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Manifestation of Archimedean forces in the interaction of the gas volume with the underlying reservoir water

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Abstract. *The paper is devoted to modeling the process of gas injection into an aquifer. The key element of the model is to take into account the Archimedes force as an external mass force acting on the gas volume from the underlying water. This force is added to the Darcy filtration law and on this basis the stabilization of the top of the gas volume during gas injection into the upper region of a horizontal aquifer is justified. The task of determining the rate of gas surfacing and spreading at the formation roof.*

Keywords. archimedean force · gas volume · gas injection · Darcy's law · gas injection and spreading · formation.

Mathematics Subject Classification (2010): 76S05, 76T10

1 Introduction

Control of filtration processes during gas injection into reservoir systems is one of the key tasks of modern underground hydro-gas dynamics. This problem has a wide range of practical applications: from the creation and operation of underground gas storage facilities in aquifers and the management of gas caps in oil fields to promising technologies for geological disposal of carbon dioxide (co2) in order to reduce greenhouse gas emissions. In all these cases, the ability to predict the behavior of the light (gas) phase after its injection into a porous medium saturated with a heavier fluid phase, usually water, is of fundamental importance.

Classical filtration models based on Darcy's law often treat the fluid as a homogeneous medium or take into account gravitational effects in the framework of standard two-phase filtration equations [2]. However, when the gas volume locally forms a macroscopic cluster (”bubble”) in the underlying water-saturated medium, a physical effect occurs that requires separate consideration. We are talking about the Archimedean force acting on a body immersed in a liquid, which can be interpreted as an external mass force in the framework

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of the continuum approach. This force is not an internal property of the phases, but arises solely as a result of their interaction in the gravitational field and has a determining effect on the vertical migration of the gas cluster.

Existing approaches to modeling gas injection do not always adequately take into account this mechanism [8-10]. Ignoring the explicit consideration of the Archimedean force as a driving factor can lead to significant errors in the forecasts of the gas front rise rate, the shape of its contact with a water-saturated reservoir, and, most importantly, the conditions for its stabilization near an impenetrable roof. Thus, there is a scientific and practical problem associated with the development of a physically sound and mathematically correct model that would integrate the mechanism of Archimedean ascent into the standard apparatus of filtration theory.

2 Problem statement.

Consider a solid body T with volume Ω and weight G_T, which is completely submerged in a liquid at rest. According to Archimedes law, a buoyant force A, equal to the weight of the liquid displaced by the body, acts on the body through its surface $\sum$ from the side of the liquid G_f. In this case, the line of action of the Archimedes force A passes through the center of mass of the displaced liquid (Fig. 2.1, b). The Archimedes force A is a hydrostatic force that occurs due to an uneven distribution of pressure in the liquid (pressure increases with depth). For a floating body, the hydrostatic lifting force of Archimedes will be determined in terms of a closed surface $\sum$ consisting of the wetted surface of the body and the cross-sectional area of the body volume by a horizontal plane coinciding with the level of the liquid at rest.

We note that the expression for the Archimedean force $\bar{A}$ is obtained by applying the Gauss-Ostrogradsky theorem from the theory of analysis of continuous functions. A solid body (like a liquid) is represented as a system of material points that continuously fill a certain part of space.

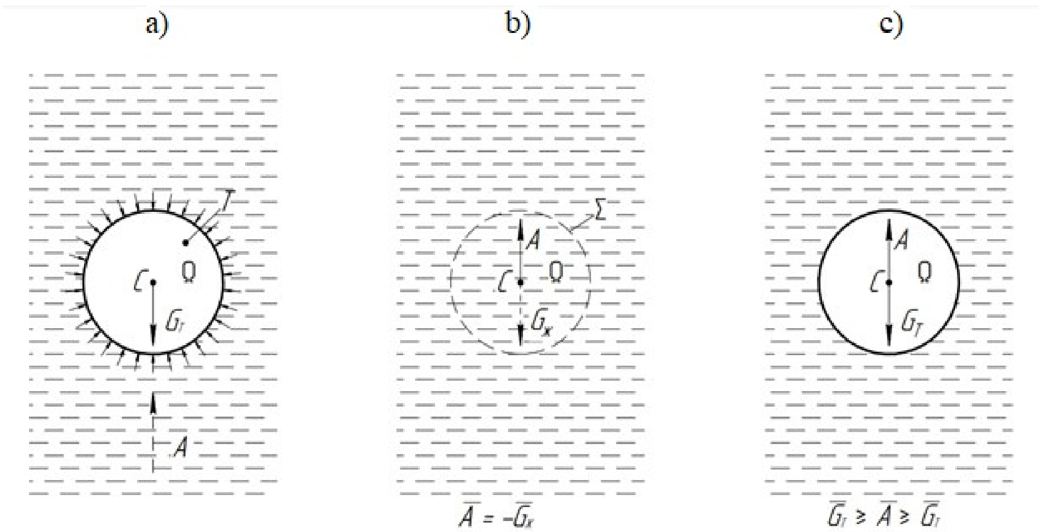

Fig. 2.1 Archimedean force acting on a body submerged in a liquid

In this case, it is necessary (Fig. 2.1, b) to mentally replace the volume of a body immersed in a liquid with the same volume of a resting liquid with density and pressure distributions satisfying the equilibrium equations. It is quite obvious that the equilibrium of the fluid surrounding the body will not be disturbed (the force A will not change). For the filled volume W of a liquid, as for a part of a single liquid space, the Gauss-Ostrogradsky formula is used, which makes it possible to convert the surface integral of the pressure P over the surface $\sum$ of the selected volume W into the volume integral of the pressure P over the same volume W. In this case, there is a transition from surface pressure forces to the category of volume (mass) forces.

The expression for the hydrostatic lifting force of Archimedes in vector form is represented as [2]:

$$\bar{A} = \int\limits_{\sum} P\bar{n}\, d\sigma = -\int\limits_{\Omega} gradP\, d\tau = -\int\limits_{\Omega} \rho\bar{F}\, d\tau. \tag{2.1}$$

If $\bar{F}$is the gravity density vector, and the z axis is directed vertically upwards, then $\bar{F} = g\bar{k}$; then

$$\bar{A} = \int\limits_{\Omega} \rho g\bar{k}\, d\tau = -\overline{G_f}, \tag{2.2}$$

where $\overline{G_f}$is the weight of the liquid contained in the volume Ω.

Note (explanation) to expression (2.1):

The equality of the first and second integrals expresses the content of the Gauss-Ostrogradsky theorem (Fig 2.1., a and b).

The equality of the second and third integrals expresses the equilibrium state of the extracted volume Ω_j (Fig. 1, b) of the liquid in accordance with the Euler equations:

$$\frac{\partial P}{\partial x} = \rho \mathrm{F}_x; \quad \frac{\partial P}{\partial y} = \rho \mathrm{F}_y; \quad \frac{\partial P}{\partial z} = \rho \mathrm{F}_z,$$

where: $\mathrm{F}_x, \ \mathrm{F}_y, \ \mathrm{F}_z$- projections of the density vector $\bar{F}$of massive forces;

ρ- density of the liquid.

Thus, the vector of the Archimedean force $\bar{A}$, as the resulting vector of surface forces acting on a body immersed in a liquid, is equal to and opposite to the vector of gravity $\overline{G_f}$ of the volume Ω of the liquid displaced by the body, i.e., the liquid mentally introduced into the surface $\sum$ of the volume Ω; $\bar{A} = \overline{G_f}$. The latter is the main vector of vertical gravity forces $\delta\overline{G_f}$ of liquid masses of elementary volumes $\delta\Omega$, i.e. $\overline{G_f} = \sum \delta\overline{G_f}$. Since the vector $\bar{A}$ is balanced by a force $\overline{G_f}$, i.e. it is balanced by a system of gravity forces $\delta\overline{G_j}$ of elementary volumes of a liquid, therefore $\bar{A}$ we will also consider the Archimedean force as the resulting vector distributed over the entire volume in the form of elementary Archimedean forces $\delta\bar{A}$ equal to and opposite to the forces of gravity $\delta\overline{G_f} \ - \delta\overline{G_f} = \delta\bar{A})$, (i.e. $\bar{A} = \sum \delta\bar{A}$.

As a result, we note that for each * element of the volume δW of a body immersed in a liquid, two forces act: $\delta\overline{G_T} \downarrow\uparrow \bar{A}$ (Fig. 1, b).

$\overline{G_T}$ - the gravity of the body, as a result of direct (direct) manifestation of the gravitational field.

$\bar{A}$ - the Archimedes force, as a result of the indirect (opposite) manifestation of the gravitational field through the fluid displaced by the body.

In addition, we note that the surface forces acting on a body immersed in a liquid during the movement of the liquid will depend and be determined not only by the hydrostatic pressure, which in the general case will be part of the total pressure [6].

All of the above regarding the Archimedean force acting on a body immersed in a liquid is fully transferred to any other type of medium (liquid, gas, placed in a denser liquid). In this case, the principal vector $\bar{A}$ of the Archimedean force acting from the side of the resting liquid on the surface of the submerged volume W of the considered liquid will be determined in the same way. The study of the manifestation of Archimedean forces in this case is of particular importance for the transition to the region of liquid and gas filtration in a porous medium, where Darcy's law is decisive [11].

To Darcy's Law.

Consider a certain volume of gas hypothetically located at the roof of an aquifer (Fig. 2.2). Under the action of Archimedean forces, gas will float up and spread out at the top of the formation, thereby changing the shape of the gas volume. We will consider the gas volume as the mass volume of gas submerged in the aquifer region of the formation and bounded from below by the contact surface with the underlying reservoir water, and from above by the surface of the formation roof [1, 13].

We will consider the manifestation of the Archimedean forces of gas ascent as external mass forces in terms of elementary Archimedean forces $\delta\bar{A}$ acting on the gas mass in each of its elementary volumes $\delta\ \Omega$. Let's select the elementary volume of the reservoir in the gas zone in the form of a rectangular parallelepiped with a volume of $\delta\Omega = dxdydz$. The gas contained in this volume is affected by gravity $\delta\overline{G_g}$and the Archimedean force $\delta\bar{A}$.

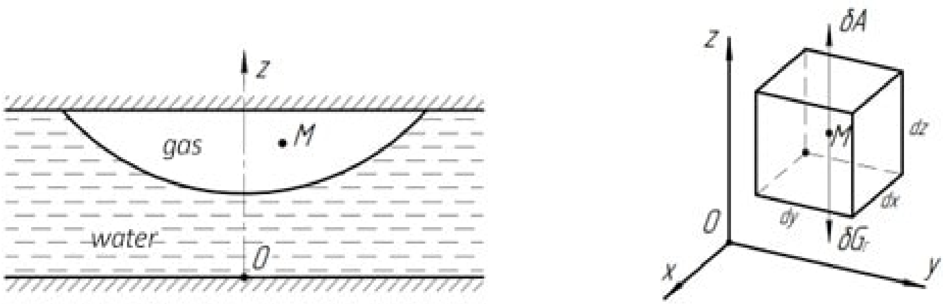

Fig. 2.2 Gas volume located at the top of a water-saturated aquifer

We have: $\delta\overline{G_g} = \gamma_\Gamma \cdot \delta\Omega = \rho_\Gamma g\ dxdydz \cdot m;$

$$\delta A = \gamma_b \cdot \delta\Omega = \rho_v g\ dxdydz \cdot m;$$

where: ρ_Γ, ρ_v, is the density of the floating gas and underlying water, respectively.

m is the porosity coefficient of the formation.

Gas gravity and Archimedean force densities:

$$z_g = \frac{\delta G_g}{\delta M} = \frac{\rho_\Gamma g\ m\delta\Omega}{\rho_\Gamma\ m\delta\Omega} = g;$$

$$z_A = \frac{\delta A}{\delta M} = \frac{\rho_2 g\ m\delta}{\rho\ m\delta} = \frac{\rho_2}{\rho} g. \tag{2.3}$$

We mark it. Darcy's law, according to which the projections of the filtration rate of an incompressible liquid (gas) on the coordinate axes (without taking into account mass forces) have the form:

$$V_x = -\frac{k}{\mu}\frac{\partial p}{\partial x},\ \ V_y = -\frac{k}{\mu}\frac{\partial p}{\partial y},\ \ V_z = -\frac{k}{\mu}\frac{\partial p}{\partial z}, \tag{2.4}$$

where k is the permeability coefficient of the porous medium;

μ- absolute viscosity coefficient of the liquid.

The Euler equations for the motion of an ideal fluid (without convective terms) in relation to a porous medium, when the friction forces of the fluid on the surface of grains composing the porous medium were classified as volume (mass) forces distributed over the entire volume of the porous medium, take the form (N. E. Zhukovsky's equations):

$$\begin{aligned} &\tfrac{\partial P}{\partial x} - \rho x_1 - \rho x_2 = 0 \\ &\tfrac{\partial P}{\partial y} - \rho y_1 - \rho y_2 = 0 \ , \\ &\tfrac{\partial P}{\partial z} - \rho z_1 - \rho z_2 = 0 \end{aligned} \tag{2.5}$$

where: $x_1,\ y_1,\ z_1$ - projections of external mass forces related to the unit mass of the liquid (density of mass drag forces).

From equations (2.3), taking into account expressions (2.2), we obtain the expression of projections of the filtration rate according to Darcy's law, taking into account mass forces

$$V_x = -\frac{k}{\mu}\left(\frac{\partial p}{\partial x} - \rho x_1\right), \ \ V_y = -\frac{k}{\mu}\left(\frac{\partial p}{\partial y} - \rho y_1\right), \ \ V_z = -\frac{k}{\mu}\left(\frac{\partial p}{\partial z} - \rho z_1\right). \tag{2.6}$$

In our case, the mass forces $\delta\overline{G_3}$ and $\delta\bar{A}$are projected only on the axis z. Therefore, the velocity expression V_z is supplemented with a term that takes into account the action of the Archimedean force:

$$V_z = -\frac{k}{\mu}\left(\frac{\partial p}{\partial z} - p_g z_g - p_g z_A\right).$$

Given (2.1), we get (the z-axis is directed upwards):

$$V_z = -\frac{k}{\mu_g}\left(\frac{\partial p}{\partial z} + \rho_g g - \rho_v g\right) = -\frac{k}{\mu_g}\left(\frac{\partial p}{\partial z} + \gamma_g - \gamma_v\right) =$$

$$= -\frac{k}{\mu_g}\left[\frac{\partial p}{\partial z} - (\gamma_v - \gamma_g)\right] = -\frac{k}{\mu_g}\left(\frac{\partial p}{\partial z} - \varDelta\gamma\right);$$

where $\gamma = \gamma_v - \gamma_g$;

$\gamma_v,\ \gamma_g$– the volume weight of water and gas, respectively.

As a result, the projections of the filtration rate (2.6) according to Darcy's law, taking into account mass forces, take the form:

$$V_x = -\frac{k}{\mu_g}\frac{\partial p}{\partial x}, \ \ V_y = -\frac{k}{\mu_g}\frac{\partial p}{\partial y}, \ \ V_z = -\frac{k}{\mu_g}\left(\frac{\partial p}{\partial z} \pm \varDelta\gamma\right).$$

$+\varDelta\gamma$ – with the axis $z$pointing downwards.

$-\varDelta\gamma$ – with the axis $z$pointing up.

If we ignore the value γ_g, since $\gamma_g \ll \gamma_v$, then the velocity projection expression V_z takes the form:

$$V_z = -\frac{k}{\mu_g}\left(\frac{\partial p}{\partial z} \pm \gamma_v\right),$$

that is, we have in Darcy's law a clear manifestation of the Archimedean force through γ_v, displacing the reservoir fluid (water).

3 To the problem of gas injection.

Let us consider in general terms the manifestation of the Archimedean force during gas injection into a horizontal partially opened aquifer (in relation to the problem of possible creation of underground gas storage facilities in such reservoirs) [7].

A horizontal aquifer with a thickness of h, which is opened to a depth of b, is injected with gas, which is incompressible under reservoir conditions (Fig. 3.1). The problem of forming a gas volume is extremely difficult, since it is associated with a mobile gas - water interface and the unknown shape of the interface itself. However, it can be assumed that in the process of injecting gas into the formation with a certain volume flow $Q = Const$ rate, it is possible to stabilize the top A of the gas volume; in this case, the injected gas will be located (pushing water) in the zone adjacent to the formation roof. We assume that when gas is injected into the reservoir, its pressure $p = P(r, z)$ in the gas zone will be known at some point in time. This pressure along the well axis $p = P(o, z)$ will decrease as you move away from the injection well. We find a condition under which gas particles at the top A of the gas volume can become stationary. Finding this condition is similar to determining the condition of immobility of the top of a water cone during oil extraction from a reservoir underlain by reservoir water [3, 5].

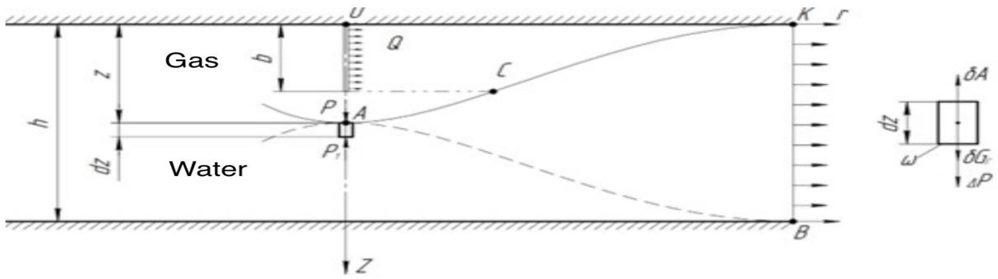

Fig. 3.1 Gas injection into a horizontal aquifer and stabilization of the gas volume top

Two assumptions for the gas injection process when the gas volume of vertex A is stabilized:

1. The moving interface of the AC will have a concave section of the ACC and a convex CS with an inflection point "*C*" at the bottom of the well.

2. In the aquifer region of the formation, a zone of stationary water will appear with its upper boundary AB moving in the form of the lowest water stream line.

Let us select at the top A of the gas volume lying on the axis of the well an elementary cylinder of a porous medium with a height dz and cross-section $d\omega$ filled with gas that has penetrated into the aquifer zone of the formation (Fig. 3). Let us consider the forces acting on this elementary volume of gas. If the pressure on the upper face of the element is denoted by $p = P(o, z)$, then the pressure on the lower face will be:

$$P' = P(o, z + dz) = P + \frac{\partial p}{\partial z} dz.$$

The resulting force due to the injection filtration field will be:

$$\Delta P = P - P' = -\frac{\partial p}{\partial z} dz d\omega \cdot m \to \dots$$

And is directed in the direction of pressure drop, i.e. down the axis z. Here m is the porosity of the formation, since gas does not occupy the entire cross-sectional $d\omega$ area, but only a part $md\omega$ of it.

At the same time, the selected element of the gas volume $\delta\Omega = dzd\omega m$ is affected vertically by the gravity of the gas $8\bar{G}_g$ and the Archimedean buoyant force $\delta\bar{A}$.

$$\delta G_g = \gamma_g dzd\omega m, \delta A = \gamma_v \cdot dzd\omega m,$$

where γ_g,γ_v, is the volume weight of gas and water, respectively. When the top A of the gas volume is stabilized, the condition of equilibrium of all vertical forces must be fulfilled

$$\frac{\partial p}{\partial z} dzd\omega m + \gamma_g dzd\omega m - \gamma_v dzd\omega m = 0,$$

i.e. $\frac{\partial p}{\partial z} - \gamma_v + \gamma_g = 0$; $\frac{\partial p}{\partial z} - \varDelta\gamma = 0$. $\frac{\partial p}{\partial z} \leq \varDelta\gamma$; $\varDelta\gamma = \gamma_v - \gamma_g$.

Thus, the movement of the gas volume vertex A will stop (stabilize) at the Z-axis point, where the pressure drop gradient $\frac{\partial p(0,z)}{\partial z}$is equal to the difference between the specific volumes of liquid and gas$\varDelta\gamma = \gamma_v - \gamma_g$.

So, the condition for stability and stability of the top of the gas volume during gas injection into the aquifer is

$$\frac{\partial p\,(0,z)}{\partial z} \leq \varDelta\gamma; \quad \varDelta\gamma = \gamma_v - \gamma_g. \tag{3.1}$$

We note that the condition of stabilization (3.2) of the gas volume vertex is obtained directly from expression (3.1) for the filtration rate according to Darcy's law, which takes into account the manifestation of the Archimedean force, that is, assuming

$$V_z = -\frac{k}{\mu_g}\left(\frac{\partial p_g}{\partial z} - \varDelta\gamma\right) = 0. \tag{3.2}$$

At the same time, we note that after gas injection into the aquifer ceases (Q=0) and reservoir pressure stabilizes, i.e., when $\frac{\partial p(0,z)}{\partial z} = 0$, the process of gas surfacing (peak A) begins under the action of Archimedean forces at a speed of

$$V_z = V_A = \frac{k}{\mu_g}\varDelta\gamma = const. \tag{3.3}$$

By way of illustration, we will reveal the condition of stabilization (3.3) of the top of the gas volume through the flow rate Q of the injected gas. At the same time, we make two assumptions that are used in the problem of fluid flow to a well that is imperfect in terms of opening degree [8]:

1. the surface of the borehole through which gas is injected is replaced by an equally large hemisphere (at b < < h), i.e. $2\pi \mathrm{r_e} b = 2\pi r_0^2$; from where $\mathrm{r_0} = \sqrt{\mathrm{r_A} b}$.

2. gas injection is considered as injection into a reservoir of infinite thickness (based on the theory of the potential of a point source in space).

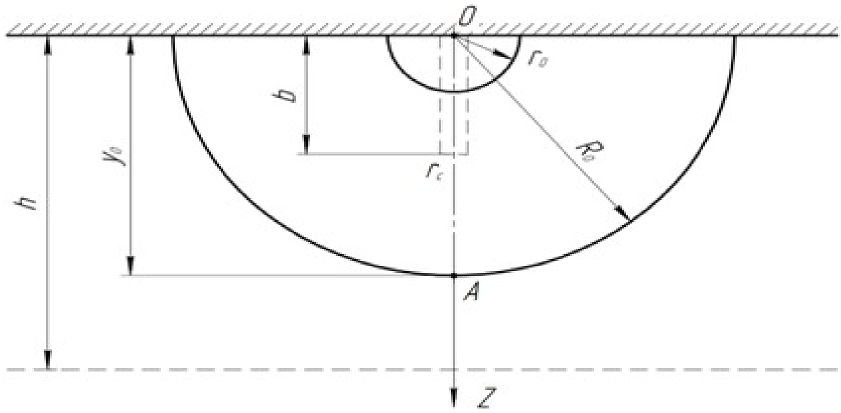

Fig. 3.2 Equivalent hemispherical model of the gas injection zone

Initial condition (3.3):

$$\frac{\partial p}{\partial z} = \Delta\gamma \ \text{ or } \ \frac{k}{\mu_g}\frac{\partial p}{\partial z} = \frac{k}{\mu_g}\Delta\gamma.$$

But $\frac{k}{\mu_g}\frac{\partial p}{\partial z} = V$- flow rate.
That is

$$V = \frac{k}{\mu_g}\Delta\gamma. \qquad (a)$$

In turn, for a source with a flow rate of *Q*, the filtration rate at the boundary of a hemisphere of radius R_0 will be:

$$V = \frac{Q}{2\pi R_0^2}. \qquad (b)$$

We equate (a) and (b).

$$\frac{\mathrm{Q}}{2\pi \mathrm{R}_0^2} = \frac{k}{\mu_g}\mathrm{H}\gamma, \ \text{where } \mathrm{R}_0 = \mathrm{y}_0$$

We get:

$$\mathrm{y}_0 = \sqrt{\frac{Q\mu_g}{2\pi k\Delta\gamma}}.$$

Let's take an example, we accept:
b=2 m; h=8 m. (b << h).
Q=50000 $\frac{m^3}{day} = 0,6\ \frac{m^3}{s}$;
$\mu_g = 1\ cPs = 10^{-3}\ \frac{N\cdot s}{m^2}$; k= 3.4 D=3.4·$1,02 \cdot 10^{-12} = 3,5 \cdot 10^{-12} m^2$

$$\Delta\rho = 980\frac{kg}{m^3};\ \Delta\gamma = \Delta\rho \mathrm{g} = 980 \cdot 9,8 = 9800\ \frac{N}{m^3}$$

$$\mathrm{r}_0 = \sqrt{\mathrm{r_A} b} = \sqrt{0,06 \cdot 2} = 0,34\ \ m.$$

We find:

$$\mathrm{y}_0 = \sqrt{\frac{Q\mu_g}{2\pi k\Delta\gamma}} = \mathrm{y}_0 = \sqrt{\frac{0,6 \cdot 10^{-3}}{2 \cdot 3,14 \cdot 3,5 \cdot 10^{-12} \cdot 9800}} = 5,5\ \ \mathrm{m}.$$

I.e. $\mathrm{y}_0 > b$ on $3,5\ m$.

4 The problem of floating and spreading gas.

Let us also consider the process of spreading of a gas volume (Fig. 4.1). Let a horizontal aquifer with thickness h contain a certain volume of injected gas at its roof. Due to the manifestation of Archimedean forces, gas will float up and spread out at the top of the formation. We proceed from the assumption of incompressibility of gas in reservoir conditions. The permeability coefficient k in the gas and water zones of the formation is the same. We will consider the process of gas surfacing and spreading as quasi-static, i.e. we assume a hydrostatic pressure distribution in the reservoir that does not correspond to the dynamic equilibrium of the gas-water system, as a result of which the system reacts to this discrepancy by shifting the interface [10].

We consider the position of an axisymmetric gas volume at a certain moment of time t, at which the pressure at any point of it will be $P(r,\ z)$. Let us find out the rates of gas ascent and spread, respectively, at contour points A and K of the gas volume (Fig. 4.2). Selecta point M at the gas volume boundary on a vertical circular section of a reservoir of arbitrary radius *r*. Two forces act on the gas boundary layer at this point: the force due to the weight of the gas column, and the counteracting (Archimedean) force due to the hydrostatic pressure of the reservoir water surrounding the gas volume.

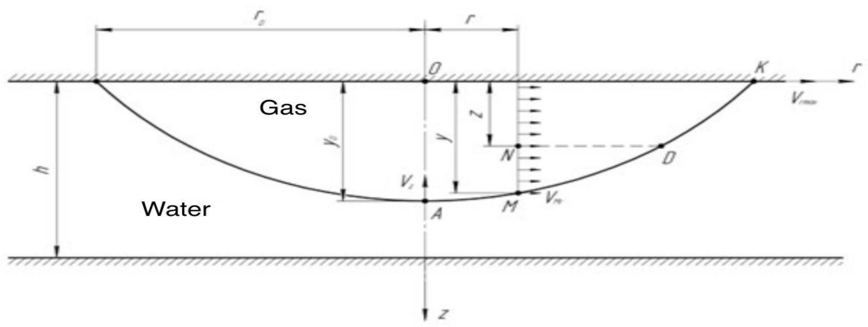

Fig. 4.1 Forces acting on a gas element at the gas–water interface

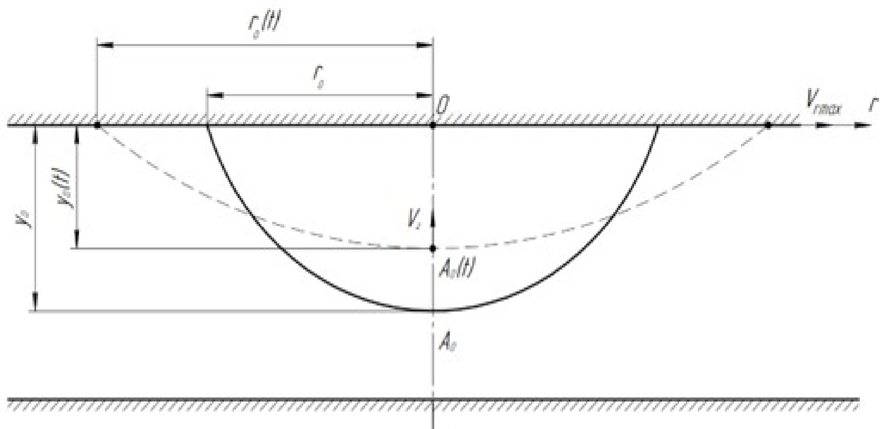

Fig. 4.2 Paraboloid model of the gas volume during buoyant rise and spreading

We denote the pressure at the roof of the aquifer through P_0. Then, taking the vertical pressure distribution as hydrostatic, we write the expression of pressure at the point M:

$$P_M = Po + \gamma_v y - \gamma_g y = P_0 + (\gamma_v - \gamma_g)\, y = P_0 + \varDelta\gamma y. \quad (4.1)$$

The pressure at any Nvertical point is similar:

$$P_N = P_0 + \varDelta\gamma \cdot z, P_N = P_m - \varDelta\gamma\,(y - z) = P_o + \varDelta\gamma y - \varDelta\gamma y + \varDelta\gamma z = P_0 + \varDelta\gamma z. \quad (4.2)$$

We find the vertical velocity of gas filtration through pressure at an arbitrary point N:

$$V_z = -\frac{k}{\mu_g}\frac{\partial P_N}{\partial z} = -\frac{k}{\mu_g}\frac{\partial}{\partial z}\,(P_o + \varDelta\gamma z) = -\frac{k}{\mu_g}\varDelta\gamma = const. \quad (4.3)$$

In other words, the rate of gas upsurge does not depend on the vertical coordinate z and is the same at all vertical points, including at M the interface point.

Therefore,

$V_{MZ} = V_{NZ} = -\frac{k}{\mu_g}\varDelta\gamma$, which corresponds to (4.1) to expression (3.3) of Darcy's law.

The horizontal velocity $V_r$of gas spreading is found through the expression of pressure (4.1) at a point M.

$$V_r = V_{Mr} = -\frac{k}{\mu_g}\frac{\partial P_M}{\partial r} = -\frac{k}{\mu_g}\frac{\partial}{\partial r}\,[P_0 + \varDelta\gamma y] = -\frac{k}{\mu_g}\varDelta\gamma\frac{\partial y}{\partial r}\ . \quad (4.4)$$

Comparing (4.4) and (4.5), we find the possibility of velocities $V_r$and V_z:

$$V_r = V_z\frac{\partial y}{\partial r}. \quad (4.5)$$

The component $\varDelta\gamma y$for the pressure at the boundary (4.1) will be called the ascending (Archimedean) pressure ($h_{bH} = \varDelta\gamma y$), which changes at the boundary of the gas volume and thereby causes the gas to spread horizontally (4.5).

We find the heads reduced to the axis $r$at points MandN

$$h_M = \frac{P_M}{\varDelta\gamma} - y = \frac{P_0}{\varDelta\gamma},\ h_N = \frac{P_N}{\varDelta\gamma} - z = \frac{P_0}{\varDelta\gamma}, \quad (4.6)$$

That is, the heads are equal (as they should be with a vertical hydrostatic pressure distribution). Therefore, the horizontal filtration rates $V_r$and are the same along the entire vertical and their value will be determined by the speed (4.7) of the boundary point Mthrough its upward pressure $\frac{\partial h_{bH}}{\partial r} = \varDelta\gamma\frac{\partial y}{\partial r}$.

To expand expression (4.7), it is necessary to have an equation of the interface$y = \mathrm{y}\,(\mathrm{r})$, which we assume based on the assumed constant shape and size of the gas volume W [12, 14]. We take the gas volume as a paraboloid of rotation:

$$y = y_0\left(1 - \frac{r^2}{r_0^2}\right). \quad (4.7)$$

Then

$$\frac{\partial y}{\partial r} = -2\frac{y_0}{r_0^2}r;\ V_r = 2\frac{k}{\mu_g}\varDelta\gamma\frac{y_0}{r_0^2}r, \quad (4.8)$$

That is V_r, it depends linearly on the coordinate r.

At

$$r = 0,\ V_r = 0; r = r_0, V_r = V_{r\,max} = 2\frac{k}{\mu_g}\varDelta\gamma\frac{y_0}{r_0}\ . \quad (4.9)$$

Taking into account (4.3) $V_r = 2V_z \frac{y_0}{r_0^2}; \quad V_{r\ max} = 2V_z \frac{y_0}{r_0}$

We also find the relationship y_0 and r_0 for (4.9) from W the gas volume expression. We find the volume of the paraboloid based on (4.7).

Select the b/m element of the paraboloid by volume $dW = ds \cdot dz = \pi r^2 \cdot dz = \pi r^2 dy \cdot y_0$.

Then:

$$W = \smallint dW = \int \pi r^2 dy.$$

From equation (4.7) we find

$$r^2 = r_0^2 \left(1 - \frac{y}{y_0}\right);$$

then

$$W = \int_0^{y_0} \pi r_0^2 \left(1 - \frac{y}{y_0}\right) dy = \pi r_0^2 \left(y_0 - \frac{1}{2} y_0\right) = \frac{1}{2} \pi r_0^2 \cdot y_0.$$

$$W = \frac{1}{2} \pi r_0^2 \cdot y_0 \tag{4.10}$$

From (4.10) we have:

$$r_0 = \sqrt{\frac{2W}{\pi y_0}}; \, y_0 = \frac{2W}{\pi r_0^2} \tag{4.11}$$

Then:

$$V_{r\ max} = 2V_z \frac{1}{r_0} \frac{2W}{\pi r_0^2} = V_z \frac{4W}{\pi r_0^3}, \tag{4.12}$$

or:

$$V_{r\ max} = 2V_z \frac{y_0}{\sqrt{\frac{2W}{\pi y_0}}} = V_z \sqrt{\frac{2\pi y_0^3}{W}}. \tag{4.13}$$

Expressions (4.11) and (4.13) determine the dependences of the radius r_0 and velocity of gas spreading $V_{r\ max}$ of a gas volume on the coordinate y_0 of its vertex A at a known volume w_0(under reservoir conditions) of the injected gas. As the gas y_0 rises (decreases), the gas spreading rate decreases $V_{r\ max}$.

$$w_0 = m\bar{\sigma}w;$$

m– gas porosity coefficient

$\bar{\sigma}$– average gas saturation of the gas volume w

By revealing the condition of stabilization (2.1) Aof the gas volume vertex, i.e.$\frac{\partial P}{\partial r} = \Delta\gamma$, we can determine the initial value of the coordinate $y_0 = y_0\,(0)$.

From expression (4.11), we find the initial value $r_0 = r_0\,(0)$ of the gas spreading radius at the top of the formation at a fixed value of the gas volume pumped into the formation w_0 by the time the top is stabilized A. At subsequent points in time (t > 0), the values of the spreading radius $r_0\,(t)$ will be determined from the same dependence (4.11) with the corresponding coordinate value $y_0 = y_0\,(t)$ determined by the pop-up speed $V_z = \frac{k\Delta\gamma}{\mu_3} = Const.$

Note that $W = \frac{w_0}{m\bar{\sigma}}$.

Find the equations of motion $y_0\,(t)$ of the vertex Aand contour K of the gas volume W. The starting time (t $= 0$) takes the moment when the vertex stabilizes A($\frac{\partial O}{\partial z} = 0$). Therefore, the initial conditions for the motion of points A and K are the parameters y_0 and

$r_0$defined by expression (4.13) in terms of the known value of the ordered gas volume w_0 at this moment [15].

The initial equations are expressions of filtration rates (4.9) and (4.13), which are related to the actual (true) speed of movement by the dependence $v_D = mv_{8AB}$

m – the porosity coefficient of the formation.

Consider the movement of a vertex A.

$$m\frac{\partial \gamma_0}{\partial t} = -\frac{k\varDelta\gamma}{\mu_g}; \frac{\partial \gamma_0}{\partial t} = -\frac{k\varDelta\gamma}{m\mu_g}. \tag{4.14}$$

We divide the variables in (4.14) and integrate them within the time range from 0 to t

$$dy_0 = -\frac{k\varDelta\gamma}{m\mu}dt; \int\limits_{y_0}^{y_o(t)} dy_0 = -\frac{k\varDelta\gamma}{m\mu}\int\limits_0^t dt.$$

We get:

$$y_o\left(t\right) - y_o = -\frac{k\varDelta\gamma}{m\mu}t, \text{ that is } y_o\left(t\right) = y_o - \frac{k\varDelta\gamma}{m\mu}t \tag{4.15}$$

We have a linear time law of vertex motion A.

Consider the motion of the contour *b* point of the gas volume

$$m\frac{dr_0}{dt} = V_z\frac{4W}{\pi r_0^3} = \frac{k\varDelta\gamma}{\mu}\frac{4W}{\pi r_0^3}$$

Separating variables

$$dr_0 = \frac{4k\varDelta\gamma \cdot W}{\pi m\mu}\textstyle\int_0^t dt; r_0^3 dr_0 = \frac{4k\varDelta\gamma}{\pi m\mu}W \cdot dt$$

Integrating:

$$\textstyle\int_{r_0}^{r_{0(t)}} r_0^3 dr_0 = \frac{4k\varDelta\gamma}{\pi m\mu}W\textstyle\int_0^t dt.$$

We get:

$$\frac{1}{4}\left[r_0^4\left(t\right) - r_0^4\right] = \frac{4k\varDelta\gamma}{\pi m\mu}Wt.$$

From where:

$$r_0\left(t\right) = \sqrt[4]{r_0^4 + \frac{16k\varDelta\gamma}{\pi m\mu}Wt}, W = \frac{1}{2}\pi r_0^2 y_0.$$

Then

$$r_0\left(t\right) = r_0\sqrt[4]{1 + \frac{16k\varDelta\gamma}{\pi m\mu r_0^4 \cdot 2}\pi r_0^2 y_0 t}$$

$$r_0\left(t\right) = r_0\sqrt[4]{1 + \frac{8k\varDelta\gamma \cdot y_0}{m\mu \cdot r_0^2} \cdot t}$$

$$r_0\left(t\right) = r_0\left[1 + \frac{8k\varDelta\gamma \cdot y_0}{m\mu \cdot r_0^2}t\right]^{\frac{1}{4}}$$

Thus , having the initial values of contour parameters r_0 and y_0 gas volume W, expressions (4.13) and (4.15) allow us to find their values $y_0\left(t\right)$ and $r_0\left(t\right)$ at subsequent points in time t > 0.

5 Conclusions

Archimedes' force acting on the volume of gas in a water-saturated porous medium is consistently represented as an external mass force distributed throughout the entire volume of gas. Including Archimedes' force in Darcy's law leads to a modified filtration rate that correctly describes the buoyant migration of gas in aquifers.

The condition for stabilizing the upper level of the gas volume during gas injection into a horizontal aquifer is determined based on the balance between the pressure gradient caused by the injection and the difference in the specific weights of water and gas. After gas injection ceases, gas rise occurs only under the action of Archimedes' forces at a constant vertical velocity independent of depth.

A quasi-static analytical model of gas distribution across the reservoir roof has been developed, allowing the geometry of the gas volume and migration dynamics to be predicted.

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