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On optimal high-speed heating of an unlimited plate with restrictions on thermal stresses

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Abstract. *The problem of optimization of axisymmetric heating of an unbounded plate in the presence of phase constraints caused by thermoelastic stresses is investigated. The statement takes into account the nonlinear dependence of the compressive and tensile strength limits of the material, as well as the thermal conductivity coefficient on temperature. The method of sequential linearization is applied to solve the formulated nonlinear problem. At each iteration, the solution of the linearized problem is found by a modified method of rotating the reference hyperplane. The results of numerical calculations confirming the efficiency of the algorithm are presented. The results obtained can be used in the design of technological processes of high-temperature processing.*

Keywords. Optimal heating control · thermoelastic stresses · strength limits · phase constraints · speed time.

Mathematics Subject Classification (2010): 49J15, 35K05

1 Introduction

The problem of controlling heating processes taking into account the stress-strain state of a material remains insufficiently studied, despite its practical significance. Most of the known works either do not consider constraints, or make simplifying assumptions about the constancy of thermophysical coefficients and the linear dependence of strength characteristics on temperature [5, 7].

In this paper, we consider a case characterized by a significant nonlinear dependence of the compressive and tensile strength limits on temperature. This factor becomes the determining factor during high-temperature heating, since the strength characteristics of the material can vary greatly [4]. Additionally, the temperature dependence of the thermal conductivity coefficient is taken into account, which leads to non-linearity of the thermal conductivity equation. Following the approach of [2], the original nonlinear equation is

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linearized, and its solution is found by the method of successive approximations. At each step of the iterative process, the solution of the linearized equation is constructed using the Fourier integral transform. As a result, at each iteration, the problem is reduced to optimal control of a linear system of ordinary differential equations with nonlinear constraints on phase variables. Such an optimal control problem is solved using the algorithm presented in [6]. The results of numerical calculations are presented.

2 Problem statement.

The process of axisymmetric external heating of an unbounded plate is described by the following equations:

$$A\rho\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\lambda\left(T\right)\frac{\partial T}{\partial x}\right), \quad x \in (0,\ \bar{x}),\ t \in (0,\bar{t}),\ 0 < \bar{t} < \infty \tag{2.1}$$

$$T\left(x,0\right) = p = const, \quad x \in [0,\bar{x}] \tag{2.2}$$

$$\lambda\left(T\right)\frac{\partial T(x,t)}{\partial x}|_{x=\bar{x}} = \alpha\left(v\left(\tau\right) - T\left(x,t\right)\right)|_{x=\bar{x}}\ t \in [0,\ \bar{t}] \tag{2.3}$$

$$\frac{\partial T(x,t)}{\partial x}|_{x=0} = 0,\ t \in [0,\ \bar{t}], \tag{2.4}$$

where t is time, x is a spatial variable in $T\left(x,tx,\ t\right)$ is temperature, c is the specific heat capacity, ρ is the material density, α is the heat transfer coefficient $\lambda\left(T\right)$ - is the thermal conductivity coefficient, $v\left(t\right)$ - is the temperature of the heating medium (control),

$$v\left(t\right)\ \in V,\ V = \ \{v = \ v\left(t\right),\ \ 0 \leq v^{-} \leq v\left(t\right) \leq v^{+},\ v\left(t\right)\ \in L_2\left[0,\ \bar{t}\right]\} \tag{2.5}$$

We assume that the coefficient of thermal conductivity is

$$\lambda\left(T\right) > 0\ , \quad 0 < \ \beta_1\ \leq\ \lambda\left(T\right) \leq \beta_2 \tag{2.6}$$

The paper considers materials that are prone to brittle fracture. During the heating process, the body experiences both tensile and compressive stresses. According to studies [7], [1], during axisymmetric heating, the maximum tensile stresses are formed on the plate axis, while compressive stresses reach the highest values on its surface. If we assume that the edges of the plate are firmly pinched, the restrictions apply. Assuming that the elastic modulus E and the coefficient of linear expansion $\alpha_{a\,T}$ are constant, and the plate edges are rigidly pinched, the problem of thermoelasticity is solved analytically in the quasi-static formulation [1].

$$\frac{\alpha_T E}{1-\nu}\left(-\,T\left(0,t\right) + \ \frac{1}{\bar{x}}\int\limits_0^{\bar{x}} T\left(x,t\right)dx\right) \leq \ \sigma_p\left(T\left(0,t\right)\right) \tag{2.7}$$

$$\frac{\alpha_T E}{1-\nu}\left(T\left(\bar{x},t\right) + \ \frac{1}{\bar{x}}\int\limits_0^{\bar{x}} T\left(x,t\right)dx\right) \leq \ \sigma_c\left(T\left(\bar{x},t\right)\right) \tag{2.8}$$

Here ν is the Poisson's ratio, σ_c and σ_p are the compressive and tensile strength limits, respectively

Problem 1. We need to find a control $v^{v0}\left(t\right) \in V$, which, if the inequalities (2.7) and (2.8) are fulfilled with a given accuracy $\varepsilon_1 \geq 0$, will translate the system (2.1), (2.3), (2.4) from position (2.2) to position $\bar{T}\left(x\right)$ with fixed accuracy $\varepsilon_2 \geq 0$ in minimum time $t^0,\ 0 \leq t^0 \leq \bar{t}$, i.e.

$$\int\limits_0^{\bar{x}} \left[T\left[x,t^0\right] - \bar{T}\left(x\right)\right]^2 dx \ \leq \varepsilon_2 \tag{2.9}$$

3 Linearization. Solving a linearized system of equations

We will search for the solution of the system of equations (2.1) - (2.4) using the method of successive approximations [2]. Consider an iterative process

$$c\rho\frac{\partial T_k(x,t)}{\partial x} - \lambda_0\frac{\partial^2 T_k(x,t)}{\partial x^2} = \lambda_0\frac{\partial}{\partial x}\left[\left(\lambda\frac{(T_{k-1}(x,t))}{\lambda_0} - 1\right)\frac{\partial T_{k-1}(x,t)}{\partial x}\right] \tag{3.1}$$

$$T_k\left(x,t\right) = p \tag{3.2}$$

$$\left[\lambda_0\frac{\partial T_{k-1}(x,t)}{\partial x} - \alpha\left(v\left(t\right) - T_k\left(x,t\right)\right)\right]_{x=\bar{x}} = \left[\left(\lambda_0 - \lambda\left(T_k\left(x,t\right)\right)\right)\frac{\partial T_{k-1}(x,t)}{\partial x}\right]_{x=\bar{x}} \tag{3.3}$$

$$\frac{\partial T_k(x,t)}{\partial x}|_{x=0} = 0, \tag{3.4}$$

where

$$\lambda_0 = \frac{\beta_1+\beta_2}{2} \tag{3.5}$$

Similarly to [2], it can be shown that the sequence $\{T_k\left(x,\ t\right)\}, \quad k = 1,\ 2, ...,$ converges to the solution $T\left(x,t\right)$ of the system of equations (2.1) - (2.4) in the space $W_2^{1,0}$.

For the convenience of further calculations, we introduce the following dimensionless variables

$$l = \frac{x}{\bar{x}}\ ,\ \theta = \alpha_T\left(T-p\right),\ \tau = \frac{\lambda_0\,t}{c\rho\bar{x}^2}\ ,\ Bi = \frac{\alpha\bar{x}}{\lambda_0}\ ,\ \bar{\theta} = \alpha_T\left(\bar{T}-p\right),\ u = \alpha_T\left(v-p\right),$$
$$\sigma_c^* = \frac{(1-\nu)\sigma_c}{E},\ \sigma_p^* = \frac{(1-\nu)\sigma_p}{E}\ ,\ u^- = \alpha_T\left(v-p\right),\ u^+ = \alpha_T\left(v^*-p\right) \tag{3.6}$$

In these variables, the system of equations (3.1) – (3.4) is written as

$$\frac{\partial\theta_k}{\partial\tau} - \frac{\partial^2\theta_k}{\partial l^2} = \frac{\partial}{\partial l}\left[\left(\frac{\lambda(\theta_{k-1})}{\lambda_0} - 1\right)\frac{\partial\theta_{k-1}}{\partial l}\right] \tag{3.7}$$

$$\theta_k\left(r,0\right) = 0 \tag{3.8}$$

$$\frac{\partial\theta_k(l,t)}{\partial l}|_{l=1} = Bi\left[u\left(\tau\right) - \theta_k\left(1,\tau\right)\right] + \frac{\lambda_0 - \lambda\left(\theta_{k-1}(1,r)\right)}{\lambda_0}\frac{\partial\theta_{k-1}(1,\tau)}{\partial l} \tag{3.9}$$

$$\frac{\partial\theta_k(l,\tau)}{\partial l}|_{l=0} = 0 \tag{3.10}$$

Restrictions on thermal stresses (2.7), (2.8) are written in the form of inequalities

$$-\theta_k\left(0,\tau\right) + \int_0^1\theta_k\left(l,\tau\right)dl \ \leq\ \sigma_p^*\left(\theta_k\left(0,\tau\right)\right) \tag{3.11}$$

$$\theta_k\left(1,\tau\right) - \int_0^1\theta_k\left(l,\tau\right)dl \ \leq\ \sigma_c^*\left(\theta_k\left(0,\tau\right)\right) \tag{3.12}$$

We will search for the solution of linear equations (3.7) – (3.10) using the Fourier integral transform [3]

$$\theta_F\left(\mu,\tau\right) = \int_0^1\theta\left(l,\tau\right)\cos\left(\mu,\ l\right)dl \tag{3.13}$$

By applying the integral transformation (3.7) to both parts of equation (3.13) and selecting μ as the root of the equation

$$\mu\sin\left(\mu\right) - Bi\cos\left(\mu\right) = 0 \tag{3.14}$$

taking into account the boundary conditions (3.9), (3.10), equation (3.7) in the images is written as

$$\frac{\partial\theta_F(\mu,\tau)}{\partial\tau} = Bi\cos\mu\left(u\left(\tau\right) + I^{k-1}\right) - \mu^2\theta_F\left(\mu,\tau\right), \tag{3.15}$$

where $I^{k-1} = \int\limits_0^1 \left(\frac{\lambda(\theta_{k-1})}{\lambda_{0-1}} - 1\right) \frac{\partial\theta_{k-1}}{\partial\, l} \frac{\sin(\mu l)}{\sin\mu} dl$

The inversion formula follows from the theory of Fourier series [3] and in this case, taking into account (3.15), is written as

$$\theta_k\left(l, t\right) = \sum_{n=1}^{\infty} D_n\, x_n^k\left(u,\ \tau\right) \cos\left(\mu_n l\right) \tag{3.16}$$

where $\mu_n,\ n = 1, 2, ...$ are the roots of equation (3.14)

$$D_n = \frac{2\ Bi}{(\mu_n^2 + Bi + Bi^2)\sin(\mu_n)} \tag{3.17}$$

$x_n^k\left(u,\ \tau\right),\ n = 1, 2, ...$ – component of the solution of the differential equation

$$\frac{dx_n^k}{d\tau} = -\mu_n^2 x_n^k + \mu_n\left(u + I_n^{k-1}\right),\quad x_n^k\left(0\right) = 0,\ n = 1, 2,\ \ldots \tag{3.18}$$

Here

$$I_n^{k-1} = \int\limits_0^1 \left(\frac{\lambda(\theta_{k-1})}{\lambda_0} - 1\right) \frac{\partial\theta_{k-1}}{\partial l} \frac{\sin(\mu l)}{\sin\mu} dl \tag{3.19}$$

For a given finite dimensionless temperature distribution $\bar\theta\left(l\right)$, the expansion holds

$$\bar\theta\left(l\right) = \sum_{n=1}^{\infty} \frac{1}{\cos(\mu_n l)^2} g_n \cos\left(\mu_n l\right) = \sum_{n=1}^{\infty} \frac{D_n g_n \cos(\mu_n l)}{\sin\mu_n}, \tag{3.20}$$

since the system of functions $\left\{\cos\left(\mu_n l\right)\right\}_{n=1}$ is orthogonal and complete in $\mathrm{L}_2\left[0, 1\right]$. Here

$$g_n = \int\limits_0^1 \bar\theta\left(l\right) \cos\left(\mu_n l\right) dl$$

Taking into account the specific form $\theta_k\left(l, t\right)$ in (3.16), the inequalities (3.11), (3.12) can be rewritten as

$$\sum_{n=1}^{\infty} A_{1n} x_n^k\ \le \sigma_p^*\left(\sum_{n=1}^{\infty} D_n x_n^k\right) \tag{3.21}$$

$$\sum_{n=1}^{\infty} A_{2n} x_n^k\ \le \sigma_c^*\left(\sum_{n=1}^{\infty} D_n x_n^k\ \cos\mu_n\right), \tag{3.22}$$

where $A_{c1n} = D_n\left(\frac{\mathrm{Dn}\ \sin\mu_n}{\mu_n} - 1\right), A_{c2n} = D_n\left(\mathrm{Dn}\ \cos\mu_n - \frac{\sin\mu_n}{\mu_n}\right)$

4 Finite-dimensional approximation

Limiting ourselves in relation (3.16) to the first N terms of the series, system (3.18) can be written as

$$\frac{dX^N}{d\tau} = -A^N X^N + B^N u + I^N,\ X^N\left(0\right)\ = 0 \tag{4.1}$$

where $X^N = \left(x_1^k,\ \ldots, x_n^k\right),\ A^N = \left(\mu_1^2,\ \ldots, \mu_n^2\right),$

$$B^N = \left(\mu_1,\ \ldots, \mu_n\right),\ \ I^N = \left(\mu_1 I_1^{k-1},\ \ldots, \mu_n I_n^{k-1}\right)^T$$

Constraints (3.20), (3.21) are rewritten as

$$C^N X^N \leq F^N \tag{4.2}$$

where,

$$C^N = \begin{pmatrix} c_{11} & c_{12} \dots & c_{1N} \\ c_{2N} & c_{22} \dots & c_{2N} \end{pmatrix}, \; F^N = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix},$$

$$F_1 = \sigma_p^* \left(\sum_{n=1}^{\infty} D_n x_n^k \right), \; F_2 = \sigma_c^* \left(\sum_{n=1}^{\infty} D_n x_n^k \cos \mu_n \right) \tag{4.3}$$

As a result, we will solve the following problem:

Problem 2. Find the control $u^0(\tau) \in U$, in which solutions of system (4.1) τ^0 will satisfy the inequality in the minimum time τ^0:

$$\sum_{n=1}^{N} \frac{\mathrm{D}_n \left[\sin \mu_n x_n^k \left(u^0, \;\; \tau^0\right) - g_n\right]^2}{Bi \; \sin \mu_n} \leq \; \varepsilon_{2,}$$

inequality (4.2) is satisfied with an accuracy ε_1 on $\left[0, \;\; \tau^0\right]$.

Here $U = \{u = u(\tau), \; 0 \leq \bar{u} \; \leq u^+, \; u(\tau) \in L_2[0, \bar{\tau}]\}$

5 Computational experiment

Problem 2 was solved at each iteration using the algorithm proposed in [6].

Table 5.1 Source data

Parameter	Name of the parameter	Meaning	Unit
ρ	material density	8130	$kg\,/m^3$
c	specific heat capacity	368	$J\;/\;kg \cdot {}^\circ\mathrm{C}$
$\bar{x}$	half the plate thickness	0,23	m
α	heat transfer coefficient	200	$\mathrm{W/(m^2 \cdot^\circ C)}$
p	initial temperature	20	°C
T	specified final distribution	920	°C
v^-	minimum value of the heating medium temperature	800	°C
v^+	maximum value of the heating medium temperature	1600	°C
$\alpha_{a\,T}$	coefficient of linear expansion	$0,18 \cdot 10^{-4}$	$1/{}^\circ\mathrm{C}$
E	elastic modulus	$145 \cdot 10^6$	Pa
ν	Poisson's ratio	0,3	is a dimensionless value

Table 5.2 Temperature dependence of the ultimate strength

Name of the parameter	Unit value	Values				
Temperature	°C	20	975	1050	1100	1150
Compressive Strength	MPa	1100	580	470	310	210
Tensile Strength	MPa	680	540	370	200	140

Table 5.3. Temperature dependence of the thermal conductivity coefficient

Temperature	°C	20	200	500	600	700	800	900	1000
Coefficient of thermal conductivity	$W/\left(m^2\cdot{}^{\circ}!\right)$	10,05	15,07	18,84	20,51	22,19	24,28	26,38	28,05

After switching to dimensionless values, the data in Table 5.2 was approximated by the functions

$$\sigma_c^* = \left(-0,\ 023e^{0,003030\theta} + 0,747\right)$$

$$\sigma_p^* = \left(-0.003e^{0.00460\theta} + 0.476\right),$$

and the data in Table 5.3 is a linear function

$$\lambda\left(\theta\right) = 10,68 + 9,74\theta$$

The linearized problem was solved for different values of N starting from N=3. At $N \geq 6$, the change in the response time became insignificant, and further calculations were performed at $N = 6$. The calculation results are shown in Fig. 5.1

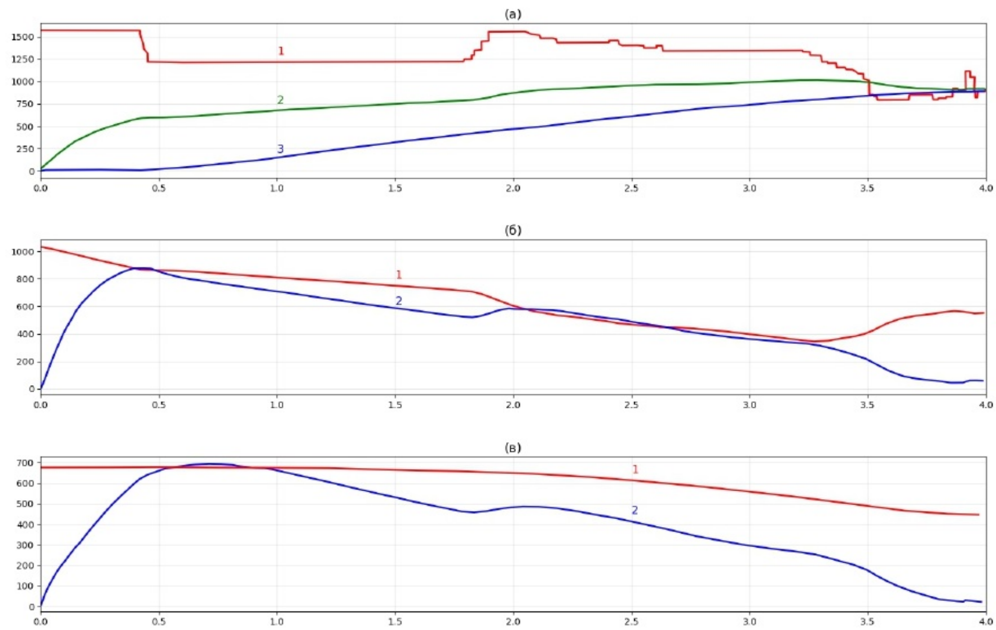

Fig. 5.1 Calculation results

Graph a) of Fig. 5.1 shows the time changes in optimal control (curve 1), surface temperature (curve 2), and plate center temperature (curve 3) under optimal heating conditions. The optimal heating time was 3.98 hours. Graph b) shows the time dependences of the compressive strength (curve 1) and compressive thermal stresses (curve 2) calculated under optimal control conditions. Accordingly, graph b) shows changes in the tensile strength (curve 1) and tensile thermal stresses (curve 2). From the analysis of graphs b) and c), it follows that the heating rate is mainly limited by the value of compressive thermal stresses. At the same time, scientific research has traditionally focused on tensile stresses.

6 Conclusions

The article offers an algorithm for optimizing axisymmetric heating of an unbounded plate, taking into account the dependence of the ultimate strength and thermal conductivity coefficient on temperature.

The developed algorithm makes it possible to form heating modes that ensure the maximum speed of the process while observing the limits on thermal stresses.

The given example confirmed the effectiveness of the proposed approach and its applicability for practical problems of high-temperature processing of materials.

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