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Some analytical solutions of thixotropic fluid flow equations

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Abstract. *The flow of thixotropic fluids is governed by complex nonlinear differential equations, making analytical solutions challenging to obtain. This study employs group analysis to construct particular invariant solutions for the one-dimensional flow equations of a thixotropic fluid. A comprehensive group classification is conducted, leading to the identification of optimal subalgebra systems and their associated invariant solutions. The study demonstrates that, under specific conditions, the governing equations allow for an expanded algebra of infinitesimal operators, enabling the derivation of exact solutions. The results include analytical expressions for velocity and structural parameter distributions, along with conditions for their applicability. Special cases where the system reduces to solvable quadratures are also examined. The findings provide insights into the mathematical properties of thixotropic fluid models and contribute to the broader understanding of structured fluid dynamics.*

Keywords. thixotropic fluids · group analysis · invariant solutions · differential equations · structured media.

Mathematics Subject Classification (2010): 76B07

1 Introduction

As is known, the flow of a thixotropic fluid is described by complex nonlinear equations. In this work, it is shown that group analysis of differential equations can be used to construct certain particular solutions. A group classification of the system of equations describing the one-dimensional flow of a thixotropic fluid is carried out. Some invariant solutions are analyzed.

1. A thixotropic fluid is understood as a medium in which an increase in shear stresses leads to a decrease in viscosity due to the destruction of the internal structure of the medium [5]. Such fluids include asphalt- and paraffin-containing oils, certain polymer solutions, clay suspensions, etc.

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To describe the flow of a thixotropic fluid with viscosity , which depends on a single dimensionless structural parameter λ, models [2, 6] have been proposed that, in the one-dimensional case, can be written as:

$$u_t = \left(\mu\left(\lambda\right) u_x\right)_x, \quad \lambda_t = \Phi\left(\lambda, u_x\right) \tag{1.1}$$

Models of this type are also used to describe the filtration of viscoelastic fluids [1].

We study the group properties [3, 4] of system (1.1).

For arbitrary functions $\mu\left(\lambda\right)$ and $\Phi\left(\lambda, u_x\right)$, system (1.1) admits a three-dimensional algebra L_3 of infinitesimal operators with a basis $X_1 = \partial/\partial t, \ \ X_2 = \partial/\partial x, \ \ X_3 = \partial/\partial u,$ corresponding to shifts in t, x, u.

We consider under what specializations of and Φ the algebra can be expanded. An analysis of the determining equations [4] for (1.1) shows that the following statement holds.

If $\mu^{'} \not\equiv 0$, $\partial\Phi/\partial u_x \not\equiv 0$, $\partial\Phi/\partial\lambda \not\equiv 0$, then the algebra expands only for one of the following sets of μ and Φ:

1 . μ is an arbitrary function, $\Phi = u_x^{\alpha} f\left(\lambda\right)$ where f are arbitrary functions, and $\alpha \neq 0$; the additional basis operator has the form:

$$X_4 = \alpha x \frac{\partial}{\partial x} + 2\alpha t \frac{\partial}{\partial t} + \left(\alpha - 2\right) u \frac{\partial}{\partial u}$$

2 . μ is an arbitrary function, $\Phi = \left(\mu^{1+\gamma}/\mu^{'}\right) F\left(u_x \mu^{\beta}\right)$, where $\partial F/\partial u_x \not\equiv 0$ is a constant; the additional operator:

$$X_5 = \frac{1-\gamma}{2} x \frac{\partial}{\partial x} - \gamma t \frac{\partial}{\partial t} + \left(\frac{1-\gamma}{2} - \beta\right) u \frac{\partial}{\partial u} + \frac{\mu}{\mu'} \frac{\partial}{\partial \lambda}$$

3 . μ is an arbitrary function, $\Phi = \left(\mu/\mu^{'}\right)\left(1 + \mu u_x^{\varepsilon}\right)$, where $\Phi = \left(\mu/\mu^{'}\right)\left(1 + \mu u_x^{\varepsilon}\right)$ is a constant; the additional operators:

$$X_6 = \varepsilon x \frac{\partial}{\partial x} + \left(\varepsilon - 2\right) u \frac{\partial}{\partial u} + 2\varepsilon \frac{\mu}{\mu'} \frac{\partial}{\partial \lambda'}, \quad X_7 = e^{-t} \frac{\partial}{\partial t} + e^{-t} \frac{\mu}{\mu'} \frac{\partial}{\partial \lambda}$$

2. We consider the invariant solutions arising from the additional basis operators specified in Case 1. For each of these cases, we derive invariant solutions corresponding to the operators included in the optimal system of subalgebras [4] that contain additional operators.

1) The optimal system of subalgebras includes the operator $X_4 + cX_3, \ c \in R$, if $\alpha \neq 2$, then $c = 0$ satisfies. The corresponding invariant solution takes the form:

$$u = c \ln t + t^{-\nu}\varphi\left(\xi\right), \ \lambda = \psi\left(\xi\right), \ \xi = xt^{-1/2}$$

where $\nu = (2-\alpha)/(2\alpha)$, φ and ψ satisfy the system of ordinary differential equations:

$$\nu\varphi + 1/2\xi\varphi' + \left(\mu\left(\psi\right)\varphi'\right)' = 1/4\, c, \ 1/2\xi\psi' + \varphi\left(\psi\right)\psi'^{\alpha} = 0$$

2) If $\gamma \neq 0$ or $\gamma \neq 1$, the operator entering the optimal system has the form $X_5 + cX_3, \ \ c \in R$, where $c = 0$ for $\sigma = \beta - \frac{1}{2}\left(1-\gamma\right) \neq 0$. The invariant solution is: $u = (c/\gamma)\ln t + \varphi\left(\xi\right), \ \mu\left(\lambda\right) = t^{-1/\gamma}\psi\left(\xi\right), \ \xi = xt^{(1-\gamma)/(2\gamma)}$, and φ and ψ satisfy the system:

$$\frac{c}{\gamma} + \frac{\sigma}{\gamma}\varphi + \frac{1-\gamma}{2\gamma}\xi\varphi' = \left(\psi\varphi;\right)', \left(1-\gamma\right)\xi\psi' = 1 + \gamma\psi^{1+\gamma}F\left(\varphi'\psi^{\beta}\right)$$

When $\gamma = 0$, the optimal system includes $X_5 + aX_1 + cX_3,\ \ a, c \in R$, where $\beta \neq 1/2$, then $c = 0$. The invariant solution is:

$$u = x^{1-2\beta}\varphi(\xi) + 2c\ln x\ , \mu(\lambda) = x^2\varphi(\xi)\ , \xi = t - 2\alpha\ \ln x$$

When $\gamma = 1$, the required operator has the form $X_5 + bX_2 + cX_3$ (with $\beta \neq 0$ when $c = 0$), and the corresponding solution is:

$$u = t^{\beta}\varphi(\xi) - c\ln t\ , \mu(\lambda) = t^{-1}\psi(\xi)\ ,\ \ \xi = x + b\ \ln t$$

The functions φ and ψ satisfy the corresponding system of differential equations in each case.

3) The operators entering the optimal system can be conveniently represented as:

a) $aX_1 + cX_3 + X_6$,

b) $bX_2 + dX_3 + X_7$,

c) $cX_3 + X_6 + X_7$,

where $a, b, c, d \in R$, and $c = 0c = 0$ when $c = 0$ at $\varepsilon \neq 2$. For each of these operators, the invariant solutions are written as:

a) $u = x^{(\varepsilon-2)/\varepsilon}\varphi(\xi) + 2c\ln x\ ,\ \ \mu(\lambda) = x^2\psi(\xi)\ ,\ \ \xi =\ x^{-2a}e^t$

b) $u = de^t + \varphi(\xi)\ ,\ \ \ \mu(\lambda) = e^t\psi(\xi)\ ,\ \ \xi =\ x - be^t$

c) $u = e^{(\varepsilon-2)t}\varphi(\xi)\ ,\ \ \mu(\lambda) = e^{2\varepsilon t}\tau\psi(\xi)\ ,\ \ \xi =\ \ln x\ - \varepsilon\tau,\ \ t = e^t$

Here, as before, the functions φ and ψ satisfy the corresponding system of differential equations.

Let us analyze some of the invariant solutions obtained in section 2. In applications, a linear kinetic equation in λ and some power of u_x is often used, i.e., an equation of the form $\lambda_t + \lambda = u_x^m$. In equations (1.1), a power dependence $\mu = \lambda^n$ is assumed. For this case, the invariant solution can be written as:

$$\mu = \lambda^n$$

$$u = (x_0 - x)^p\varphi(t)\ ,\ \ \lambda = (x_0 - x)^{2/n}\psi(t)\ ,\ \ p = (2/(mn)) + 1,\ x_0 > 0$$

In the case $n = -1$, the system of differential equations for φ and ψ is integrated in quadratures, and its solution is written as:

$$t = t_0 + \ln\left(1 + l\int_{\varphi_0}^{\varphi}\exp\left[q\left(\varphi^m - \varphi_0^m\right)\right]d\varphi\right) \tag{1.2}$$

$$\psi = \psi_0 exp\left[q\left(\varphi^m - \varphi_0^m\right)\right]\left(1 + l\int_{\varphi_0}^{\varphi}\frac{exp\left(q\varphi^m\right)}{\varphi}d\varphi\right)^{-1}$$

where

$$q = \frac{(m-2)^{m-1}}{2(m-1)m^{m-1}}, l = \frac{\psi_0 m^2}{(m-1)(m-2)}.$$

For a thixotropic medium, m must be positive. Then, from equation (1.2), it can be seen that if at the initial moment there was a zone with a destroyed structure $(0 \leq x \leq x_0)$, then at later times the disturbance, under a given boundary condition at $x = 0$, will not spread beyond $x = x_0$.

The solvability of the system of equations for φ and ψ in quadratures for the operator $bX_2 + dX_3 + X_7$ allows us to explicitly write the corresponding invariant solution:

$$u = de^t + b\psi^{-1}(\xi) + c_3,\ \ \mu(\lambda) = e^t\psi(\xi)$$

$$\psi = exp\left[-1/2db^{-1}(\xi + c_1)^2\right]\left(b\int exp\left[-1/2\ db^{-1}(\xi + c_1)^2\right]d\xi + c_2\right)$$

$$\xi = x - be^t$$

where $c_1,\ c_2,\ c_3$ are arbitrary constants.

In cases where the system of differential equations for φ and ψ cannot be solved in quadratures, numerical methods and methods from the qualitative theory of ordinary differential equations may be used to analyze the corresponding invariant solutions.

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