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Method for solving inverse coefficient problems of nonlinear flow based on sensitivity theory

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Abstract. *The paper considers the inverse coefficient problem of the theory of nonlinear filtration, which arises when modeling the movement of liquids and gases in porous media. The piezoelectric conductivity coefficient is assumed to be an unknown function of pressure and is approximated by a polynomial with unknown parameters. To identify the coefficients of the model, the sensitivity theory method is used in combination with iterative procedures for minimizing the residual functional. Equations for sensitivity functions are obtained and a stable finite-difference scheme for their numerical solution is proposed. The results of numerical experiments confirming the effectiveness of the proposed approach are presented.*

Keywords. nonlinear flow · sensitivity theory · finite difference methods · iterative algorithms · porous media

Mathematics Subject Classification (2010): 35R30, 65M32

1 Introduction

In the problems of oil and gas mechanics and filtration theory, the correct description of the processes of movement of liquids and gases in porous media plays an essential role. Practical interpretation of field data requires knowledge of reservoir and filtration media parameters, which, as a rule, cannot be measured directly and must be restored based on pressure observations. In this regard, it becomes necessary to solve the inverse problems of determining the coefficients of mathematical filtering models [1, 3, 4].

In most cases, flow of real fluids and gases is described by nonlinear parabolic equations, the coefficients of which depend on pressure. One of the key parameters of such models is the piezoelectric conductivity coefficient, which determines the nature of pressure propagation in the medium. The unknown functional dependence of this coefficient significantly complicates the analysis and numerical solution of the corresponding problems, especially in conditions of limited and noisy experimental data.

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An effective tool for solving inverse coefficient problems is sensitivity theory, which makes it possible to establish a relationship between variations in model parameters and changes in observed quantities. The use of sensitivity functions in combination with iterative methods for minimizing residual functionals makes it possible to stably identify model parameters and evaluate their impact on solving a direct problem. However, the practical implementation of such methods requires the development of stable numerical schemes for the joint solution of the initial filtration equations and sensitivity equations. [5, 9, 10].

This paper is devoted to the development and research of a numerical method for solving the inverse coefficient problem of nonlinear filtration theory [10, 11]. The piezoelectric conductivity coefficient is considered as a polynomial function of pressure with unknown parameters to be determined from experimental data. The sensitivity theory method is used to identify the coefficients, and the numerical solution is implemented on the basis of finite difference schemes and iterative algorithms. Additionally, the problem of choosing the optimal degree of a polynomial based on minimizing the functional of empirical risk is considered.

2 Formulation of the inverse coefficient filtering problem

When solving many problems of oilfield mechanics, it becomes necessary to determine reservoir and fluid parameters based on current field information, that is, to solve inverse problems for determining the parameters of the filtration process model.

In most cases, filtration of real liquids and gas in reservoirs is described by nonlinear equations of parabolic type with corresponding initial and boundary conditions:

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(k(p)\frac{\partial p}{\partial x}\right), \tag{2.1}$$

$$p(0,\ x) = f_0(x), \tag{2.2}$$

$$p(t,0) = f_1(t), \tag{2.3}$$

$$\frac{\partial p\,(t,\ l)}{\partial x} = 0. \tag{2.4}$$

To solve the inverse problem (2.1) - (2.4), additional information is needed , which can be used as pressure measurements on$x = l$, i.e.$p(t,\ l) = p_1(t)$, as well as setting the type of function $K(p)$.

3 Method for identifying coefficients based on sensitivity theory

To solve the formulated problem, the sensitivity theory method is used. To do this, the function $K(p)$ is represented as a polynomial with unknown parameters α_i:

$$K(p) = \sum_{i=0}^{N} \alpha_i p^i \tag{3.1}$$

Parameters α_i are evaluated using the condition:

$$\sum_{j=1}^{n} \left(P_{ij} - P_{ij}^{*}\right)^2 \to \min, \tag{3.2}$$

Where P_{1j} are discrete values $P(t,l) = P_1(t)$ at a given time t_j.

P_{1j^*} - calculated values $P(t,l)$ based on the model (2.1) – (2.4).

Then, according to the sensitivity theory [2, 6], the parameters α_i are estimated using the following iterative procedure:

$$\alpha_i^{(j+1)} = \alpha_i^{(j)} + \Delta\alpha_i^{(j)} \tag{3.3}$$

Where $\Delta\alpha_i^{(j)}$ are defined from the system of algebraic equations:

$$C\Delta\alpha_i^{(j)} = P, \tag{3.4}$$

where,

$C = (u_i \cdot u_j)$ - matrix of sensitivity functions,

$P = |u_i \cdot \Delta P_1|$ - vector,

$\Delta\alpha = |\Delta\alpha_i|$ - vector parameters.

$$(u_i \cdot u_j) = \sum_{k=1}^{n} u_{ik} \cdot u_{jk} \tag{3.5}$$

$$(u_i \cdot \Delta P_1) = \sum_{k=1}^{n} u_{ik} \cdot \Delta P_{1k} \tag{3.6}$$

$$\Delta P_{1k} = P_{1k} - \Delta P_{1k}^{*} \tag{3.7}$$

u_i - sensitivity functions:

$$u_i = \frac{\partial P}{\partial \alpha_i}. \tag{3.8}$$

The sensitivity functions are determined by differentiating equation (2.1) with respect to the corresponding parameter α_i, which results in a system of equations with respect u_i to:

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x}\left[K(P,\alpha_i)\frac{\partial u_i}{\partial x}\right] + \frac{\partial}{\partial x}\left[\frac{\partial K(P,\alpha_i)}{\partial P}\frac{\partial P}{\partial x}u_i\right] + \frac{\partial}{\partial x}\left[\frac{\partial K(P,\alpha_i)}{\partial \alpha_i}\frac{\partial P}{\partial x}\right]; i = \overline{0,N} \tag{3.9}$$

with zero initial and boundary conditions.

Given that $K(P,\alpha_i)$ is a polynomial of the form (3.1), we obtain that

$$\frac{\partial K(P,\alpha_i)}{\partial \alpha_i} = P^i,$$

then

$$\frac{\partial K(P,u_i)}{\partial \alpha_i} \cdot \frac{\partial P}{\partial x} = P^i\frac{\partial P}{\partial x} = \frac{1}{i+1}\frac{\partial P^{i+1}}{\partial x}; i = \overline{0,N}. \tag{3.10}$$

Taking into account (3.10), we simplify (3.9) and write it in the following form:

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x}\left[K(P,\alpha_i)\frac{\partial u_i}{\partial x}\right] + \frac{\partial}{\partial x}\left[\frac{\partial K(P,\alpha_i)}{\partial x}u_i\right] + \frac{1}{i+1}\frac{\partial}{\partial x}\left(P^i\frac{\partial P}{\partial x}\right); i = \overline{0,N} \tag{3.11}$$

In the described procedure (3.3) – (3.9), the values u_{ik} are the values of the sensitivity function for i the second parameter at $x = l$ discrete time t_k points .

4 Numerical implementation and optimal model selection

Thus, to solve the set inverse problem, it is necessary to calculate the functions $P(l,t)$ and $u_i(l,t)$ from the equations (2.1) – (2.4), (3.11).

These equations are quasilinear and efficient methods for solving them are the finite difference method [8]. To solve them, we used a two-layer six-point implicit scheme with accuracy $O\left(h^2+\tau\right)$.

We apply this scheme to the following problem obtained from (2.1) – (2.4) by maintaining dimensionless variables:

$$\frac{\partial y}{\partial \tau}=\frac{\partial}{\partial \xi}\left[K^*\left(y,c_i\right)\frac{\partial y}{\partial \xi}\right] \tag{4.1}$$

$$y(0,\xi)=1-\xi;\;\;0\leq\xi\leq 1 \tag{4.2}$$

$$y(\tau,0)=1;\;\;0\leq\tau\leq 1 \tag{4.3}$$

$$\frac{\partial y(\tau,l)}{\partial \xi}=0, \tag{4.4}$$

where y_i^j are the values y on j the nth time layer t_j at i the nth node ξ_i in the variable ξ.

The boundary conditions are taken in general form:

$$y_0=\chi_1 y_1+\nu_1;\;\;y_n=\chi_2 y_{n-1}+\nu_2 \tag{4.5}$$

from which, when choosing different values $\nu_i;\;\chi_i$, conditions of the 1st or 3rd kind are obtained.

When $\chi_1=0;\;\nu_1=1;\;\chi_2=1;\;\nu_2=0$ conditions (4.5) in the difference form correspond to conditions (4.3), (4.4).

Since the coefficients $K^*(y)$ in equation (4.1) are variables, then α_i in equation (4.4), according to the balance method, they must satisfy the following relation:

$$\alpha_i=\frac{1}{\frac{1}{h}\int_{\xi_{i-1}}^{\xi_i}\frac{dx}{K^*(x)}}, \tag{4.6}$$

or $\alpha_i(y)=K_{i-1}$

By linear approximation of the value y between grid nodes, we get:

$$\alpha_i(y)=K^*\left(\frac{y_{i-1}+y_i}{2}\right) \tag{4.7}$$

The applied scheme (4.4) is nonlinear y_i^{j+1} , so an iterative run-through method is used to solve it, which consists in the following.

Equation (4.4) is replaced by an algebraic one:

$$\overset{s}{A_i}\overset{s+1}{y_{i-1}}-\overset{s}{C_i}\overset{s+1}{y_i}+\overset{s}{B_i}\overset{s+1}{y_{i+1}}=-\overset{s}{F}{}_i^{j}, \tag{4.8}$$

where

$$\overset{s}{A_i}=\sigma\gamma\alpha_i\binom{s}{y};\;\overset{s}{B_i}=\overset{s}{A_{i+1}};$$

$$\overset{s}{C_i}=\overset{s}{A_i}+\overset{s}{A_{i+1}}+1;$$

$$\overset{S}{F}{}_i^j = \gamma\left(1-\sigma\right)\alpha_i\begin{pmatrix}S\\y\end{pmatrix}\underset{i-1}{\overset{S}{y}} - \left[\gamma\left(1-\sigma\right)\left(\alpha_{i+1}\right)\begin{pmatrix}S\\y\end{pmatrix} + \alpha_i\begin{pmatrix}S\\y\end{pmatrix} - 1\right]\overset{S}{y}_i +$$
$$+\gamma\left(1-\sigma\right)\alpha_{i+1}\begin{pmatrix}S\\y\end{pmatrix}\underset{i+1}{\overset{S}{y}}, \tag{4.9}$$
$$\gamma = \frac{\theta}{h^2}$$

where S is the iteration number.

As a zero approximation, we usually take values y_i^j from the previous layer, assuming $\overset{0}{y}_i = y_i^j$.

On the zero-time layer, the values $\overset{0}{y}_i$ according to the initial condition are determined from the expression

$$\underset{i}{\overset{0}{y}} = 1 - ih, \quad i = \overline{0,\ n}$$

Problem solving (4.8), (4.9), (4.5) the relative y_i^{j+1} value is found by the run method and is searched in the form:

$$y_i = \alpha_{i+1}y_{i+1} + \beta_{i+1}, \quad i = \overline{0, n-1} \tag{4.10}$$

where the unknown parameters α_{i+1} and β_{i+1} are determined from the relations:

$$\alpha_{i+1} = \frac{\overset{S}{B}_i}{\underset{i}{\overset{S}{C}} - \overset{S}{A}_i\,\alpha_i}; \beta_{i+1} = \frac{\underset{i}{\overset{S}{A}}\beta_i + \underset{i}{\overset{S}{F}}}{\underset{i}{\overset{S}{C}} - \underset{i}{\overset{S}{A}}\alpha_i}; i = \overline{1,\ n-1} \tag{4.11}$$

The parameters $\underset{i}{\overset{S}{A}},\ \underset{i}{\overset{S}{B}},\ \underset{i}{\overset{S}{C}},\ \underset{i}{\overset{S}{F}}$ are determined by the relations (4.9). To end an iteration, use the condition

$$\max_{1\le i\le n-1}\left|\overset{S+1}{y_i} - \underset{i}{\overset{s}{y}}\right| < \varepsilon$$

then proceed to calculating the values y_i on the next time layer.

When defining the parameters from (4.11), the boundary condition on is used $x = 0$, from which we obtain

$$\alpha_1 = \kappa_1; \quad \beta_1 = \nu_1 \tag{4.12}$$

By (4.11) and (4.12) are determined $\alpha_i, \beta_i(i = 1, n)$, then from (4.10) in reverse order are determined y_i using the second boundary condition:

$$y_n = \frac{\nu_2 + x_2\beta_n}{1 - \kappa_2\alpha_n} \tag{4.13}$$

The above scheme is stable under $\sigma \ge \frac{1}{2} - \frac{h^2}{4C_2\theta}$ the condition $0 < C_1 \le a(y) \le C_2$.

Similarly to the previous one, we present a scheme for calculating sensitivity functions by parameter C_g in grid nodes $\omega_{hx\theta}$.

The finite-difference analog of equation (3.11) corresponding to the implicit scheme is written as follows:

$$\frac{u_i^{j+1} - u_i^j}{\tau} = \frac{1}{h^2}\left[a_{i+1}\left(y_{i+1}^{j+1}\right)\left(\sigma u_{i+1}^{j+1} + \left(1-\sigma\right)u_{i+1}^j - \sigma u_i^{j+1} - \left(1-\sigma\right)u_i^j\right) - \right.$$

$$-a_i\left(y_i^{j+1}\right)\left(\sigma u_i^{j+1}\left(1-\sigma\right)u_i^j-\sigma u_{i-1}^{j+1}-\left(1-\sigma\right)u_{i-1}^j\right)\Big]+$$

$$+\frac{1}{h^2}\left[b_{i+1}\left(y^{j+1}\right)\left(\sigma u_{i+1}^{j+1}+\left(1-\sigma\right)u_{i+1}^j\right)-b_i\left(y^{j+1}\right)\left(\sigma u_i^{j+1}+\left(1-\sigma\right)u_i^j\right)\right]+$$

$$+\frac{1}{h^2\left(g+1\right)}\left[\left(y^g\right)_{i+1}^{j+1}\left(y_{i+1}^{j+1}-y_i^{j+1}\right)-\left(y^g\right)_i^{j+1}\left(y_i^{j+1}-y_{i-1}^{j+1}\right)\right] \tag{4.14}$$

To calculate the values of functions u_i^{j+1}on a new $(j+1)$ time layer, the scheme (4.8) – (4.13) described earlier is used, in which the values $A_i,\;\; B_i,\;\; C_i,\;\; F_i$ are calculated using the following formulas:

$$\overset{S}{\underset{i}{A}}=\gamma\sigma a_i\begin{pmatrix}S\\y\end{pmatrix};\;\; \overset{S}{B_i}=\gamma\sigma\left[a_{i+1}\begin{pmatrix}S\\y\end{pmatrix}+b_{i+1}\begin{pmatrix}S\\y\end{pmatrix}\right];$$

$$\overset{S}{\underset{i}{C}}=\gamma\sigma\left[a_{i+1}\begin{pmatrix}S\\y\end{pmatrix}+a_i\begin{pmatrix}S\\y\end{pmatrix}+b_i\begin{pmatrix}S\\y\end{pmatrix}\right]+1$$

$$F_i^S=\gamma\left(1-\sigma\right)a_i\begin{pmatrix}S\\y\end{pmatrix}u_{i-1}-\left[\gamma\left(1-\sigma\right)\left(a_{i+1}\begin{pmatrix}S\\y\end{pmatrix}+\right.\right.$$

$$\left.\left.y+a_i\begin{pmatrix}S\\y\end{pmatrix}+b_i\begin{pmatrix}S\\y\end{pmatrix}\right)+1\right]u_i^j+\gamma\left(1-\sigma\right) \tag{4.15}$$

$$\left[a_{i+1}\begin{pmatrix}S\\y\end{pmatrix}+b_{i+1}\begin{pmatrix}S\\y\end{pmatrix}\right]u_{i+1}^j+\frac{\gamma}{g+1}\varphi_i,$$

where $b_i\left(y\right)=a_i\begin{pmatrix}S\\y\end{pmatrix}-a_{i-1}\begin{pmatrix}S\\y\end{pmatrix}$

$$\varphi_i=\left(y_{i+1}^{j+1}\right)^{g+1}+\left(y_i^{j+1}\right)^g\left(y_{i-1}^{j+1}\right)-\left(y_{i+1}^{j+1}\right)^g\left(y_i^{j+1}\right)-\left(y_i^{j+1}\right)^{g+1},\; g=\overline{0,N} \tag{4.16}$$

y_i^j - already calculated function values y in (i,j) the grid node $\omega h\times\theta$

Zero initial and boundary conditions for all sensitivity functions lead to the fact that conditions (4.12) and (4.13) will correspond to the values $\nu_1=0;\;\; x_1=0;\; x_2=0;\;\; \nu_2=0$.

In an iterative algorithm for calculating unknown coefficients C_i by (3.2) – (3.7), ((2.4)) the values of the and functions y_n^S and $\left(u_n^j\right)C_i$ taken on all time layers $j=\overline{0,m}$ at the last point $i=n$ are used.

Due to the fact that the nonlinear piezoconductivity function $K^*\left(y\right)$ is given as a polynomial of degree N (**??**), the problem of determining the degree of the polynomial arises.

The degree of the polynomial for which the functional $J\left(K\right)$ takes the smallest value is chosen [7]:

$$J\left(K\right)=\frac{I_M\left(\alpha\right)}{1-\sqrt{\frac{\left(K+1\right)\left(l\frac{l}{K+1}+1\right)-\ln\eta}{l}}} \tag{4.17}$$

Where $I_M\left(\alpha\right)$ is the empirical risk functional that depends on the values of the coefficients of the polynomial calculated using the above method for a fixed degree of the polynomial K.

$$I_M\left(\alpha\right)=\frac{1}{n}\sum_{i=1}^{n}\left(y_{ij}^*-\sum_{i=1}^{K}C_jy_i^j\right)/\sigma_i^2. \tag{4.18}$$

Here:

n – the number of experimentally observed values of the studied quantity y_{ij}^*;

σ_i^2 – measurement variance y_i.

$1-\eta$ - the probability with which the estimate (**??**) is valid.

The coefficients of the model problem are calculated:

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left((1+y)\frac{\partial y}{\partial x}\right) \tag{4.19}$$

$$\begin{cases} y(x,0) = 1-x \\ y(0,t) = 1 \\ \frac{\partial y(1,t)}{\partial x} = 0 \end{cases} \tag{4.20}$$

For the direct task, the values were calculated $y(t,x)$. The values $y(t,1)$ were taken as experimental. Then, according to the described method for determining the coefficients from the sensitivity functions, the coefficients C_i of the polynomial were determined $K^*(y) = \sum_{i=0}^{K} C_i y^i$, where K the following values were taken:

$$K=0: \quad K^* = C_0,$$

$$K=1: \quad K^* = C_0 + C_1 y,$$

$$K=2: \quad K^* = C_0 + C_1 y + C_2 y^2,$$

and the corresponding values $J(K)$for (4.17).

The degree of the polynomial and the corresponding coefficients for which the value $J(K)$ is minimal are assumed to be true.

5 Conclusions

The paper investigates the inverse coefficient problem of nonlinear filtration associated with the restoration of the dependence of the piezoelectric conductivity coefficient on pressure in porous media. To identify unknown parameters, a method based on sensitivity theory and iterative minimization of the residual functional is proposed.

The piezoelectric conductivity coefficient is approximated by a polynomial function of pressure, which makes it possible to reduce the inverse problem to determining a finite number of parameters. Equations for sensitivity functions are obtained and a stable implicit finite difference scheme is developed for the combined numerical solution of the direct problem and sensitivity equations.

A criterion for choosing the optimal degree of a polynomial based on minimizing the empirical risk functional is proposed, ensuring the stability and adequacy of the approximation. Numerical experiments on a model problem confirm the effectiveness of the method and its applicability for interpreting experimental data in filtration problems.

The results obtained can be used in solving applied inverse problems of oil and gas mechanics and in identifying parameters of nonlinear parabolic models with coefficients depending on the state of the medium.

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