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On the wall effect in the flow of a gas-liquid mixture in pipes

Robert Z. Nurgaliyev · Olga G. Kantor · Gulnar M. Salmanova

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Abstract. *The paper considers the flow of a gas-liquid mixture in a horizontal pipe, considering the influence of gas concentration on density and viscosity. The problem is approached using a variational approach based on the Pontryagin maximum principle. The optimal distribution of gas concentration across the pipe section has been determined, minimizing the kinetic energy of the flow. It is demonstrated that at specific ratios of gas and liquid viscosities, a phenomenon known as the wall effect occurs - the formation of a gas layer adjacent to the pipe wall. This layer acts as a "gas bearing", reducing the hydraulic resistance of the flow. The findings provide a theoretical framework for the experimentally observed increase in the flow rate of the gas-liquid mixture at pressures close to the nucleation pressure.*

Keywords. gas–liquid flow · wall effect · two-phase mixture · optimal control · Pontryagin principle · pipe hydraulics

Mathematics Subject Classification (2010): 76T10, 76D05

1 Introduction

It is known that when various suspensions move in a liquid stream, so-called wall-to-wall or axial effects can occur [12]. Such effects are related to the relative movement of the suspension particles and the carrier medium and consist in the fact that the suspension particles migrate to the walls or to the axis of the pipe. Experimental detection of various hydrodynamic effects was obtained in [6, 7]. The axial effect has been established in theoretical works [9, 10]. In [3], the existence of a wall effect during the movement of solid particles

Robert Z. Nurgaliyev
Associations of Small Oil Companies of the Republic of Bashkortostan, Ufa, Bashkortostan, Russian Federation,
E-mail: nuracaro@yandex.ru

Olga G. Kantor
Ufa State Petroleum Technological University, Ufa, Bashkortostan, Russian Federation,
E-mail: o_kantor@mail.ru

Gulnar M. Salmanova
Karabakh University MSE AR, Khankendi, Azerbaijan

suspended in a viscous liquid was proved. In this paper, the motion of a gas-liquid mixture in a horizontal pipe is investigated. The effect of the amount of gas on the dynamics is reduced to the functional dependence of the density and viscosity of the mixture on the gas concentration. The problem is considered in a variational formulation, with the control being the distribution of concentration over the pipe section.

The optimal concentration distribution is found from the condition of a minimum of kinetic energy, which is converted into internal energy. The optimal distribution found indicates the presence of a gas layer at the pipe wall, i.e., there is a wall effect [2, 13]. The appearance of such a "gas bearing" leads to a decrease in the hydraulic resistance of the mixture, which could be explained by the experimentally detected significant increase in the flow rate of the gas-liquid stream at pressure levels close to the nucleation pressure [1, 4, 14].

2 Variational formulation and optimal control (Pontryagin's principle)

Consider the pressure of a gas-liquid mixture in a circular tube of radius R. The gas concentration is denoted by C. The gas-liquid mixture will be considered as a continuous, compressible liquid medium, the motion of which is described by the Navier-Stokes equation. The viscosity η_1 and density ρ of this medium at each point with cylindrical coordinates $(x,\ r_1)$ are functions.

We will consider the motion to be steady and having axial symmetry. We denote by $v,\ w,\ p$, respectively, the velocity in the direction of the x-axis, the velocity in the direction perpendicular to the x-axis, and the pressure of the mixture. Under the assumptions made:

$$u \equiv u\left(r_1\right),\ c \equiv c\left(r_1\right),\ 0 \le r_1 \le R,\ w \equiv 0,\ \frac{\partial P}{\partial x} = -k\left(k > 0\right) \tag{2.1}$$

Consider an arbitrary cylinder whose axis coincides with the x axis, the length of the generatrix is L, and the radius of the base is R. From the condition of the balance of forces of friction and pressure, as well as from the condition of adhesion on the pipe wall, we have:

$$k\pi L r_1^2 + 2\pi L r_1 \frac{\partial u}{\partial r_1}\eta_1\left(c\right) = 0,\ \ u\left(R\right) = 0 \tag{2.2}$$

From (2.2) we find

$$u\left(r_1\right) = -\frac{k}{2}\int\limits_R^{r_1}\frac{\xi d\xi}{\eta\left(c\right)}; m = -\int\limits_0^R \pi k\rho\left(c\right) r_1 \int\limits_R^{r_1}\frac{\xi d\xi}{\eta\left(c\right)} dr \tag{2.3}$$

where m is the mass flow rate of the gas–liquid mixture. We assume that the amount of gas in the cross section is equal to the known constant E, i.e.

$$2\pi\int\limits_0^R c\left(r\right) r dr = E \tag{2.4}$$

Consider the problem of determining a piecewise continuous and bounded on $[0, R]$ function $c\left(r\right),\ 0 \le c\left(r\right) \le 1$, which maximizes m under conditions (2.2), (2.4). note that maximizing mass consumption is equivalent to minimizing kinetic energy.

After entering the dimensionless values:

$$r = \frac{r_1}{R},\ \eta = \frac{\eta_1\left(c\right)}{\eta_1\left(0\right)}, \rho = \frac{\rho_1\left(c\right)}{\rho_1\left(0\right)},\ M = \frac{-m\cdot\eta_1\left(0\right)}{\pi\cdot k\cdot R^4\rho_1\left(0\right)},\ \ F = \frac{D}{2\pi R^2}$$

we get the following optimal control problem

$$M = \int_0^1 r\rho(c) \int_1^r \frac{\xi d\xi}{\eta(c)} \cdot min \tag{2.5}$$

when limited

$$F = \int_0^1 rc(r)\, dr \tag{2.6}$$

The constraint $0 \le c \le 1$ implies that in order for the problem to have a solution, it is necessary that $0 \le F \le 1/2$.

Similarly, as in [8], we formulate the problem in a convenient way for applying the Pontryagin maximum principle [11]. Let $\{x_0(r),\ x_1(r)\}$ be the solution of the following system

$$\frac{dx_1}{dr} = f_1(r,c) \equiv c(r)\, r, \ \ x_1(0) = 0 \tag{2.7}$$

$$\frac{dx_0}{dr} = f_0(r,c) \equiv r \cdot \rho(c) \cdot \int_1^r \frac{\xi d\xi}{\eta(c)}, \ \ x_0(0) = 0 \tag{2.8}$$

Find a control $c(r)$ under the influence of which the phase point $(x_1,\ r)$ will move from point $(0,\ \ 0)$ to point $(F,\ 1)$, while the functional will take the smallest value.

The Hamilton-Pontryagin functional [11] of the formulated problem has the form:

$$H = \psi_0 f_0 + \psi_1 f_1, \tag{2.9}$$

where (ψ_0, ψ_1) is the solution of the following conjugate system:

$$\frac{\partial \psi_0}{\partial r} = -\frac{\partial H}{\partial x_0} = 0,\ \frac{\partial \psi_1}{\partial r} = -\frac{\partial H}{\partial x_1},\ \psi_0(0) = -1. \tag{2.10}$$

Hence, we have

$$\psi_0 = -1, \ \ \psi_1 = A = Const.$$

Considering in (2.9) we get

$$H = -r \cdot \rho(c) \cdot \int_1^r \frac{\xi d\xi}{\eta(c)} + A \cdot r \cdot c. \tag{2.11}$$

For any $r \in [0;1]$, $A \in [0;\infty]$ considering $c \in [0;1]$ as a numerical function, we find $c(r;A) \in [0;1]$, which maximizes the function (2.11). The same c will be the maximum for the function:

$$h(c) = \frac{\rho(c)}{2\mu(c)}\left(1 - r^2\right) + A \cdot c.$$

The optimal value of the constant A is found from the condition:

$$F = \int_0^1 c(r, A^*)\, rdr. \tag{2.12}$$

Note that the existence of a solution to the formulated problem follows from physical considerations. Therefore, it follows from the Pontryagin maximum principle that the solution will be among the found controls $c\,(r, A^*)$.

We assume that there is a wall effect if for the solution $c\,(r)$ there exists $\varepsilon > 0$, such that $(r) \equiv 1$, for $1 - \varepsilon \le r \le 1$.

The following theorem is valid.

Let $1 - \varepsilon \le r \le 1$ be a continuous function on $[0; 1]$ and $f^{'}\,(1)$ exists. Then for any $F \in \left(\frac{c_1(0)}{2}\ ;\ \frac{1}{2}\right]$ there is a wall effect, where:

$$c_1\,(0) = \underset{\xi\in[0;1]}{sup}\left\{f\,(\xi) = \max_{0\le x\le 1} f\,(x)\right\} \tag{2.13}$$

Proof. We denote

$$\tilde{h}\,(c) = \frac{h\,(c)}{1 - r^2} = f\,(c) + \frac{A}{1 - r^2}c \equiv f\,(c) + z \cdot c, \tag{2.14}$$

where $z = \frac{A}{1-r^2}$. Let $c_1\,(z) \in [0; 1]$ represent the maximum of the function $\tilde{h}\,(c)$ (the existence of such a point follows from the Weierstrass theorem). Since,

$$c_1\left(\frac{A}{1 - r^2}\right) = c\,(r, A)\,, \tag{2.15}$$

then from (2.12) we get

$$F = \int\limits_0^1 c\,(r, A)\,rdr = \int\limits_0^1 c_1\left(\frac{A}{1 - r^2}\right) rdr = \int\limits_A^\infty c_1\,(z)\,\frac{A}{2z^2}dz = \frac{A}{2}\int\limits_0^\infty \frac{c_1\,(z)}{z^2}dz. \tag{2.16}$$

The function $c_1\,(z)$ is monotonically increasing and $0 \le c_1\,(z) \le 1$. Therefore, for any $A > 0$, there is an improper integral $\int\limits_A^\infty \frac{c_1(z)dz}{z^2} < \infty$.

At any point $A \in (0,\ \infty)$, the function $\int\limits_A^\infty \frac{c_1(\ z)\ dz}{z^2}$ is continuous, as can be seen from the following estimate:

$$\left|\int\limits_A^{A+\ \varepsilon} \frac{c_1\,(\ z)\ dz}{z^2}\right| \le \left|\frac{1}{A} - \frac{1}{A+\ \varepsilon}\right| \to 0,\ \text{for}\ \ \varepsilon \to 0. \tag{2.17}$$

Thus

$$\varPhi\,(\ A) = \frac{A}{2}\int\limits_A^\infty \frac{c_1\,(\ z)}{z^2}\ dz \tag{2.18}$$

a continuous function on $(0,\ \infty)$. Calculate $\lim\limits_{A\to 0}\ \varPhi\,(\ A)$

$$\lim_{A\to 0}\ \varPhi\,(\ A) = \lim_{z\to 0}\frac{c_1\,(z)}{2} \tag{2.19}$$

Since $f\,(0) = 1/2$, there exists a $\delta > 0$ such that for $z \le \delta,\ c_1\,(z) \equiv c_1\,(0)$ where $c_1\,(0)$ is the point representing the maximum of the function f.

$$\lim_{A\to 0}\ \varPhi\,(\ A) = c_1\,(0)\,/2$$

It is clear from the form of the function $\tilde{h}(c)$ that there exists $\bar{A} < \infty$ such that for any $A \geq \bar{A}$, $c_1(z) \equiv 1$. Let A be the smallest of such numbers.

Then

$$\Phi(A) = \frac{A}{2}\int_{A}^{\bar{A}} \frac{c_1(z)}{z^2}dz + \frac{A}{2\bar{A}}\,,\ \ \Phi\left(\bar{A}\right) = \frac{1}{2} \tag{2.20}$$

It is clear that $c_1(0)/2 \leq 1/2$. due to the continuity of Φ, it follows that for any $F \in [c_1(0)/2; 1/2]$ equation $\Phi\left(\bar{A}\right) = F$ has a solution of $0 \leq A \leq \bar{A}$. In this case, $A > 0$ if $F \neq c_1(0)/2$. We prove that for any $A > 0$ there exists such an $\varepsilon_A > 0$, that for any $r \in [1-\varepsilon_A; 1]$, $c(r) \equiv 1$, where $c(r)$ provides the maximum. By the assumption of the theorem $\left|f'(1)\right| < \infty$, i.e. it is possible to choose such an integer $n > 1$, which means that for any $c \in [0; 1]$:

$$f(c) < f(1)\,c + n\cdot(1-c)\,, \tag{2.21}$$

obviously, otherwise there is a sequence $\{c_n\}$ such that:

$$f(c_n) \geq f(1)\cdot c_n + n\cdot(1-c_n)\,,\ \ n = 1,\ 2,\ ... \tag{2.22}$$

Choose $\{\{n_k\}\,,\ k = 1, 2, \ldots,\ \ that \lim\limits_{n\to\infty} c_{n_k} = c_\infty \in [0; 1]$.

Since $1 - c_{n_k} \leq \frac{f\left(c_{n_k}\right)}{n_k} - \frac{f(1)}{n_k}\cdot c_{n_k}$, we proceed to the limit at $k \to \infty$

$$1 - c_\infty \leq 0,\ c_\infty = 1.$$

Then from (2.22) we have

$$\frac{f(c_n) - f(1)}{c_n - 1} \leq -\left(n - f(1)\right). \tag{2.23}$$

If in (2.23) we go to the limit at $n \to \infty$, we get a contradiction with the assumption made $|f'(1)| < \infty$, which proves the validity of (2.21). Considering (2.21) from (2.11), we have

$$h(c) \leq \left[f(1) + \frac{A}{1-r^2}\right]\cdot c + n(1-c) = \left[f(1) + \frac{A}{1-r^2} - r\right]c + n \tag{2.24}$$

It can be seen from (2.24), that $r^2 > 1 - A/\left(n - f(1)\right)$, $c(r) \equiv 1$ (we assume that $n > f(1)$).

Thus, for any $F \in (c_1(0)/2;\ 1/2)$ of the equation $\Phi(A) = F$ has a solution $A_* > 0$, for which there is a wall effect. The theorem has been proved.

Example:

Let $\rho(c) = 1 - c$, $\eta(c) = \varepsilon/(\varepsilon + c)$, where $\eta(1) = \frac{\varepsilon}{1+\varepsilon}$ is the ratio of the viscosity of the gas to the viscosity of the liquid.

The function $h(c)$ from (2.11) has the form:

$$h(c) = \frac{1}{2\varepsilon}(1-c)(c+\varepsilon)\left(1-r^2\right) + A\cdot c.$$

An elementary analysis shows that

$$c(r, A) = \begin{cases} \frac{1-\varepsilon}{2} + \frac{\varepsilon\cdot A}{1-r^2},\ 0 \leq r\langle\sqrt{1-\frac{2\varepsilon A}{1+\varepsilon}} \\ 1,\ \ \sqrt{1-\frac{2\varepsilon A}{1+\varepsilon}}\langle r \leq 1 \end{cases}. \tag{2.25}$$

The optimal value of parameter A is the solution of the equation

$$\phi(A) = \frac{1-\varepsilon}{4} + \frac{\varepsilon}{2}A - \frac{\varepsilon}{2}A\,ln\left(\frac{2\varepsilon A}{1+\varepsilon}\right) = F \qquad (2.26)$$

Equation (2.26) has a solution $A \in [0;(1+\varepsilon)/2\varepsilon]$ for any $F \in \left[\frac{1-\varepsilon}{4};\frac{1}{2}\right]$. For $F = \frac{1-\varepsilon}{4}$, $A = 0$ and as can be seen from $c(r,\ A) \equiv (1-\varepsilon)/2$, i.e. there is no wall effect. However, for any $F \in \left(\frac{1-\varepsilon}{4}\ ;\ \frac{1}{2}\right]$, $A \in \left(0\ ;\ \frac{1+\varepsilon}{2\varepsilon}\right]$ and parietal effect. If $F \in \left[0;\ \frac{1-\varepsilon}{4}\right)$, then equation (**??**) has no solution $A \in \left[0;\ \frac{1+\varepsilon}{2\varepsilon}\right]$.

How easy it is to check $c_1(0) = \frac{1-\varepsilon}{2}$.

Thus, in the course of the conducted research, a mathematical model of a gas-liquid mixture flow in a horizontal pipe has been developed, where the density and viscosity depend on the gas concentration. It was found, that the variational problem of optimal distribution of gas concentration over the pipe section is posed and solved using the Pontryagin maximum principle. The optimal distribution is found, which minimizes the kinetic energy of the flow, which turns into internal energy [5, 15]. Theoretically, it is shown that the optimal solution corresponds to the existence of a gas layer at the pipe wall - the wall effect. The conditions under which the effect exists are derived: it manifests itself if the ratio of the viscosity of the gas to the viscosity of the liquid exceeds a certain threshold value. An analytical example has been carried out, demonstrating that at certain parameters of viscosity and gas concentration, the wall effect is realized, while at others it disappears.

3 Conclusions

Spontaneous formation of a wall gas layer caused by internal hydrodynamic processes is possible in gas-liquid mixtures.

The resulting gas layer plays the role of a “gas bearing”, which reduces friction between the flow and the walls of the pipe.

The presence of a wall effect leads to a decrease in hydraulic resistance and, consequently, to an increase in the consumption of the mixture.

The theoretical results are consistent with experimental observations of an increase in flow rate at pressures close to the nucleation pressure.

The findings can be used to explain and predict the behavior of two-phase flows in pipeline systems, as well as to optimize transport processes in the chemical and oil and gas industries.

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