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Speed-optimal heating of an unbounded plate taking into account phase restrictions

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Abstract. *The problem of optimal heating of a brittle plate is studied, taking into account the limitations of thermal stress and maximum temperature, as well as the dependence of the thermal conductivity coefficient and the limits of brittle strength on temperature. An iterative method for searching optimal control is developed and implemented in the form of a computer program, where the dependence of brittle strength limits on temperature is approximated by exponential functions and the dependence of the thermal conductivity coefficient on temperature by a linear function. The results of computational experiments are presented.*

Keywords. optimal heating · thermal stress · speed · unbounded plate · body temperature.

Mathematics Subject Classification (2010): 74F05

1 Introduction

Problem Formulation. Consider the problem of axisymmetric heating of an unbounded plate with external heat sources. The heating process is described by the following equations:

$$c\rho\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda\left(t\right)\frac{\partial T}{\partial x}\right),\ \ 0 < x < \overline{x},\ \ 0 < t \leq \bar{t} < \infty, \tag{1.1}$$

$$T\left(x,0\right) = T^0 = const,\ \ x \in \left[0,\overline{x}\right] \tag{1.2}$$

$$\lambda\left(t\right)\frac{\partial T\left(\overline{x,\ t}\right)}{\partial x} = \alpha\left(v\left(t\right) - T\left(\overline{x},t\right)\right),\ t \in \left[0,\bar{t}\right] \tag{1.3}$$

$$\frac{\partial T\left(0,t\right)}{\partial x} = 0,\ t \in \left[0,\bar{t}\right] \tag{1.4}$$

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where $T = T(x,t)$ is the temperature, t is time, c is the heat capacity coefficient, ρ is the density, α is the heat exchange coefficient, $v(t)$ is the external environment temperature, and $\nu(t) \in V,\ V = \{v = v(t),\ v(t) \in L_2[0,\bar{t}],\ 0 \le \overline{v}(t) \le v(t) \le v^+\}$.
Let $\lambda(t) > 0$, have a bounded derivative, and satisfy:

$$0 < \beta_1 \le \lambda(t) \le \beta_2 \tag{1.5}$$

Under these conditions, the system of equations (1.1)-(1.4) has a generalized solution from the space $V_2^{1,0}(G)$, where $G = \{(x,t) : x \in (0,\overline{x}), t \in (0,\bar{t})\}$ [1].
Restrictions on thermal stresses, in the considered case, are written in the following form

$$\frac{\alpha\tau E}{1-\nu}\left(-T(0,t) + \frac{1+3\Gamma}{\overline{x}}\int_0^{\overline{x}} T(\xi,t)d\xi - \frac{6\Gamma}{\overline{x}^2}\int_0^{\overline{x}} \xi T(\xi,t)d\xi\right) \le \sigma_p(T(0,t)) \tag{1.6}$$

$$\frac{\alpha\tau E}{1-\nu}\left(-T(l,t) + \frac{1-3\Gamma}{\overline{x}}\int_0^{\overline{x}} T(\xi,t)d\xi - \frac{6\Gamma}{\overline{x}^2}\int_0^{\overline{x}} \xi T(\xi,t)d\xi\right) \le \sigma_c(T(\overline{x},t)) \tag{1.7}$$

Here α_T is the coefficient of linear expansion, E is the simplicity modulus, v is the Poisson's ratio, $\Gamma \in [0, 1]$ characterizes the degree of pinching from rotations of the plate edges, $\sigma_p(T)$ and $\sigma_c(T)$ are the tensile and compressive strength limits, respectively.
The limit on maximum temperature in the case of external heating is reached at the surface and has the following form.

$$T(\overline{x},t) \le T^*,\ 0 \le t \le \bar{t} \tag{1.8}$$

Problem. Find a control $v^0(t) \in V$, which translates the system (1.1) - (1.4) to a given finite position $\hat{T}(x)$ with fixed accuracy in minimum time $t^0 \le \bar{t}$

$$\int_0^{\overline{x}} \left(T\left(x,t^0,v^0\right) - \hat{T}(x)\right)^2 \le \varepsilon \tag{1.9}$$

such that inequalities (1.6) - (1.8) are satisfied for all $t \in [0,\ \bar{t}]$.

2 Linearization

The solution of the non-linear system of equations (1.1)-(1.4) with fixed control will be sought using the method of successive approximations outlined in [2]. For the system of equations (1.1)-(1.4), we set up the following iterative process:

$$c\rho\frac{\partial T_{k+1}}{\partial t} - \lambda_0\frac{\partial^2 T_{k+1}}{\partial x^2} = \frac{\partial}{\partial x}\left[(\lambda(T_k) - \lambda_0)\frac{\partial T_k}{\partial x}\right] \tag{2.1}$$

$$T_{k+1}(x,0) = T^0,\ x\,[0,\overline{x}] \tag{2.2}$$

$$\lambda_0\frac{\partial T_{k+1}(\overline{x},t)}{\partial x} - \alpha\left(v(t) - T_{k+1}(\overline{x},t)\right) = [\lambda_0 - \lambda(T_k)]\frac{\partial T_k(\overline{x},t)}{\partial x} \tag{2.3}$$

$$\frac{\partial T_{k+1}(0,t)}{\partial x} = 0, \tag{2.4}$$

where

$$\lambda_0 = (\beta_1 + \beta_2)/2 \tag{2.5}$$

The following theorem holds. Let the function $\lambda(T)$ be positive, satisfy relation (1.5), and have a bounded derivative on T. Then, for any fixed control $v(t)$ in $Vv(t)$ in V, the solutions $Tk+1T\ k+1$ of the system of equations (2.1)-(2.4) converge to the solution of the system of equations (1.1)-(1.4) in the corresponding norm.

Thus, at each step of the iterative process, the problem is reduced to finding optimal control when the process is described by the linear system (2.1)-(2.4) with nonlinear constraints (1.6)-(1.8).

3 Solving the speed-optimal heating problem with linear state equations and phase constraints

For convenience of further calculations, the system of equations (2.1)-(2.4) and the constraints (1.6)-(1.8) are rewritten in dimensionless variables. Then, using finite integral Fourier transformations, the solution to the obtained system is written as a series:

$$\theta\left(r,\tau,u\right)=\sum_{n=1}^{\infty}D_nx_n^k\left(u,\tau\right)cos\left(\mu_nr\right), \tag{3.1}$$

where

$$D_n=\frac{2B_i}{\left(\mu_n^2+B_i^2+B_i\right)cos\left(\mu_n\right)}. \tag{3.2}$$

$$\mu_n\geq 0,\ n=1,2,\ldots$$

$$B_i\ cos\left(\mu_n\right)-\mu_n sin\left(\mu_n\right)=0$$

And $x_n^k\left(u,\tau\right),\ \ n=1,2,\ldots$ omponents of the solution vector of the infinite system of differential equations

$$\frac{\partial x_n^{(k)}}{\partial x}=-\mu_n^2x_n^{(k)}+\mu_n^2\left(u+I_n^{(k-1)}\right),\ \ I_n^{(k-1)}=$$

$$=\int_0^1\left(\frac{\lambda\left(\theta_{k-1}\right)}{\lambda_0}-1\right)\frac{\partial\left(\theta_{k-1}\right)}{\partial r}\frac{sin\left(\mu_nr\right)}{sin\left(\mu_n\right)}dr \tag{3.3}$$

$x_n^k=0,\ n=1,2,\ldots,$ where $B_i=\alpha\overline{x}/\lambda_0$ – Bias criterion.

Constraints (1.5)-(1.8) will be written in the form of inequalities

$$\sum_{n=1}^{\infty}C_{in}x_n^{(k)}-l_i\leq 0,\ i=\overline{1,3}, \tag{3.4}$$

where $l_1=\sigma_p\left(\theta\left(0,\tau\right)\right),\ l_2=\sigma_c\left(\theta\left(1,\tau\right)\right),\ l_3=\theta^*$

Thus, the original problem at each k-th iteration is equivalent to the following: find the control $u^{(k)}\left(\tau\right)$ that translates the system (3.3) from a zero initial position to the final position $\left(\hat{x}_1^{(k)},\ \ldots,x_1^{(k)},\ldots\right)$ in minimum time while satisfying the constraints (3.4).

The infinite-dimensional optimal control problem is replaced by a N-members of finite-dimensional one, which is solved at each iteration using a modified method of rotation of the reference hyperplane [4].

4 Computational experiment results

The proposed approach to solving nonlinear heat conduction problems with phase constraints was tested on the following problem: heating an unbounded plate made of the alloy JC6U with a thickness of h from an initial temperature of 20°C to a final (uniform across the section) temperature of 920°C in minimal time, considering thermal stress and surface temperature constraints, which should not exceed 1100°C. The material JC6U is brittle.

Initial data: $\alpha = 0,0153m^2/h$; $\lambda = 23W/(m{\cdot}deg)$; $\alpha = 200W(m{\cdot}deg)$; $\alpha_T =$ 0,18·$10^{-4}1/deg$; $E = 0,145{\cdot}10^{12}N/m^2$; $v = 0,3$.

The temperature of the heating environment varied between 800°C and 1600°C. The dependence of the strength limits on temperature was given in a table, and after conversion to dimensionless values, it was approximated using the least squares method with nonlinear relationships.

Table 1. Temperature dependence of tensile strengths

Temperature, ^oC	20	975	1050	1100	1150
Tensile strength, compression, MPa	1500	700	470	310	210
Tensile strength, tensile, MPa	980	540	370	200	140

and after the transition to dimensionless values was approximated using the least squares method by nonlinear relations:

$$\sigma_c(\theta) = \left(-0,023e^{0,003030\theta} + 0,747\right),\ \sigma_p(\theta) = \left(-0,003e^{0,004600\theta} + 0,476\right)$$

The dependence of the thermal conductivity coefficient on temperature was approximated by the linear function $\lambda(\theta) = 10,68 + 9,74\theta$.

At each iteration step, the problem was solved for N=6N = 6. The required accuracy was achieved after 7 iterations. The optimal heating time was 3.98 hours.

The dependence of the thermal conductivity coefficient on temperature after the transition to dimensionless values was approximated by a linear function $\lambda(\theta) = 10.68 + 9.74\theta$.

At each k-th step of the iterative process the problem was solved for N=6. The required accuracy of the thermal problem solution was achieved in 7 iterations. The time of optimal heating in terms of speed was obtained as 3.98 h.

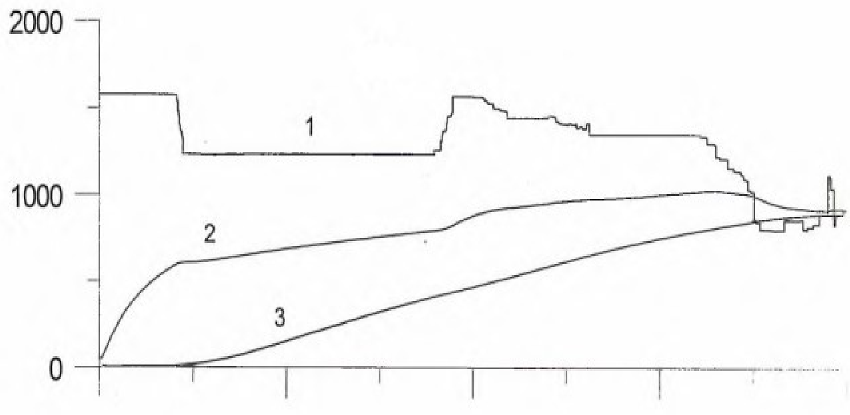

Fig. 1 Graphs for control and body temperature

The graph (Fig. 1) shows in dimensional units:

1. Time dependence of the optimal control. The optimal control has 135 switching operations;

2. Dependence of body surface temperature on time;

3. Dependence of the body center temperature on time.

Note also that in the scientific literature, tensile thermal stresses are usually considered to be active In the considered case, the heating rate is limited by compressive stresses [5]. The choice of the dimensionality of the finite-dimensional system $N = 6$ is due to the convergence of the series included in the constraints on thermal stresses.

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