Tran. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mechanics, **46** (7), 3–8 (2026);
https://doi.org/10.30546/2706-7734.46.7.2026.08

On bubble liquid filtration problem in case of stability loss

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Received: 20.12.2025 / Revised: 07.02.2026 / Accepted: 12.05.2026

Abstract. *The article investigates nonlinear filtration of a bubbly liquid in a porous medium under conditions of stability loss. The gas–liquid mixture is treated as an ideal compressible barotropic medium, which allows the governing equations to be reduced to a nonlinear filtration equation analogous to polytropic gas flow in porous media. Special attention is paid to boundary pressure variation in a blow-up mode. Using a self-similar transformation, the problem is reduced to a nonlinear ordinary differential equation for the pressure profile. An exact solution is obtained for a special case, describing finite localization of the pressure disturbance. It is shown that the solution behavior depends strongly on the boundary pressure growth rate. Localized, expanding, and boundary blow-up regimes are identified. The results contribute to the theory of nonlinear gas–liquid filtration and may be useful for modeling instability phenomena in oil and gas reservoirs.*

Keywords. bubbly liquid · nonlinear filtration · porous medium · self-similar solution · stability loss · pressure localization

Mathematics Subject Classification (2010): 35K55, 35B44, 76S05

1 Introduction

In oil and gas production, there often arise nonhomogeneous systems with complex rheophysical properties. The motion of such systems is accompanied with nonlinear nonequilibrium effects that originate from both these systems (which can be gas condensate, gas–liquid, viscous-elastic, viscous-plastic, polymer, etc.) and the interaction between them

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and the porous medium. Therefore, the hydrodynamic exploration of these systems is difficult because it is impossible to take into consideration all processes going on during the motion. The existing models allow solving the problem of the stability of motion for complicated systems, diagnosing a change of state, predicting changes in observed values with respect to time and parameters, detecting and predicting new effects inherent in the considered systems in other modes. In [1], mathematical models are constructed for the three-phase filtration of sparkling water, oil and free gas in a porous medium with consideration of the growth of small bubbles in the pores, formation of a free gas phase, slipping effect, pore compressibility and non-isothermal nature of the process.

In [2, 3], stability conditions for stationary filtration modes for carbonated liquids are obtained when the pressure is below the saturated pressure in case of nonmonotonic dependence of relative phase permeability of the liquid on gas saturation. Two-phase filtration flows are hard to explore because it is problematic to construct the corresponding physical model for them. Besides, the related calculations are also too bulky due to the different scales of phase filtration processes. In [4], a universal calculation method is presented which allows constructing the van Genuchten mathematical model. An example of two-phase filtration in a planar domain is given.

The tendency of gas bubbles to fragmentation behind the water front due to the Rayleigh–Taylor or Kelvin–Helmholtz instability is treated in [5]. In the first case, the criterion for bubble instability is a Bond number, while in the second case the criterion is a Weber number. It is established in [5] that, for the same initial state of bubble system, the bubble collapse due to the Rayleigh–Taylor instability may be caused by the incident pressure wave with a smaller amplitude than that due to the Kelvin–Helmholtz instability. The possibility of bubble fragmentation in a constricting channel by small amplitude pressure waves, which do not change the gas phase dispersion in a cylindrical channel, is proved. The maximum threshold dispersion of the mixture is found at which no bubble fragmentation by the small amplitude waves occurs.

In [6, 7], Morenko studied the impact of the shape of a single bubble at initial time on the rise dynamics in a stagnant viscous liquid. Elliptic bubbles with different compression ratios have been considered. Mathematical modeling of this process is based on the VOF (volume of fluid) method, which allows tracing the evolution of the interface. The results of test problems are compared to those obtained by other authors. Shapes of interface area, velocity fields, and vorticity fields occurring during the process of bubble rise in a gravitational field are obtained in [3, 7]. The main attention was paid to the unsteady motion mode. It was shown that the velocity of bubble rise had one or two local maxima depending on the properties of the medium. The author proved that the bubbles stretched vertically at the initial moment rise faster than those stretched horizontally.

In view of the above-said, the problems considered in this work and the methods used to treat them are highly relevant both from scientific and practical points of view.

2 Mathematical formulation of the bubble-liquid filtration problem

Under some physical assumptions, the liquid containing gas bubbles (i.e., a bubble liquid) can be considered as an ideal compressible barotropic medium. In other words, if we neglect the liquid phase compressibility and the capillary pressure, then the equation of state for the bubble liquid has the following form [8]:

$$P(\rho) = \frac{\alpha_{20} P_0}{\alpha_{10}} \frac{\rho}{\rho_1^0 - \rho}. \tag{2.1}$$

Here P_0 is a fixed pressure of the mixture, α_{20} is an initial gas volume, α_{10} is an initial liquid volume, ρ_1^0 is a true density of a liquid, and P, ρ are the pressure and the density of the mixture, respectively.

From (2.1) we obtain

$$\rho = \frac{\rho_1^0 P}{\beta + P}, \qquad \beta = \frac{\alpha_{20} P_0}{\alpha_{10}}. \tag{2.2}$$

Substituting (2.2) in the continuity equation and Darcy's law

$$m\frac{\partial P}{\partial t} + \frac{\partial (Pv)}{\partial x} = 0, \qquad v = -\frac{k}{\mu}\frac{\partial P}{\partial x}, \tag{2.3}$$

we get

$$\frac{\partial \Phi(P)}{\partial t} = \frac{k}{m\mu}\frac{\partial}{\partial x}\left(\Phi(P)\frac{\partial P}{\partial x}\right), \tag{2.4}$$

$$\Phi(P) = \frac{P}{\beta + P}. \tag{2.5}$$

The function $\Phi(P)$ is convex in the domain and can be approximated by the power function

$$\Phi(P) \approx P^{1/n}. \tag{2.6}$$

By choosing the number n sufficiently great, we can make formulas (2.5) and (2.6) sufficiently close to each other on every finite interval.

In view of (2.6), equation (2.4) becomes

$$\frac{\partial P^{1/n}}{\partial t} = D\frac{\partial}{\partial x}\left(P^{1/n}\frac{\partial P}{\partial x}\right), \qquad 0 \ll x < \infty, \quad n \gg 1, \tag{2.7}$$

where $D = k/(m\mu)$, k is permeability, m is porosity, μ is viscosity of the bubble mixture, and P is pressure of the mixture. Equation (2.7) coincides with the filtration equation for a polytropic gas in a porous medium [9].

Self-similar solutions for equation (2.7) with no blow-up mode on the boundary have been obtained in [12,13]. In this work, the self-similar solutions for equation (2.7) are obtained in the case where the pressure on the reservoir boundary varies in a blow-up mode.

3 Self-similar solution under boundary blow-up conditions

Let the pressure on the boundary of the semi-infinite reservoir vary as follows:

$$P(0,t) = A(T_f - t)^{\alpha}, \qquad \alpha < 0, \quad -\infty < t < T_f. \tag{3.1}$$

Then problem (2.7)–(3.1) has a self-similar solution of the form

$$P(x,t) = A(T_f - t)^{\alpha} f(\xi), \qquad \xi = \frac{x}{\sqrt{AD}\,(T_f - t)^{(\alpha+1)/2}}. \tag{3.2}$$

The function f satisfies the equation

$$\frac{d}{d\xi}\left(f^{1/n}\frac{df}{d\xi}\right) + \frac{\alpha}{n} f^{1/n} - \frac{\alpha+1}{2n}\,\xi f^{1/n-1}\frac{df}{d\xi} = 0 \tag{3.3}$$

and the conditions

$$f(0) = 1, \qquad f(\xi) \to 0 \quad \text{as} \quad \xi \to \infty. \tag{3.4}$$

In the special case $\alpha = -1$, the exact solution of problem (3.3)–(3.4) is

$$\begin{aligned} f(\xi) &= \left(1 - \frac{\xi}{\xi_0}\right)_+^2, \\ \nu = (1+n)^{-1}&, \qquad \gamma = (1+n)n^{-2}, \\ \xi_0 &= 2^{1/2}(1+\nu)^{1/2}\gamma^{-1/2}(1-\nu)^{-1}. \end{aligned} \tag{3.5}$$

where

$$(\chi)_+ = \begin{cases} \chi, & \chi > 0, \\ 0, & \chi < 0. \end{cases}$$

Passing to dimensional quantities, for the distribution of pressure in the reservoir we obtain

$$P(x,t) = A(T_f - t)^{-1}\left(1 - \frac{x}{L}\right)_+^2, \qquad L = \xi_0\sqrt{AD}. \tag{3.6}$$

The solution (3.6) is different from zero only within the finite localization domain $0 \leq x < L$, and is identically zero outside of it. The depth of localization L depends on the parameters that characterize the properties of the medium and the boundary mode of gas injection into the reservoir. For $t \to T_f$, solution (3.6) is unboundedly increasing within the localization domain, while staying zero for $x \geq L$. Following the terminology of [6, 10], we call distribution (3.6) an S-regime.

4 Qualitative analysis of filtration regimes and pressure localization

In general, the qualitative behavior of the solution of equation (3.3) and that of solution (3.2) change dramatically depending on whether $\alpha < -1$, $\alpha = -1$, or $-1 < \alpha < 0$. This can be foreseen in the structure of the self-similar variable ξ. Indeed, by definition, the half-width of a pressure wave is equal to

$$\ell(t) = \xi_\phi\sqrt{AD}\,(T_f - t)^{(\alpha+1)/2}, \tag{4.1}$$

where ξ_ϕ is defined by the condition $f(\xi_\phi) = 0$.

As seen from (3.1), $\alpha = -1$ corresponds to the S-regime, $\alpha < -1$ to the HS-regime, and $-1 < \alpha < 0$ to the LS-regime.

In general, the behavior of the integral curves of (3.3) can be studied by reducing this equation to one of first order. Equation (3.3) is invariant with respect to the similarity transformations

$$\xi^* = \lambda\xi, \qquad f^* = \lambda^2 f, \qquad \lambda > 0, \tag{4.2}$$

and therefore allows a reduction of order by means of the change of variables

$$\eta = \frac{f}{\xi^2}, \qquad F = \frac{f'}{\xi}. \tag{4.3}$$

Substituting (4.3) in (3.3), we obtain the first-order equation

$$\eta(F - 2\eta)\frac{dF}{d\eta} + \eta F + \frac{1}{n}F^2 + \frac{\alpha}{n}\eta - \frac{\alpha+1}{2n}F = 0. \tag{4.4}$$

For small η, which corresponds to small f, (4.4) has the local form

$$\frac{dF}{d\eta} \simeq \frac{\alpha+1}{2}, \tag{4.5}$$

with an integral

$$F(\eta) = \frac{\alpha+1}{2}\eta + C. \tag{4.6}$$

Expression (4.6) implies that the integral curves of (4.4) on the right half-plane are divided into two classes by the separatrix

$$F(\eta) = \frac{\alpha+1}{2}\eta. \tag{4.7}$$

If $\alpha < -1$, then the motion of the phase point along separatrix (4.7) occurs in the direction of the singular point $(0,0)$. Along the separatrix we have

$$\frac{dF}{d\eta} = -\frac{1}{2}|1+\alpha|, \qquad \alpha < -1. \tag{4.8}$$

Expression (4.8) implies that when F varies from the finite value $F_0 < \infty$ to zero, the variable η varies within the finite interval

$$\frac{\mathrm{const} - F_0}{0.5|1+\alpha|} < \eta < \frac{\mathrm{const}}{0.5|1+\alpha|}.$$

Thus, ξ also varies within a finite interval along the separatrix in the neighborhood of the singular point. Further, from (4.3) it follows that ξ stays finite as $f \to 0$, i.e., the integral curve of equation (4.4) corresponding to separatrix (4.7) intersects the axis at the finite point ξ_ϕ, and the slope of the tangent line at the point of intersection is equal to

$$f'(\xi_\phi) = 0. \tag{4.9}$$

Thus, the solution of (3.3), which consists of the part of the integral curve with slope (4.9) for $\xi \leq \xi_\phi$ and is equal to zero for $\xi > \xi_\phi$, also satisfies the continuity condition for filtration flow. Indeed, the liquid filtration flow, with accuracy up to a constant factor, is equal to $f^{1/n}f'$, and $f^{1/n}f' \to 0$ as $f \to 0$.

The self-similar coordinate of the front is defined numerically by means of the shooting method [13–15].

Thus, when $\alpha = -1$, the liquid injected into the reservoir is localized in a finite domain. The size of this domain and the pressure inside it are defined by formula (3.6) (S-regime). When $\alpha < -1$, there exists a finite front of the liquid, which extends to infinity in blow-up mode (HS-regime). When $-1 < \alpha < 0$, there occurs an LS-blow-up regime localized only on the boundary.

Conclusions

1. The nonlinear problem of filtration of a bubbly liquid in a porous medium is investigated using simplifications that make it possible to consider a gas–liquid mixture as an ideal compressible medium whose properties depend on pressure.
2. After applying the power approximation, it was found that the resulting equation has the same structure as the classical filtration equation for a polytropic gas in a porous medium.
3. In a special case, an exact solution was found for the similarity parameter. This solution describes the pressure distribution in a limited area of the reservoir.
4. Three main modes were identified: a localized mode, a mode with an expanding front that behaves like an explosion, and a mode with a localized explosion at the boundary.
5. The results obtained can be useful for the theoretical analysis of instability during filtration of gas–liquid mixtures and modeling of pressure fluctuations in oil and gas reservoirs containing dispersed gas phases.

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