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Reducing the inverse problem for a one nonlinear equation of vibrations of thin plate to an optimal control problem and its investigation

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Abstract. *The paper deals with an inverse problem of determining the right-hand side of the linear equation of oscillations of thin plates. The problem is reduced to the optimal control problem. Existence of the optimal control proved. Differentiability of the functional is studied. Necessary condition of optimality is derived.*

Keywords. thin plate · inverse problem · optimal control · adjoint problem · optimality conditions.

Mathematics Subject Classification (2010): 49N45

1 Introduction

Systems described by fourth other differential equations are often arise in mechanics, physics and applied problems. Therefore, studying of optimal control problems related to fourth order partial differential equations is of great importance in different fields. These equations are often used in creating dynamic and high accuracy models. These equations can describe the properties (for example, free vibrations in mechanical systems, elastic motions, etc.) of various physical systems very accurately. The control of prosses described by these equations enables to control real systems very accurately and to optimization them.

The study of optimal control problems for fourth order partial differential equations is very important in terms of development of modern technologies, control of complex systems and ensuing efficient safe utilization. Such studies have wide applications in various fields including aerospace, robotics, transport, biomedical engineering and energy fields. The studies lead to the development of the best control systems and technologies and in future this will enable to create efficient and safe systems. It is known that vibrations of thin plates are also described by fourth order partial differential equations. Therefore, the

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study of optimal control problems for the equation of vibrations of thin plates is of great theoretical and practical interest [3].

There are various methods for finding the coefficients and the right hand sides of fourth order equation. One of these methods is to approach these problems as an inverse problem, to reduce them to optimal control problems and solve them by means of the methods of this theory. This method is called the variational or optimization method. Such problems began to be studied since the end of the XX century and are currently being intensively studied [3-10].

2 Statement of the problem

Our needs to find the pair of functions $(u, \upsilon) \in U \times U_{ad}$ from the relations

$$\rho\frac{\partial^2 u}{\partial t^2} + \Delta\left(D\Delta u\right) + (1-\nu)\left(2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial^2 D}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 u}{\partial x^2}\right) +$$

$$+u^3 = \upsilon\left(x,y\right) f\left(t\right), (x,y,t) \in Q, \tag{2.1}$$

$$u(x,y,0) = f_0(x,y),\ \frac{\partial u(x,y,0)}{\partial t} = f_1(x,y), (x,y) \in O, \tag{2.2}$$

$$u(0,y,t) = 0,\ \frac{\partial u(0,y,t)}{\partial x} = 0,\ \ 0 \le y \le b,\ 0 \le t \le T,$$

$$u(a,y,t) = 0,\ \frac{\partial u(a,y,t)}{\partial x} = 0,\ \ 0 \le y \le b,\ 0 \le t \le T,$$

$$u(x,0,t) = 0,\ \frac{\partial u(x,0,t)}{\partial y} = 0,\ 0 \le x \le a, 0 \le t \le T,$$

$$u(x,b,t) = 0,\ \frac{\partial u(x,b,t)}{\partial y} = 0,\ 0 \le x \le a,\ 0 \le t \le T, \tag{2.3}$$

$$\int_0^T K(x,y,t)u(x,y,t)dt = g(x,y), \tag{2.4}$$

where $(x,y) \in \Omega = \{(x,y) : 0 < x < a,\ 0 < y < b\}, t \in (0,T), Q = \Omega \times (0,T), \rho(x,y)$ is a density of the mass at the point (x,y), $h(x,y)$ is the heath thickness of the plate in the point (x,y), $u(x,y,t)$ - is deflection of the plate in the point (x,y) at the moment t, Δ is Laplace operator with respect to x, y, $D = \frac{Eh^3}{12(1-\nu^2)}$- cylindrical rigidity, ν $\left(0 < \nu < \frac{1}{2}\right)$-Poisson's coefficient, $E > 0$-Young's modulus,

$$U = \left\{u|\, u(x,y,t) \in C([0,T]\,; \overset{\circ}{W}{}_2^2(\Omega)), \frac{\partial u}{\partial t} \in C([0,T]\,; L_2\left(\Omega\right))\right\},$$

$U_{ad} = \{\nu|\, \upsilon(x,y) \in L_2(\Omega) : \mu_0 \le \upsilon(x,y) \le \mu_1 a.\text{e.on } \Omega\}$, $f\left(t\right) \in L_2\left(0,T\right)$, $f_0(x,y) \in \overset{\circ}{W}{}_2^2(\Omega)$, $f_1(x,y) \in L_2(\Omega)$, $K(x,y,t) \in L_\infty(Q)$, $g(x,y) \in L_2(\Omega)$ are given functions, $h(x,y)$ -is sufficiently smooth given function, a, b, T are given positive numbers, μ_0, μ_1 given numbers.

As a generalized solution for the problem (2.1)-(2.3) for each function $\upsilon(x,y)$ from $L_2(\Omega)$ we consider the function $u(x,y,t) \in U$ such that for any $\forall \eta(x,y,t) \in U, \eta(x,y,T) = 0$ the integral identity

$$\int_Q \left[-\rho \frac{\partial u}{\partial t} \cdot \frac{\partial \eta}{\partial t} + D\Delta u \Delta \eta + (1-\nu)\left(2\frac{\partial^2 D}{\partial x \partial y}\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 D}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 u}{\partial x^2} \right) \eta \right] dxdydt -$$

$$- \int_\Omega \rho f_1(x,y)\eta(x,y,0)dxdy + \int_Q u^3 \eta dxdydt = \int_Q \upsilon(x,y) f(t) \eta dxdydt. \tag{2.5}$$

is fulfilled.

This problem we reduce to the following optimal control problem: to find the minimum of the functional

$$J_0(\upsilon) = \frac{1}{2}\int_\Omega \left[\int_0^T K(x,y,t)u(x,y,t,\upsilon)dt - g(x,y) \right]^2 dxdy, \tag{2.6}$$

subject to (2.1)-(2.3). The function $\upsilon(x,y)$ is called a control. By $u = u(x,y,t,\upsilon)$ we denote the generalized solution of the problem (2.1)-(2.3) corresponding to the control $\upsilon(x,y)$.

We regularize the problem (2.1)-(2.3), (2.6) by the following way: instead of the functional (2.6) consider the next one

$$J_a(\upsilon) = J_0(\upsilon) + \frac{a}{2}\int_\Omega \upsilon^2(x,y)dxdy, \tag{2.7}$$

where $a > 0$ is a positive number.

Let's assume that by any fixed control $\upsilon(x,y)$ boundary problem (2.1)-(2.3) has unique generalized solution from U.

3 Existence of the optimal control

Theorem 1. Under the imposed conditions on the problem data, there exists an optimal control in problem (2.1)-(2.3), (2.7).

Proof. Let's $\{\upsilon_n\} \in U_{ad}$ be a minimizing sequence, i.e.

$$\lim_{n\to\infty} J_a(\upsilon_n) = \inf_{\upsilon \in U_{ad}} J_a(\upsilon).$$

It is clear, that

$$\|\upsilon_n\|_{L_2(\Omega)} \le const. \tag{3.1}$$

Taking into account, for solutions of problem (2.1)-(2.3) corresponding to υ_n, we obtain the estimation

$$\|u_n\|_{\overset{\circ}{W}{}_2^2(\Omega)} + \left\| \frac{\partial u_n}{\partial t} \right\|_{L_2(\Omega)} \le const, \quad \forall t \in [0,T]. \tag{3.2}$$

By virtue of (3.1) and (3.2), property of weak compactness in he Hilbert spaces and imbedding theorem [13], it is possible to consider, that as $n \to \infty$

$\upsilon_n \to \upsilon_0$ weakly in $L_2(\Omega)$,
$u_n \to u_0$ weakly in $L_6(Q)$,
$u_n \to u_0$ a.e.on Q,
$u_n^3 \to u_0^3$ weakly in $L_2(Q)$,
$u_n \to u_0$, $\frac{\partial u_n}{\partial x} \to \frac{\partial u_0}{\partial x}$, $\frac{\partial u_n}{\partial y} \to \frac{\partial u_0}{\partial y}$ strongly in $L_2(Q)$,

$\frac{\partial^2 u_n}{\partial x^2} \to \frac{\partial^2 u_0}{\partial x^2}$, $\frac{\partial^2 u_n}{\partial x \partial y} \to \frac{\partial^2 u_0}{\partial x \partial y}$, $\frac{\partial^2 u_n}{\partial y^2} \to \frac{\partial^2 u_0}{\partial y^2}$ weakly in $L_2(Q)$.

Considering these relations, in the definition of the generalized solution for the problem (2.1)-(2.3), by $\upsilon = \upsilon_n$, $u = u_n$, passing to limit as $n \to \infty$ we have

$$\int_Q \left[-\rho \frac{\partial u_0}{\partial t} \cdot \frac{\partial \eta}{\partial t} + D\Delta u_0 \Delta \eta + (1-\nu) \left(2\frac{\partial^2 D}{\partial x \partial y} \frac{\partial^2 u_0}{\partial x \partial y} - \frac{\partial^2 D}{\partial x^2} \frac{\partial^2 u_0}{\partial y^2} - \frac{\partial^2 D}{\partial y^2} \frac{\partial^2 u_0}{\partial x^2} \right) \eta \right] dxdydt-$$

$$-\int_\Omega \rho \varphi_1(x,y) \eta(x,y,0) dxdy + \int_Q u_0^3 \eta dxdydt = \int_Q \eta\,(x,y)\, f\,(t)\, \eta dxdydt.$$

Therefore,

$$\lim_{n\to\infty} J_a(\upsilon_n) = \inf_{\upsilon \in U_{ad}} J_\alpha(\upsilon) = J_\alpha(\upsilon_0).$$

It shows, that $\upsilon_0(x,y)$ provides the minimum to functional (2.7), i.e. is an optimal control.

The theorem is proved.

4 Differentiability of the functional (2.7) and necessary and sufficient optimality conditions

Let us introduce the adjoint to (2.1)-(2.3), (2.7) problem for the given control $\upsilon(x,y) \in L_2(\Omega)$:

$$\rho \frac{\partial^2 \psi}{\partial t^2} + \Delta\,(D\Delta\psi) + (1-\nu) \left[2\frac{\partial^2}{\partial x \partial y} \left(\frac{\partial^2 D}{\partial x \partial y} \psi \right) - \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 D}{\partial y^2} \psi \right) - \frac{\partial^2}{\partial y^2} \left(\frac{\partial^2 D}{\partial x^2} \psi \right) \right] +$$

$$+3u^2\psi = -K(x,y,t) \left[\int_0^T K(x,y,t) u(x,y,t) dt - g(x,y) \right], (x,y,t) \in Q, \qquad (4.1)$$

$$\psi(x,y,T) = 0, \rho \frac{\partial \psi(x,y,T)}{\partial t} = 0, (x,y) \in \Omega, \qquad (4.2)$$

$$\psi(0,y,t) = \psi(a,y,t) = 0, \frac{\partial \psi(0,y,t)}{\partial x} = \frac{\partial \psi(a,y,t)}{\partial x} = 0, 0 \le y \le b,\ 0 \le t \le T,$$

$$\psi(x,0,t) = \psi(x,b,t) = 0, \frac{\partial \psi(x,0,t)}{\partial y} = \frac{\partial \psi(x,b,t)}{\partial y} = 0,\ 0 \le x \le a,\ 0 \le t \le T. \qquad (4.3)$$

From the conditions imposed on the data of the problem (2.1)-(2.3), (2.7) follows that this ad joint problem has unique generalized solution from the space $W_2^{2,1}(Q)$ [11].

To derive the necessary conditions for optimality in the considered problem we take two arbitrary admissible controls $\upsilon(x,y)$ and $\upsilon(x,y) + \delta\upsilon(x,y)$. The corresponding solutions of problem (2.1)-(2.3) are denoted by $u(x,y,t;\upsilon)$ and $u(x,y,t;\upsilon+\delta\upsilon) \equiv u(x,y,t;\upsilon) + du(x,y,t)$. Then $\delta u(x,y,t) = u(x,y,t;\upsilon+\delta\upsilon) - u(x,y,t;\upsilon)$ is a solution of the boundary value problem

$$\rho \frac{\partial^2 (\delta u)}{\partial t^2} + \Delta(D\Delta(\delta u)) + (1-\nu) \left[2\frac{\partial^2 D}{\partial x \partial y} \frac{\partial^2 (\delta u)}{\partial x \partial y} - \frac{\partial^2 D}{\partial x^2} \frac{\partial^2 (\delta u)}{\partial y^2} - \right.$$

$$\left. \frac{\partial^2 D}{\partial y^2} \frac{\partial^2 (\delta u)}{\partial x^2} \right] + 3(u + \theta\delta u)^2 \delta u = f\,(t)\, \delta\upsilon, \qquad (4.4)$$

$$\delta u(x,y,0) = 0, \frac{\partial(\delta u(x,y,0))}{\partial t} = 0, \qquad (4.5)$$

$$\delta u(0,y,t)=\delta u(a,y,t)=0,\frac{\partial(\delta u(0,y,t))}{\partial x}=\frac{\partial(\delta u(a,y,t))}{\partial x}=0,$$

$$\delta u(x,0,t)=\delta u(x,b,t)=0,\frac{\partial\left(\delta u\left(x,0,t\right)\right)}{\partial y}=\frac{\partial\left(\delta u\left(x,b,t\right)\right)}{\partial y}=0, \tag{4.6}$$

$0\le\theta\le 1$.

Let's show that

$$\|\delta u\|_{\overset{\circ}{W}{}_2^2(\Omega)}+\left\|\frac{\partial\delta u}{\partial t}\right\|_{L_2(\Omega)}\le C\|\delta\upsilon\|_{L_2(\Omega)},\forall t\in[0,T]. \tag{4.7}$$

For this purpose, we use Faedo-Galerkin's method. Take the basis $\{\omega_i(x,y)\}_{i=1}^{\infty}$ from $\overset{\circ}{W}{}_2^2(\Omega)$ where the system $\{\omega_i(x,y)\}_{i=1}^{\infty}$ is orthonormal in $L_2(\Omega)$ and the approximate solution for the problem (4.4)-(4.6) search in the form

$$\delta u^N(x,y,t)=\sum_{i=1}^{N}c_i^N(t)\omega_i(x,y)$$

from the equalities

$$\int_\Omega\rho\frac{\partial^2\delta u^N}{\partial t^2}\omega_j(x,y)dxdy+\int_\Omega D\Delta\delta u^N\Delta\omega_j(x,y)dxdy+$$

$$+(1-\nu)\int_\Omega\left(2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2\delta u^N}{\partial x\partial y}-\frac{\partial^2 D}{\partial x^2}\frac{\partial^2\delta u^N}{\partial y^2}-\frac{\partial^2 D}{\partial y^2}\frac{\partial^2\delta u^N}{\partial x^2}\right)\omega_j(x,y)dxdy+$$

$$+3\int_Q\theta^2\left(\delta u^N\right)^3\omega_j\left(x,y\right)dxdydt+$$

$$+3\int_Q\left[\left(u+\theta\delta u^N\right)^2\delta u^N-\theta^2\left(\delta u^N\right)^3\right]\omega_j\left(x,y\right)dxdydt=$$

$$=\int_\Omega f\left(t\right)\delta\upsilon\omega_j(x,y)dxdy,\ \ 1\le j\le N, \tag{4.8}$$

$$c_i^N(0)=0,\ \frac{d}{dt}c_i^N(t)\bigg|_{t=0}=0.$$

Both sides of (4.8) multiply by $\frac{d}{dt}c_j^N(t)$ and sum over j from 1 to N. Then we get

$$\frac{1}{2}\frac{d}{dt}\int_O\left[\rho\left(\frac{\partial\delta u^N}{\partial t}\right)^2+D\left(\Delta\delta u^N\right)^2+\frac{3\theta^2}{4}\left(\delta u^N\right)^4\right]dxdy=$$

$$=-(1-\nu)\int_\Omega\left(2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2\delta u^N}{\partial x\partial y}-\frac{\partial^2 D}{\partial x^2}\frac{\partial^2\delta u^N}{\partial y^2}-\frac{\partial^2 D}{\partial y^2}\frac{\partial^2\delta u^N}{\partial x^2}\right)\frac{\partial\delta u^N}{\partial t}dxdy-$$

$$-3\int_Q\left[\left(u+\theta\delta u^N\right)^2\delta u^N-\theta^2\left(\delta u^N\right)^3\right]\frac{\partial\delta u^N}{\partial t}dxdydt+$$

$$+\int_\Omega f\left(t\right)\delta\upsilon\frac{\partial\delta u^N}{\partial t}dxdy.$$

If to integrate this equality over t by the imposed conditions we get

$$\int_{\Omega}\left[\left(\frac{\partial\delta u^N(x,y,t)}{\partial t}\right)^2+\left(\Delta\delta u^N(x,y,t)\right)^2+\frac{3\theta^2}{4}\left(\delta u^N\right)^4\right]dxdy\le$$
$$+C\int_0^t\int_{\Omega}\left[\left|\frac{\partial^2\delta u^N}{\partial x^2}\right|^2+\left|\frac{\partial^2\delta u^N}{\partial y^2}\right|^2+\left|\frac{\partial^2\delta u^N}{\partial x\partial y}\right|^2\right]dxdyds+$$
$$+C\int_0^t\int_{\Omega}\left|\frac{\partial\delta u^N(x,y,t)}{\partial t}\right|^2dxdyds+$$
$$+C\int_0^t\int_{\Omega}\left|\delta u^N\right|\left|\frac{\partial\delta u^N(x,y,t)}{\partial t}\right|dxdyds+$$
$$+C\int_0^t\int_{\Omega}\left|\delta u^N\right|^2\left|\frac{\partial\delta u^N(x,y,t)}{\partial t}\right|dxdyds+C\int_0^t\int_{\Omega}|\delta\upsilon|^2|f(t)|^2dxdyds\le$$
$$\le C\int_0^t\int_{\Omega}\left|\delta u^N\right|^2dxdyds+C\int_0^t\int_{\Omega}\left|\delta u^N\right|^4dxdyds+C\int_0^t\int_{\Omega}\left|\frac{\partial\delta u^N(x,y,t)}{\partial t}\right|^2dxdyds+$$
$$+C\int_0^t\int_{\Omega}\left[\left|\frac{\partial^2\delta u^N}{\partial x^2}\right|^2+\left|\frac{\partial^2\delta u^N}{\partial y^2}\right|^2+\left|\frac{\partial^2\delta u^N}{\partial x\partial y}\right|^2\right]dxdyds\le$$
$$\le C\int_0^t\int_{\Omega}\left[\left(\delta u^N(x,y,s)\right)^2+\left(\frac{\partial\delta u^N(x,y,s)}{\partial t}\right)^2+\right.$$
$$+\left(\frac{\partial\delta u^N(x,y,s)}{\partial x}\right)^2+\left(\frac{\partial\delta u^N(x,y,s)}{\partial y}\right)^2+$$
$$\left.+\left(\frac{\partial^2\delta u^N(x,y,s)}{\partial x^2}\right)^2+\left(\frac{\partial^2\delta u^N(x,y,s)}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\delta u^N(x,y,s)}{\partial y^2}\right)^2\right]dxdyds+$$
$$+C\int_0^t\int_{\Omega}\left|\delta u^N\right|^4dxdyds+C\|\delta\upsilon\|^2{}_{L_2(\Omega)},\forall t\in[0,T],$$

where by C the constants not depending on the estimating quantities and admissible controls are defined.

Due to equivalency of the norms in $\overset{\circ}{W}{}_2^2(\Omega)$ we obtain

$$\int_{\Omega}\left[\left|\delta u^N(x,y,t)\right|^4+\left(\delta u^N(x,y,t)\right)^2+\left(\frac{\partial\delta u^N(x,y,t)}{\partial t}\right)^2+\right.$$
$$\left.+\left(\frac{\partial\delta u^N(x,y,t)}{\partial x}\right)^2+\left(\frac{\partial\delta u^N(x,y,t)}{\partial y}\right)^2+\left(\Delta\delta u^N(x,y,t)\right)^2\right]dxdy\le$$
$$\le C\int_0^t\int_{\Omega}\left[\left|\delta u^N(x,y,t)\right|^4+\left(\frac{\partial\delta u^N(x,y,s)}{\partial t}\right)^2+\left(\frac{\partial\delta u^N(x,y,s)}{\partial x}\right)^2+\left(\frac{\partial\delta u^N(x,y,s)}{\partial y}\right)^2+\right.$$
$$\left.+\left(\frac{\partial^2\delta u^N(x,y,s)}{\partial x^2}\right)^2+\left(\frac{\partial^2\delta u^N(x,y,s)}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\delta u^N(x,y,s)}{\partial y^2}\right)^2\right]dxdyds+C\|\delta\upsilon\|^2{}_{L_2(\Omega)}.\tag{4.9}$$

Following to known inequality [13]

$$\int_{\Omega}\left[\left(\frac{\partial^2\delta u^N}{\partial x^2}\right)^2+\left(\frac{\partial^2\delta u^N}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\delta u^N}{\partial y^2}\right)^2\right]dxdy\le$$

$$\le\int_{\Omega}\left[\frac{\partial^2\delta u^N}{\partial x^2}+\frac{\partial^2\delta u^N}{\partial y^2}\right]^2dxdy$$

from (4.9) we have

$$\int_{\Omega}\left[\left|\delta u^N(x,y,t)\right|^4+\left(\delta u^N(x,y,t)\right)^2+\left(\frac{\partial\delta u^N(x,y,t)}{\partial t}\right)^2+\right.$$

$$+\left(\frac{\partial\delta u^N(x,y,t)}{\partial x}\right)^2+\left(\frac{\partial\delta u^N(x,y,t)}{\partial y}\right)^2+\left(\frac{\partial^2\delta u^N(x,y,t)}{\partial x^2}\right)^2+$$

$$\left.+\left(\frac{\partial^2 du^N(x,y,t)}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\delta u^N(x,y,t)}{\partial y^2}\right)^2\right]dxdy\le$$

$$\le C\int_0^t\int_{\Omega}\left[\left(\delta u^N(x,y,s)\right)^4+\left(\delta u^N(x,y,s)\right)^2+\left(\frac{\partial\delta u^N(x,y,s)}{\partial t}\right)^2+\right.$$

$$+\left(\frac{\partial\delta u^N(x,y,s)}{\partial x}\right)^2+\left(\frac{\partial\delta u^N(x,y,s)}{\partial y}\right)^2+\left(\frac{\partial^2\delta u^N(x,y,s)}{\partial x^2}\right)^2+$$

$$\left.+\left(\frac{\partial^2\delta u^N(x,y,s)}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\delta u^N(x,y,s)}{\partial y^2}\right)^2\right]dxdyds+C\|\delta v\|^2_{L_2(\Omega)}.$$

Application of the Gronwall's lemma leads to

$$\int_{\Omega}\left[\left|\partial\delta^N\right|^4+\left(\delta u^N(x,y,t)\right)^2+\left(\frac{\partial\delta u^N(x,y,t)}{\partial t}\right)^2+\left(\frac{\partial\delta u^N(x,y,t)}{\partial x}\right)^2\right.$$

$$+\left(\frac{\partial\delta u^N(x,y,t)}{\partial y}\right)^2+\left(\frac{\partial^2\delta u^N(x,y,t)}{\partial x^2}\right)^2+$$

$$\left.+\left(\frac{\partial^2\delta u^N(x,y,t)}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\delta u^N(x,y,t)}{\partial y^2}\right)^2\right]dxdy\le$$

$$\le C\|\delta v\|^2_{L_2(\Omega)},\ \forall t\in[0,T]. \tag{4.10}$$

From this we get

$$\left\|\delta u^N\right\|_{\overset{\circ}{W}{}_2^2(\Omega)}+\left\|\frac{\partial\delta u^N}{\partial t}\right\|_{L_2(\Omega)}\le C\|\delta v\|^2_{L_2(\Omega)},\forall t\in[0,T]. \tag{4.11}$$

As follows from this inequality from the sequence $\left\{\delta u^N(x,y,t)\right\}$ one can chose a subsequence (which is also denoted by$\left\{\delta u^N(x,y,t)\right\}$) that converges weakly in U to some function $\delta u(x,y,t)$ by $N\to\infty$.

Then, by the weak lower semi-continuity of the norm in the Banax space (4.11) implies estimate (4.7).

Theorem 2. Let's the conditions of the Theorem 1 be satisfied. Then functional (2.7) is continuously Frechet differentiable on U_{ad} and its differential in the point $\forall \upsilon \in U_{ad}$ at the increment $\delta\upsilon \in L_2(\Omega)$, $\upsilon + \delta\upsilon \in U_{ad}$ is defined by the expression

$$\left\langle J_\alpha'(\upsilon), \delta\upsilon \right\rangle = \int_\Omega \left[\alpha\upsilon\left(x,y\right) - \int_0^T f\left(t\right)\psi(x,y,t)dt \right] \delta\upsilon dxdy.$$

Proof. Let's calculate the increment of the functional $J_\alpha(\upsilon)$:

$$\Delta J_\alpha\left(\upsilon\right) = J_\alpha\left(\upsilon + \delta\upsilon\right) - J_\alpha\left(\upsilon\right) =$$

$$= \frac{1}{2}\int_\Omega \left(\int_0^T K\left(u+\delta u\right)dt - g\left(x,y\right) \right)^2 dxdy - \frac{1}{2}\int_\Omega \left(\int_0^T Kudt - g\left(x,y\right) \right)^2 dxdy +$$

$$+ \frac{\alpha}{2}\int_\Omega \left[(\upsilon+\delta\upsilon)^2 - \upsilon^2 \right] dt =$$

$$= \int_\Omega \left[\left(\int_0^T Kudt - g\left(x,y\right) \right) \int_0^T K\delta udt \right] dxdy + \alpha\int_O \upsilon\delta\upsilon dt + R_1, \tag{4.12}$$

here

$$R_1 = \frac{1}{2}\int_\Omega \left(\int_0^T K\delta udt \right)^2 dxdy + \frac{\alpha}{2}\int_\Omega (\delta\upsilon)^2 dt$$

is remainder term.

Taking into account (4.7), we obtain

$$R_1 \le C\|\delta\upsilon\|^2_{L_2(\Omega)}. \tag{4.13}$$

Since δu is a generalized solution of the problem (4.4)-(4.6), for arbitrary function $\eta(x,y,t) \in U$, $\eta(x,y,T) = 0$,

$$\eta(0,y,t) = 0, \frac{\partial\eta(0,y,t)}{\partial x} = 0, \eta(x,0,t) = 0, \ \frac{\partial\eta(x,0,t)}{\partial y} = 0,$$

$$\eta(a,y,t) = 0, \frac{\partial\eta(a,y,t)}{\partial x} = 0, \ \eta(x,b,t) = 0, \frac{\partial\eta(x,b,t)}{\partial y} = 0.$$

is valid integral identity

$$\int_Q \left\{ -\rho\frac{\partial(\delta u)}{\partial t}\frac{\partial\eta}{\partial t} + D\Delta(\delta u)\Delta\eta + (1-\nu)\left[2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2(\delta u)}{\partial x\partial y} - \frac{\partial^2 D}{\partial x^2}\frac{\partial^2(\delta u)}{\partial y^2} - \right.\right.$$

$$\left.\left. \frac{\partial^2 D}{\partial y^2}\frac{\partial^2(\delta u)}{\partial x^2} \right]\eta + 3(u+\theta\delta u)^2\delta u\eta \right\} dxdydt = \int_Q f(t)\delta\upsilon\eta dxdydt. \tag{4.14}$$

Similarly, since $\psi(x,y,t)$ is a solution of the problem (4.1)-(4.3), for any function $\chi(x,y,t) \in U$, $\chi(x,y,0) = 0$,

$$\chi(0,y,t) = 0, \frac{\partial\chi(0,y,t)}{\partial x} = 0, \chi(x,0,t) = 0, \frac{\partial\chi(x,0,t)}{\partial y} = 0,$$

$$\chi(a,y,t) = 0, \frac{\partial\chi(a,y,t)}{\partial x} = 0, \chi(x,b,t) = 0, \frac{\partial\chi(x,b,t)}{\partial y} = 0$$

we have

$$\int_Q \left\{ -\rho \frac{\partial \psi}{\partial t}\frac{\partial \chi}{\partial t} + D\Delta\psi\Delta\chi + (1-\nu)\left[2\frac{\partial^2 D}{\partial x \partial y}\psi \frac{\partial^2 \chi}{\partial x \partial y} - \frac{\partial^2 D}{\partial x^2}\psi \frac{\partial^2 \chi}{\partial y^2} - \right.\right.$$

$$\left.\left. - \frac{\partial^2 D}{\partial y^2}\psi \frac{\partial^2 \chi}{\partial x^2}\right] + 3u^2\psi\chi \right\} dxdydt =$$

$$- \int_Q K(x,y,t)\left[\int_0^T K(x,y,t)u(x,y,t)dt - g(x,y)\right]\chi dxdydt. \tag{4.15}$$

If in (4.14) to take $\psi(x,y,t)$ instead of $\eta(x,y,t)$, and in (4.15) to take $\delta u(x,y,t)$ instead of $\chi(x,y,t)$ and subtract (4.14) from (4.15) we obtain

$$\int_\Omega \left(\int_0^T Kudt - g(x,y)\right) \int_0^T K\delta u dxdydt = \int_Q 3\theta \left[2u\delta u + \theta(\delta u)^2\right] \delta u \psi(x,y,t)dxdydt -$$

$$- \int_Q f(t)\psi(x,y,t)\delta \upsilon\,(x,y)\,dxdydt. \tag{4.16}$$

Then from (4.12) and (4.16) follows

$$\Delta J_\alpha(\upsilon) = \alpha \int_\Omega \upsilon\delta\upsilon dt - \int_Q f(t)\psi(x,y,t)\delta\upsilon\,(x,y)\,dxdydt + R, \tag{4.17}$$

here

$$R = R_1 + R_2,$$

$$R_2 = \int_Q 3\theta \left[2\theta\delta u + \theta\,(\delta u)^2\right] \delta u\psi(x,y,t)dxdydt.$$

Taking into account (4.7), we obtain

$$|R_2| \le C\|\delta\upsilon\|^2_{L_2(\Omega)}. \tag{4.18}$$

Then from formula for increment of the functional (4.17) follows that differential of functional (2.7) is calculate by formula

$$\left\langle J'_\alpha(\upsilon), \delta\upsilon \right\rangle = \int_\Omega \left[\alpha\upsilon\,(x,y) - \int_0^T f\,(t)\,\psi(x,y,t)dt\right] \delta\upsilon dxdy. \tag{4.19}$$

Then as follows from (4.19) the gradient of the functional has a from

$$gradJ_\alpha\,(\upsilon) = \alpha\upsilon\,(x,y) - \int_0^T f\,(t)\psi(x,y,t)dt.$$

Thus due to known theorem from [14, pp. 28] in order to the control function $\upsilon_*(x,y)$ was optimal, it is necessary fulfillment of the inequality

$$\int_\Omega \left[\alpha\upsilon_*\,(x,y) - \int_0^T f(t)\psi(x,y,t)dt\right] (\upsilon(x,y) - \upsilon_*(x,y))\,dxdy \ge 0 \forall \upsilon \in U_{ad}. \tag{4.20}$$

Thus the following theorem is proved.

Theorem 3. Let's the conditions of the Theorem 1 be satisfied. Then for the optimality of the control $\upsilon_* \in U_{ad}$ in problem (2.1)-(2.3),(2.7) it is necessary fulfilment of the inequality for arbitrary $\upsilon = \upsilon(x,y) \in U_{ad}$, here $u_*(x,y,t)$ and $\psi_*(x,y,t)$ are solutions of problems (2.1)-(2.3), (4.1)-(4.3) correspondingly.

References

1. Alifanov O.M., Artyukhin E.A., Rumyantsev S.V. Extremal methods for solving ill-posed problems and their applications to inverse heat transfer problems, Moscow, Nauka, 1988, 288 p.
2. Arman J.-L.P. Applications of the Optimal Control Theory for the Distributed Parameter Systems to the Problems of Structural Optimization, Moscow, Mir, 1977, 144 p.
3. Guy Bouchitte, Ilaria Fragala. Optimality Conditions for Mass Design Problems and Applications to Thin Plates. Archive for Rational Mechanics and Analysis. Vol. 184. Issue 2, pp.257-284, May 2007.
4. Daqun Tong and Robert L. Williams II, Sunil K. Agrawal. Optimal shape control of composite thin plates with piezoelectric actuators. Journal of Intelligent Material Systems and Structures. Vol. 9, pp.458-467, June, 1998.
5. V.S. Deineka. Optimal control of the dynamic state of a thin compound plate. Cybernetics and Systems Analysis. Vol. 42, No 4, pp. 151-175, 2006.
6. Kabanikhin S.I. Inverse and Ill-posed problems, Novosibirsk, 2009, 457 p.
7. Komkov V. The Theory of Optimal Control of Damping Vibrations of Simple Elastic Systems, Moscow, Mir, 1975, 160 p.
8. Ladijenskaya O.A. Boundary Problems of Mathmatical Physics, Moscow, Nauka, 1973, 408 p.
9. Lions J.-L. Optimal Control of the Systems Described by the Partial Differential Equations, Moscow, Mir, 1972, 416 p.
10. Lions J.-L. Some Methods of Solving Non-Linear Boundary Value Problems. M.: Mir, 1972, 588 p.
11. Maria L. Blanton & Ibrahim S. Sadek. Optimal active pointwise of thin plates via state-control parametrization. International Journal of Systems Science. Vol. 25. Issue 11, pp.2001-2014, 1994.
12. Quliyev H.F., Seyfullayeva Kh.I. Optimal control problem for the equation of vibrations for thin plate // News of Baku University, Physico-mathematical sciences, ¹3, 2013, pp.64-73.
13. Sadek S., Adali S., Sloss J.M., Bruchjr J.C. Vibration damping of a thin plate by optimal open-and closed-loop control forces // Journal of the Franklin Institute Pergamon Presa ple. Vol.329, No.2, pp.207-214, 1992. Printed in Great Britain.
14. Vasilyev F.P. Methods for Solution of the Extremal Problems, Moscow, Nauka, 1981, 399 p.
15. Zhang X., Zhang J. The hybrid control of vibration of thin plate with active constrained damping layer // Applied Mathematics and Mechanics. English Edition, Vol.19, No.12, pp.1119-1134, Dec. 1998.