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Some estimates for degenerates' elliptic equations of the second order

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Abstract. *In this paper nondivergence second order degenerated linear elliptic equations are considered. The weight function from the classes of Mackenhaupt. An infinite domain which have conic points at boundary the behavior of solutions problem of Dirichlet are studied.*

Keywords. Nondivergence · degenerate · linear elliptic equations · the behavior of solutions

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1 Introduction

Let $\Omega \subset R^n, n = 2$ be an infinite domain with the boundary $\partial\Omega$ have conic points. We are considering the following Dirichlet problem

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u(x) = 0 \tag{1.1}$$

$$u|_{\partial\Omega} = 0. \tag{1.2}$$

We suppose for $x \in \Omega$, $a_{ij}(x)$ coefficients for $x \in \Omega$, $\xi \in R^n$ coefficients $a_{ij}(x)$ - is a real symmetric matrix and satisfying following conditions: $a_{ij}(x) = a_{ij}(x), \quad i, j = \overline{1, n}$ and condition is fulfilled

$$c\sum_{i=1}^{n} \omega(x)\xi^2 \leq \sum_{i=1}^{n} a_{ij}(x)\xi_i\xi_j \leq c^{-1}\sum_{i=1}^{n} \omega(x)\xi_i^2, \tag{1.3}$$

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where, $c \in (0,1]$ is a constant and $\omega(x) \in A_p$ - weight function from classes of Mackenhaupt, $1 < p < \infty$. The conditions for coefficients $b_i(x)$, $i = \overline{1,n}$ and $c(x)$ we give later. Before we suppose this coefficient are bounded, and $c(x) \leq 0$.

Uniformly elliptic equations of second order in non-divergent form are investigated in papers [1], [4]. Elliptic equations in divergent form are studied in [5]. In [2], solutions to elliptic equations in unbounded domains are analyzed, and certain estimates for these solutions are obtained. Krylov and Safonov, in their works [3], [6], study solutions of such problems using probabilistic methods.

In this paper we degenerated non-divergence elliptic equations in infinity domains are investigated.

Let's the coefficients $b_i(x)\ \ i = \overline{1,n}$ satisfies condition (where $b(x) = (b_i(x), ..., b_n(x))$

$$(b(x), x - x_0) = 0 \tag{1.4}$$

for all $x \in \Omega$, $x^0 \neq 0$ and situated outside of domain. We introduce some ellipsoid

$$E_R^{x_0}(p) = \left\{ x : \sum_{i=1}^{n} \frac{(x_i - x_{i0})^2}{R^\alpha \omega(x)} < pR^2 \right\}, \tag{1.5}$$

where $R > 0,\ p > 0,\ \alpha \in (-2,\ 0]$. Later we take $R \geq 1$.

The operator corresponding to equation (1.1) we define the operator L and also is consider narrow operator $L' = L * c(x)$. Now we give some auxiliary results

Lemma 1. Let $z \in \partial E_R^0(\alpha) \cap \overline{\Omega},\ x_0(z) \in \partial E_R^z(\alpha) \cap \partial E_R^0(\beta)$, where $\beta > \alpha$. Then x_0 situated is outside of Ω.

Let satisfy the following conditions

$$\begin{gathered} (b_R(x),\ x - x_0) \leq 0, \\ -C_0 \leq C(x) \leq 0 \end{gathered}$$

where $b_R(x) = R^\alpha\,(b_1(x)\ ,\ \dots\ ,\ b_n(x))$, C_0 - is a positive constant.

We define

$$G_S^{(R)}(x) = \left(\sum_{i=1}^{n} \frac{(x_i - x_{i0})}{R^\alpha \omega(x)} \right)^{-S/2}.$$

Lemma 2. Let satisfy conditions (1.3), (1.5).

Then $x_0 \in \bigcup\limits_{\Omega \cap \partial E_R^0(\beta)} x_0\,(z)$ for there exists such S that for any

$$L' G_S^{(R)}(x) = 0.$$

Proof.

$$L' G_S^R(x) = S\rho^{-(S+2)} \left(\frac{(S+2)\,C}{\rho^2} \sum_{i=1}^{n} \frac{\omega(x)}{R^a} \frac{(x_i - x_{i0})^2}{R^a} - C^{-1} \frac{\omega(x)}{R^a} \right),$$

where $S = \left(\sum\limits_{i=1}^{n} R^{-2} \omega^{-1}(x)\,(x_i - x_{i0})^2 \right)^{1/2}$.

Consequently

$$L' G_S^R(x) \geq S\rho^{-(S+2)} \left[(S+2)\,C - nC^{-1} \right].$$

Lemma is proved.

If we take

$$z \in \Omega \cap \partial E_R^0(\beta); x_0 = x_0(r), x \in \Omega \cap E_R^0(2\beta)$$

$$g_S^{(R)}(x) = \beta R^S G_S^{(R)}(x),$$

then

$$g_S^{(R)}(x) \leq 1.$$

Let $\Omega' = \Omega \cap E_R^0(\beta)$.

Theorem 1. Let $z \in \Omega \cap \partial E_R^0(\beta)$, and $u(x)$ the positive solution of equation (1.1) – (1.2) in Ω',

And conditions (1.3) – (1.5) are fulfilled. Then there exists a constant $\mu > 0$ such that

$$\sup_{\Omega'} \omega(x) u \geq (1+\mu) \sup_{\Omega' \cap E_R^0(\beta)} \omega\ \ (x) u. \tag{1.6}$$

Proof. Let $\sup\limits_{\Omega'} u^2 = M$. We introduce function $U(x) = M\left[1 - g_S^{(R)}(x) + \sup\limits_{\Omega'} g_S^{(R)}(x)\right]$.
We easy can show $L'u^2(x) \geq 0$. Also $L'(V(x) - u^2(x)) \leq 0$ in Ω', $\left(U(x) - u^2(x)\right) \geq 0$.

By maximum principle at $\partial\Omega' \backslash \partial\Omega$

$$V(x) \geq u^2(x) \text{ in } \Omega'$$

and

$$\sup_{\Omega'} u^2 \leq M\left[1 - \left(\inf_{\Omega'} g_S^{R(x)} - \sup_{\Omega'} g_S^{R(x)}\right)\right].$$

Let $x \in \bar{\Omega}' \cap \partial E_R^0(\beta)$. Then $\sup\limits_{\Omega'} g_S^{(R)} \leq \xi^{-S}\beta^S$.

Therefore we get

$$\sup_{\Omega'} u^2 \leq M\left[1 - \beta^S\right].$$

Lemma is proved from this inequality.

Theorem 2. Let in Ω coefficients of the operator L satisfying the conditions (1.3) – (1.5).

Then, solution of problem $u(x)$ either equal to zero in Ω, or $\lim\limits_{r\to\infty} r^{-2}M(r) > 0$, where $M(r) = \sup\limits_{\Omega \Delta \partial E_R^0(\beta)} (\omega(x)u(x))$, $\alpha > 0$.

Proof. We note, that α depends only of parameters of problem. From Theorem 1 we have

$$\sup_{\Omega \cap E_R^0(\beta)} \omega(x) u \geq (1+\eta) \sup_{\Omega \cap E_R^0(\beta)} \omega(x) u.$$

With some calculations for $M(r)$ we have

$$M(r) \geq (1+\eta) \sup_{\Omega \cap E_R^0(\beta^{-m_0})} (\omega(x)u(x)) \geq (1+\eta)^{m-m_0} u(x)\omega(x),$$

for sufficient big r.

Then we get

$$r^{-\alpha} M(r) \geq c.$$

From applying the maximum principle, we obtain that require.

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