) are obtained as follows:

$$T = \frac{T_0 - T_i}{\log\left(1 + \frac{h}{R}\right)} \log\frac{r}{R} + T_i. \tag{2.3}$$

According to the well-known procedures described, for instance in [13], the thermo-stresses and displacements in the layers are determines through the following expressions:

$$u_r^0 = \frac{1+\nu}{1-\nu}\alpha\frac{1}{r}\left(\int_R^r T(\eta)\eta d\eta\right) + A_1 r + \frac{A_2}{r}, \sigma_{rr}^0 =$$

$$= -\frac{\alpha E}{1-\nu}\frac{1}{r^2}\left(\int_R^r T(\eta)\eta d\eta\right) + \frac{E}{1+\nu}\left(\frac{A_1}{1-2\nu} - \frac{A_2}{r^2}\right),$$

$$\sigma_{\theta\theta} = \frac{\alpha E}{1-\nu}\frac{1}{r^2}\left(\int_R^r T(\eta)\eta d\eta\right) -$$

$$-\frac{\alpha ET}{1-\nu}+\frac{E}{1+\nu}\left(\frac{A_1}{1-2\nu}-\frac{A_2}{r^2}\right), \sigma_{zz}=\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right). \tag{2.4}$$

where α, ν and E are the coefficients of linear thermal expansion, Poisson's ratio and modulus of elasticity of the cylinder material, respectively; $A_1$and A_2 are the unknown constants which are determined from the boundary conditions $\sigma_{rr}^0\big|_{r=R}=0$ and $\sigma_{rr}^0\big|_{r=R+h}=0$. In this way, we determine completely the initial thermo-stresses in the bi-layered cylinder under consideration.

Now we assume that after appearing of the foregoing initial thermo-stresses in the cylinder, the torsional waves start to propagate in this cylinder and it is required to determine how these inhomogeneous initial stresses act on the dispersion of these waves. It is evident that this determination can be modelled by utilizing the equations and relations of the TDLTEWISB.

We attempt to formulate the corresponding problem and first of all, assuming the satisfaction of the relations $u_\theta=u_\theta(r,z,t)$, $u_r=0$, $u_z=0$ (where u_r, u_θ and u_z are the components of the displacement vector) for the torsional waves we obtain the following linearized wave propagation equation [1, 2].

$$\frac{\partial t_{r\theta}}{\partial r}+\frac{1}{r}\left(t_{r\theta}+t_{\theta r}\right)+\frac{\partial t_{z\theta}}{\partial z}=\rho\frac{\partial^2 u_\theta}{\partial t^2}. \tag{2.5}$$

In (2.5) the following notation is used:

$$t_{r\theta}=\sigma_{r\theta}+\sigma_{rr}^0\frac{\partial u_\theta}{\partial r}, t_{\theta r}=\sigma_{r\theta}-\sigma_{\theta\theta}^0\frac{u_\theta}{r}, t_{z\theta}=\sigma_{z\theta}+\sigma_{zz}^0\frac{\partial u_\theta}{\partial z}. \tag{2.6}$$

We add to the foregoing equations of motion the linearized elasticity relations

$$\sigma_{r\theta}=\mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right), \sigma_{\theta z}=\mu\frac{\partial u_\theta}{\partial z}. \tag{2.7}$$

Also, we add the following boundary and contact conditions to the foregoing equations (2.5) – (2.7).

$$t_{r\theta}|_{r=R}=0,\ t_{r\theta}|_{r=R+h}=0. \tag{2.8}$$

This completes the formulation of the torsional wave propagation problem. The main difference of the present problem from the problems considered in the papers [11, 12] consists of the of the inhomogeneous initial thermo-stresses which cannot be classified as the initial stresses caused by the "dead" or "following" forces as it was made in the papers [11] and [12], respectively.

3 Method of solution

Using the expressions in (2.6) and in (2.7) we can rewrite the equation of motion (2.5) in the $u_\theta(r,z,t)$displacement term as follows:

$$\mu\left(\frac{\partial^2 u_\theta}{\partial r^2}-\frac{u_\theta}{r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{\partial^2 u_\theta}{\partial z^2}\right)+\frac{\partial}{\partial r}\left(\sigma_{rr}^0\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r}\sigma_{rr}^0\frac{\partial u_\theta}{\partial r}-$$
$$\frac{1}{r}\sigma_{\theta\theta}^0\frac{u_\theta}{r}+\sigma_{zz}^0\frac{\partial^2 u_\theta}{\partial z^2}=\rho\frac{\partial^2 u_\theta}{\partial t^2}. \tag{3.1}$$

Employing the presentation $u_\theta=U(r)\cos(kz-\omega t)$ and substituting this presentation into equation (3.1) the following equation is obtained for the unknown function $U(r)$

$$\mu\left(\frac{d^2U}{dr^2}-\frac{U}{r^2}+\frac{1}{r}\frac{dU}{dr}-k^2U\right)+\frac{d}{dr}\left(\sigma_{rr}^0(r)\frac{dU}{dr}\right)+\frac{1}{r}\sigma_{rr}^0(r)\frac{dU}{dr}-$$

$$-\frac{1}{r}\sigma^0_{\theta\theta}(r)\frac{U}{r}-k^2\sigma^0_{zz}(r)U=-\omega^2\rho U. \tag{3.2}$$

To find the analytical solution to the equation (3.2) for the case under consideration is impossible. Therefore, according to [14, 15], for the solution to this equation we employ the discrete-analytical solution method the employing of which is based on the reducing the differential equation (3.2) with variable coefficients to the series corresponding equations with constant coefficients. Note that the detail describing of this method is given in the papers [14, 15], therefore, here we indicate some principal moments of the application of this method.

Note that for application of this method the region $[R, R+h]$ occupied by the cylinder are divided into a certain number of the corresponding sub-regions. These sub-regions can be presented as $[R+(n-1)h/N)$, $(R+nh/N)]$, where $n=1,2,...,N$; the number N is determined in the solution procedure from the convergence requirement of the numerical results. After determination of these sub-regions, it is assumed that within each of them the inhomogeneous initial stresses determined through the expressions (2.3) and (2.4) are homogeneous ones and the values of these stresses are determined through the following expressions.

In the n-th sub-region:

$$\sigma^0_{rr}(r)\approx\sigma^0_{rr}(r_n),\sigma^0_{\theta\theta}(r)\approx\sigma^0_{\theta\theta}(r_n),\sigma^0_{zz}(r)\approx\sigma^0_{zz}(r_n),$$
$$r_n=R+(n-1)h/N+h/(2N). \tag{3.3}$$

Introducing the upper index n for indicating the belonging of the values to the $n-th$ sub-region of cylinder, it can be written the following boundary and contact conditions on the boundaries and on the contact surfaces between neighboring sub-regions.

$$T^1_{r\theta}\big|_{r=R}=0,\ T^1_{r\theta}\big|_{r=R+h/N}=T^2_{r\theta}\big|_{r=R+h/N},\ U^1_{\theta}\big|_{r=R+h/N}=$$
$$U^2_{\theta}\big|_{r=R+h/N},...,\ T^{n-1}_{r\theta}\big|_{r=R+(n-1)h/N}=T^n_{r\theta}\big|_{r=R+(n-1)h/N},$$
$$U^{n-1}_{\theta}\big|_{r=R+(n-1)h/N}=U^n_{\theta}\big|_{r=R+(n-1)h/N},...,\ T^{(2)N}_{r\theta}\Big|_{r=R+h}=0. \tag{3.4}$$

Thus, according to the replacements in (3.3) and (3.4), equation (3.2) can be rewritten as follows:

$$\frac{d^2U^n_\theta}{d(kr)^2}+\frac{1}{kr}\frac{dU^n_\theta}{d(kr)}-\frac{\alpha^n}{(kr)^2}U^n_\theta+\left(\frac{c^2}{(c_2)^2}\left(1+\frac{\sigma^0_{rr}(r_n)}{\mu}\right)^{-1}-\beta^n\right)U^n_\theta=0. \tag{3.5}$$

where

$$c=\frac{\omega}{k},c_2=\sqrt{\frac{\mu}{\rho}},\alpha^n=\left(1+\frac{\sigma^0_{\theta\theta}(r_n)}{\mu}\right)\left(1+\frac{\sigma^0_{rr}(r_n)}{\mu}\right)^{-1},$$
$$\beta^n=\left(1+\frac{\sigma^0_{zz}(r_n)}{\mu}\right)\left(1+\frac{\sigma^0_{rr}(r_n)}{\mu}\right)^{-1}. \tag{3.6}$$

Using the notation

$$r_{1n}=kr\sqrt{\frac{c^2}{(c_2)^2}\left(1+\frac{\sigma^0_{rr}(r_n)}{\mu}\right)^{-1}-\beta^n} \tag{3.7}$$

equation (3.6) can represented as follows:

$$\frac{d^2U^n_\theta}{dr^2_{1n}}+\frac{1}{r_{1n}}\frac{dU^n_\theta}{dr_{1n}}+\left(1-\frac{\alpha^n}{r^2_{1n}}\right)U^n_\theta=0. \tag{3.8}$$

Finally, using the notation

$$\gamma^n = \sqrt{\alpha^n} \tag{3.9}$$

the following solutions to equation (3.8) are found.

$$U_\theta^n = \begin{cases} A_1^n J_{\gamma^n}(r_{1n}) + A_2^n Y_{\gamma^n}(r_{1n}), \ \text{if } r_{1n}^2 > 0, \\ A_1^n I_{\gamma^n}(|r_{1n}|) + A_2^n K_{\gamma^n}(|r_{1n}|), \ \text{if } r_{1n}^2 < 0. \end{cases} \quad j = 1, 2, \tag{3.10}$$

where $J_\delta(x)$ and $I_\delta(x)$ ($Y_\delta(x)$ and $K_\delta(x)$) are the Bessel and modified Bessel function of the first (second) order.

Substituting the solutions in (3.10) into the elasticity relations in (2.7) we obtain the following expression for the stresses which enters into the boundary and contact conditions.

$$T_{r\theta}^n = \mu \left\{ A_1^n \left[\left(1 + \frac{\sigma_{rr}^0}{\mu}\right) \frac{dr_{1n}}{dr} \frac{1}{2} \left(J_{\gamma^n - 1}(r_{1n}) - J_{\gamma^n + 1}(r_{1n})\right) - \frac{1}{r} J_{\gamma^n}(r_{1n}) \right] + \right.$$

$$\left. A_2^n \left[\left(1 + \tfrac{\sigma_{rr}^0}{\mu}\right) \tfrac{dr_{1n}}{dr} \tfrac{1}{2} \left(Y_{\gamma^n - 1}(r_{1n}) - Y_{\gamma^n + 1}(r_{1n})\right) - \tfrac{1}{r} Y_{\gamma^n}(r_{1n}) \right] \right\}, \text{if} r_{1n}^2 > 0,$$

$$T_{r\theta}^n = \mu \left\{ A_1^n \left[\left(1 + \frac{\sigma_{rr}^0}{\mu}\right) \frac{d\,|r_{1n}|}{dr} \frac{1}{2} \left(I_{\gamma^n - 1}(|r_{1n}|) + I_{\gamma^n + 1}(|r_{1n}|)\right) - \frac{1}{r} I_{\gamma^n}(|r_{1n}|) \right] + \right.$$

$$A_2^n \left[- \left(1 + \frac{\sigma_{rr}^0}{\mu}\right) \frac{d\,|r_{1n}|}{dr} \frac{1}{2} \left(K_{\gamma^n - 1}(|r_{1n}|) + \right.\right.$$

$$\left.\left.\left. + K_{\gamma^n + 1}(|r_{1n}|)\right) - \frac{1}{r} K_{\gamma^n}(|r_{1n}|) \right] \right\}, \ \text{if } r_{1n}^2 < 0. \tag{3.11}$$

Substituting the expressions in (3.10) and (3.11) into the boundary and contact conditions in (3.5) we obtain the system of homogeneous linear algebraic equations with respect to the unknown constants A_1^n and A_2^n ($n = 1, 2, ..., N$). Equating to zero the determinant of the coefficient matrix of these system of equations it is obtained the dispersion equation

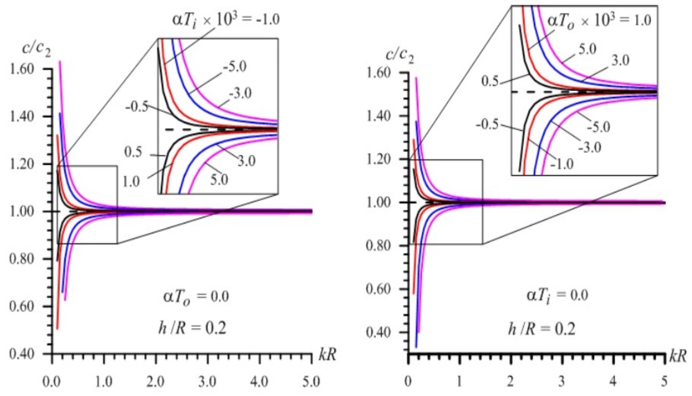
$$\det(a_{nm}(c/c_2, kR, T_o, T_i, h/R)) = 0, n; m = 1, 2, ..., 2N. \tag{3.12}$$

We do not give here the explicit expressions of the components a_{nm} of the matrix (a_{nm}) because these can be easily determined from the expressions in (3.10) and (3.11).

The solution to the dispersion equation in (3.12) is found numerically by employing the well-known bi-section method.

4 Numerical results and discussions

Under obtaining numerical results it is assumed that $N = 25$ and the values c/c_2 are determined from the dispersion equation (3.12) for each value of kR for selecting αT_i, $\alpha T_o$and $h/R = 0.1$. According to the reference [16], for many metal materials, for instance, steel, aluminum, tungsten and others the values of αT_i and αT_ocan be equal to 0.000; 0.0005; 0.001; 0.003 and 0.005. At the same time, we will also assume that $h/R = 0.10$, if otherwise not specified. Consequently, the obtained numerical results will have the universal character without concretization of the cylinder's material.



a b

Fig. 2. The influence of the heating and cooling of the inner (a) and outer (b) surfaces of the hollow cylinder on the dispersion curves of the lowest first mode of the torsional waves

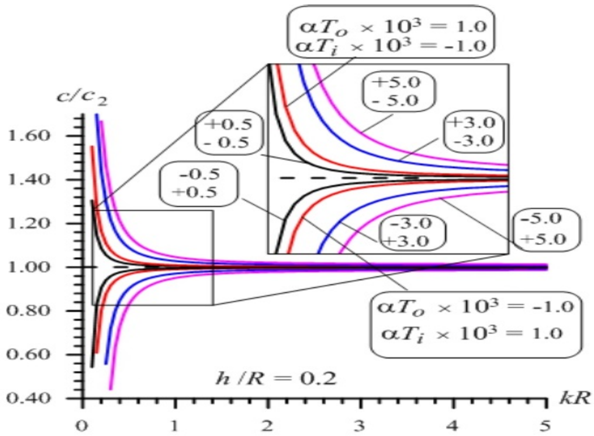


Fig. 3. The influence of the simultaneous heating of the inner surface and cooling of the outer surface, as well as the simultaneous cooling of the inner surface and heating of the outer surface on the dispersion curves of the lowest first mode of the torsional waves

Note that here we will consider only numerical results related to the first lowest mode which is non-dispersive one in the case where the initial non-homogeneous thermo-stresses in the hollow cylinder are absent. This is also because in the case under consideration the influence of the initial non-homogeneous thermo-stresses on the dispersion curves of the higher modes has an only quantitative character the magnitude of this influence is insignificant than that which is observed in the lowest first mode.

Thus, we analyze the aforementioned numerical results and begin this analysis with the graphs given in Fig. 2. These graphs illustrate the dispersion curves of the lowest first mode in the case where the initial non-homogeneous thermo-stresses in the hollow cylinder appear, as a result, of the heating or of the cooling only of the inner (Fig. 2a), as well only of the outer (Fig. 2b) surface of the hollow cylinder. The influence of the initial inhomogeneous thermo-stresses caused by the simultaneous cooling of the inner and of the heating of the outer, as well as simultaneous heating of the inner and of the cooling of the outer surfaces on the cylinder on the dispersion curves of the first mode, is illustrated by the graphs given in Fig. 3.

It follows from the foregoing results that, as a result of the initial inhomogeneous thermo-stresses the non-dispersive lowest first mode of the torsional wave in the hollow cylinder becomes dispersive mode. Moreover, it follows from the foregoing results that the heating (cooling) of the inner surface and the cooling (heating) of the outer surface of the cylinder causes to be the torsional wave propagation velocity less (more) than c_2 which is the shear wave propagation velocity in the cylinder material and is the wave propagation velocity in the first non-dispersive torsional mode under absent of the initial inhomogeneous initial stresses in the cylinder. Also, these results show that the magnitude of the aforementioned “less” and “more” increase with the absolute values of the αT_i and αT_o. According to Fig. 3, this magnitude becomes more significantly under the simultaneous heating of the inner surface and cooling of the outer surface, as well as the simultaneous cooling of the inner surface and heating of the outer surface.

Besides all of these, the foregoing results show that the new dispersive mode which appears as a result of the inhomogeneous initial thermo-stresses has not the low-wave number limit values. In other words, the dispersion curves related to this mode has cut off frequencies (cut off wavelength) in the cases where the inner surface is cooled and outer surface is heated (inner surface is heated and the outer surface is cooled). For a more clear illustration of the mentioned cut of frequencies and cut off wavelength in Fig. 4 and 5 the dispersion curves are presented in the near vicinity of these frequencies and wavelength. It follows from these curves that an increase in the absolute values of the temperature of the inner and outer surfaces of the hollow cylinder causes to increase the values of the cut off frequencies and cut off wavelength.

Note that the foregoing results agree with the results obtained in the paper [11] in the qualitative sense and coincide with the corresponding classical results obtained in the case where the initial inhomogeneous thermo-stresses are absent in the cylinder. The mentioned agreement and coinciding can be taken as arguments for trustiness of the used solution method and calculation algorithm. Moreover, note that the results given in Figs. 4 and 5 illustrate the change of the cut off frequencies and cut off wavelength in the qualitative sense only. For a more clear and digital illustration of these changes are presented in Tables 1 and 2, which show the values of the dimensionless cut off frequencies $\omega_{ctf}R$ and dimensionless cut off wavelength $k_{ctf}R$ for various values of the ratio h/R. It follows from these tables that a decrease in the values of the ratio h/R causes to increase in the values of the aforementioned cut off frequencies and cut off wavelength.

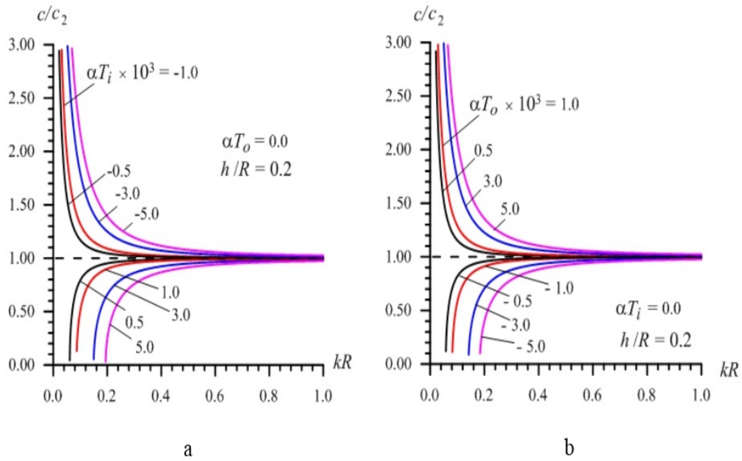


Fig. 4. The influence of the heating and cooling of the inner (a) and outer (b) surfaces of the hollow cylinder on the cut off frequencies and cut off wavelength of the dispersion curves of the lowest first mode

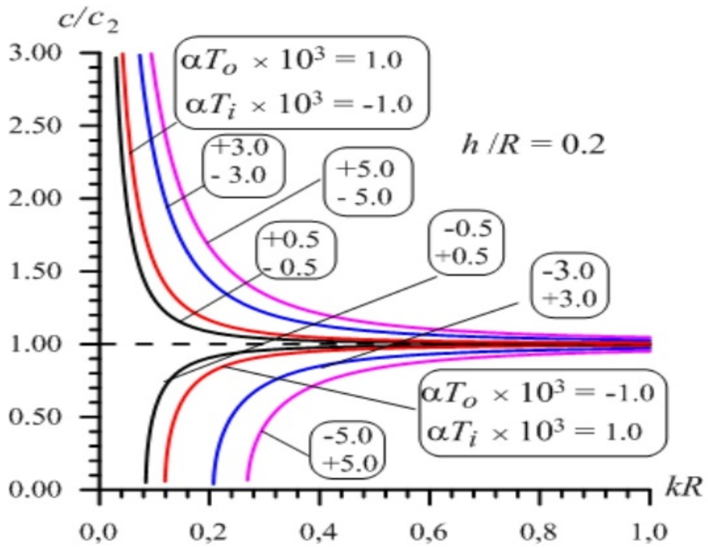


Fig. 5. The influence of the simultaneous heating of the inner surface and cooling of the outer surface, as well as the simultaneous cooling of the inner surface and heating of the outer surface on the cut off frequencies and cut off wavelength of the dispersion curves of the lowest first mode

Table 1. The values of the dimensionless cut off wavelength $k_{ct.of}R$ and dimensionless cut off frequencies $\omega_{ct.of}R$ obtained for various h/R in the case where only the inner or only the outer surface of the cylinder is cooled or is heated.

$\alpha T_i * 10^3 \left(\alpha T_o * 10^3\right)$		h/R		
		0.2	0.1	0.05
0.5 (0.0)	$k_{ct.of}R$	0.061	0.086	0.120
1.0 (0.0)		0.087	0.121	0.170
3.0 (0.0)		0.150	0.211	0.297
5.0 (0.0)		0.194	0.273	0.387
-0.5 (0.0)	$\omega_{ct.of}R$	0.064	0.090	0.127
-1.0 (0.0)		0.091	0.128	0.179
-3.0 (0.0)		0.158	0.221	0.311
-5.0 (0.0)		0.204	0.286	0.401
0.0 (0.5)	$\omega_{ct.of}R$	0.061	0.088	0.124
0.0 (1.0)		0.086	0.124	0.176
0.0 (3.0)		0.149	0.215	0.306
0.0 (5.0)		0.193	0.277	0.395
0.0 (-0.5)	$k_{ct.of}R$	0.058	0.083	0.118
0.0 (-1.0)		0.082	0.118	0.168
0.0 (-3.0)		0.142	0.205	0.292
0.0 (-5.0)		0.184	0.265	0.381

It should be noted that the foregoing numerical results are obtained in the case where $N = 15$ which indicate the number of sub-regions into which is divided the hollow cylinder. For illustration of the convergence of the obtained numerical results with respect to this number in Table 3 it is presented a certain numerical results obtained for various N from which follow higher order convergence of the results with the number N.

Table 2. The values of the dimensionless cut off wavelength $k_{ct.of}R$ and dimensionless cut off frequencies $\omega_{ct.of}R$ for $h/R = 0.2$ in the case where the inner and outer surfaces of the cylinder is cooled (is heated) or is heated (is cooled) simultenaously

$\frac{\alpha T_i * 10^3}{\alpha T_o * 10^3}$							
$\frac{0.5}{-0.5}$	$\frac{1.0}{-1.0}$	$\frac{3.0}{-3.0}$	$\frac{5.0}{-5.0}$	$\frac{-0.5}{0.5}$	$\frac{-1.0}{1.0}$	$\frac{-3.0}{3.0}$	$\frac{-5.0}{5.0}$
$k_{ct.of}R$				$\omega_{ct.of}R$			
0.084	0.119	0.207	0.269	0.089	0.125	0.218	0.281

Table 3. The values of $D_I = \left(c/c_2|_{N=10} - c/c_2|_{N=5} \right) * 10^7$ and $D_{II} = \left(c/c_2|_{N=15} - c/c_2|_{N=10} \right) * 10^7$ calculated for the hollow cylinder under various values of kRin the case where $\alpha \mathrm{T}_i * 10^3 = 5.0$, $\alpha \mathrm{T}_o = 0.0$ and $h/R = 0.05$

	kR		
	0.42	0.46	0.50
D_I	23	15	11
D_{II}	4	3	2

This completes the analyses of the numerical results.

5 Conclusions

Thus, in the present paper, it has been studied the influence of the inhomogeneous initial thermal stresses caused by the heating and cooling of the inner and outer surfaces of the single-layered hollow cylinder on the torsional wave propagation in this cylinder. This study is made within the scope of the second version of the small initial deformation theory of the TDLTEWISB, according to which, the initial thermo-stresses are determined within the scope of the classical uncoupled linear theory of thermo-elasticity. For the solution to the corresponding eigenvalue problem, it is used the discrete analytical method. The numerical results of the dispersion curves are presented and discussed. It is established that as a result of the initial inhomogeneous thermo-stresses in the cylinder the new dispersive mode appears. The influence of the problem parameters on this new dispersive mode is studied and as a result of this study the corresponding conclusions are made. These conclusions are formulated in the text of the paper and therefore are not given here again.

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