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Invariant solutions of the nonstationary laminar flow equation flows of non-Newtonian fluids in pipes

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Abstract. *This article investigates non-stationary and stationary laminar flows of non-Newtonian viscous fluids in cylindrical pipes using group analysis methods. A generalized formulation of the equation of motion is considered, taking into account the nonlinear rheological friction law, which leads to a modified form of the classical flow equation. A group classification of the basic differential equation is performed, and invariant solutions corresponding to extended symmetry operators, both point and tangential, are obtained. These solutions describe physically significant flow regimes, including those with singularities, flow in pipes with permeable walls, and flow in channels with variable geometry. In addition, invariant solutions are obtained for non-isothermal motion of a viscous fluid with temperature-dependent viscosity at high Peclet numbers. The results show that theoretical group methods provide an effective basis for constructing accurate and semi-analytical solutions to complex non-Newtonian flow problems and provide valuable insight into filtration and heat transfer processes.*

Keywords. non-Newtonian fluids · pipe flow · differential equations · rheological models · heat transfer

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1 Introduction

The rapid development of the petrochemical industry and the widespread introduction of petroleum and plastics have advanced the task of studying the regularities of the movement

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of such liquids. Many materials flows under certain conditions and exhibit nonlinear viscous properties [22, 24]. Various technological processes in the chemical, petroleum, and food industries are associated with the flow of such non-Newtonian materials, which can include non-Newtonian liquids: viscoelastic liquids, polymer solutions, liquid crystals, suspensions, and various plastics.

Pumping a non-Newtonian fluid is characterized by a constant power consumption, since as a result of intensive mechanical action, it acquires properties that do not depend on the duration of pumping.

Since non-Newtonian fluids have a large apparent viscosity, they are usually characterized by laminar motion. At the same time, under certain conditions, the laminar motion of non-Newtonian fluids can turn into a turbulent one. However, determining the transition conditions is a very difficult task.

Studies of the flow of non-Newtonian fluids are mostly experimental in nature, and a mathematical theory of a non-Newtonian fluid, similar to that created for a Newtonian fluid, has not been constructed [17].

To describe the laws of motion of non-Newtonian fluids, the Navier-Stokes equations cannot be used, since the apparent viscosity depends on the velocity of motion, and, consequently, not only the velocity, but also the apparent viscosity will change over the flow cross-section. The derivation of equations of motion for non-Newtonian fluids is given in the literature [3].

The available methods for calculating the flow of non-Newtonian fluids mainly relate only to the stationary regime. The rapid development of computer technology and mathematical modeling over the past sixty years has led to a large number of papers devoted to the numerical study of the motion of viscous media and media with rheologically complex behavior. The study of such flows is of important practical interest, since many real fluids do not obey Newton's rheological law. In particular, when processing polymer compositions by casting, liquid media are characterized by non-Newtonian rheological properties [9, 15].

Accounting for the non-Newtonian behavior of a fluid requires considerable effort for the successful implementation of computational technologies. Data on the rheological and thermophysical properties of non-Newtonian fluids are presented in [6, 8, 11]. The theory and methods for calculating laminar flow and heat transfer of high-viscosity non-Newtonian fluids in round tubes with variable physical properties of the fluid are described, taking into account the influence of motion energy dissipation.

The authors of [19] conducted a study for a power-law fluid, during which parametric calculations were performed for varying the Reynolds number, the degree of pipe expansion, and the nonlinearity index of the fluid, and the dependences of local hydraulic resistance on the defining parameters of the problem were constructed.

In [14], the stationary laminar flow of non-Newtonian fluids in curved channels (bends and turns) is considered, and a mathematical model is presented, as well as the results of numerical studies. A comparative analysis of hydraulic resistance in curved channels for pseudoplastic, Newtonian, and dilatant fluids is performed.

In [11], the problem of heat transfer in a non-Newtonian fluid flowing through a round tube in a stabilized laminar regime is formulated.

Laminar flows of a non-Newtonian fluid in flat channels and in round pipes are realized in many technical applications. In particular, when processing polymer materials in a fluid state, flows occur between parallel planes and in circular pipes in the elements of technological equipment [6].

Various methods for calculating laminar flow of rheologically stable liquids through straight circular pipes are presented in [7, 13]. For engineering calculations, it is advisable to apply a universal method and invariant solutions suitable for all liquids.

In this paper, we consider an approach for obtaining partial solutions of the laminar flow equation for non-Newtonian fluids in pipes based on group analysis of differential equations. Invariant solutions with respect to point and tangent transformations satisfying the natural boundary condition are given.

2 Invariant and tangent symmetries of equations describing laminar and filtration flow of non-Newtonian media

The problem can be formulated as the following equation for the flow velocity w:

$$\omega_t = \nu \left(\omega_{rr} + r^{-1}\omega_r\right) + \rho^{-1} f(t)$$

where $-\partial p/\partial z = f(t)$ is the given law of change of the differential pressure; p, v - is the density and kinematic viscosity of the liquid, respectively. This equation has been studied by many authors [11, 16, 18] under various laws of time dependence of the pressure drop.

For liquids with non-Newtonian properties [26], a similar problem is formulated as follows:

$$\omega_t = \Phi'(\omega_r)\,\omega_{rr} + r^{-1}\Phi(\omega_r) + \rho^{-1} f(t), \tag{2.1}$$

where Φ is a function that characterizes the law of friction of a non-Newtonian fluid.

As a replacement

$$\omega = u + \frac{1}{\rho}\int\limits_0^t f(t)\,dt,$$

equation (2.1) reduces to the equation

$$u_t = \Phi'(u_r)\,u_{rr} + r^{-1\Phi}(u_r) \tag{2.2}$$

The aim of this work is to obtain some partial solutions of equation (2.2), and hence (2.1), by studying the group properties of this equation [21].

In the case of an arbitrary dependence $\Phi = \Phi(u_r)$, equation (2.2) admits a three-dimensional algebra L_3 of infinitesimal operators with a basis $X_1 = \partial/\partial t,\quad X_2 = \partial/\partial u$ corresponding to shifts in t and u, and $X_3 = r\partial/\partial r + +2t\partial/\partial t + u\partial/\partial u$ corresponding to a self-similar solution.

A group classification of equation (2.2) by function Φ (up to equivalence transformations [21]) leads to the following result. The expansion of the algebra L_3 occurs only in the following specializations $\Phi(u_r)$ (cases $\Phi' \equiv 0,\quad \Phi' \equiv 1$ are excluded):

1 $\Phi(u_r) = exp(u_r)$; additional basis operator

$$X_4 = r\partial/\partial r + (u + 2r)\,\partial/\partial u$$

2 $\Phi(u_r) = u_r^{\lambda}$; additional basis operator

$$X_5 = (\lambda - 1)\,r\partial/\partial r + (\lambda + 1)\,u\partial/\partial u$$

3 $\Phi(u_r) = u_r^{-1}$. We obtain an infinite-dimensional group containing in addition X_1 to , X_2, X_3, X_5 (for $\lambda = -1$) additional operators

$$X_6 = 8t^2\partial/\partial t + r\left(u^2 - 2t\right)\partial/\partial r + 8ut\partial/\partial u$$

$$X_7 = ru\partial/\partial r + 4t\partial/\partial u, \quad X_\infty = \omega r^{-1}\partial/\partial r$$

where the function ω satisfies the equation $\omega_t + \omega_{uu} = 0$.

Let us consider invariant solutions of equation (2.2) associated with the appearance of these additional symmetries and corresponding to natural boundary conditions

$$w|_{r=R} = 0, \;\; w_r|_{r=0} = 0 \tag{2.3}$$

where R is the radius of the pipe (we assume later $R = 1$).

Consider an invariant solution for the operator

$$X_5 + \alpha X_3 = \delta r\partial/\partial r + 2\alpha t\partial/\partial t + \sigma u\partial/\partial u$$

$$\delta = \lambda + \alpha - 1, \;\; \delta = \lambda + \alpha + 1$$

At $\alpha = 0$ the same time, we get a solution in the form

$$w = \beta\left(1 - r^{\gamma}\right)(t_0 - t)^{1/(1-\lambda)}, \; \beta = (\gamma\,(3\lambda - 1))^{1/(1-\lambda)}, \; \gamma = (\lambda + 1)\,/\,(\lambda - 1)$$

which corresponds $f(t) = \rho\,(\lambda - 1)^{-1}(t_0 - t)^{-\lambda/(\lambda-1)}$ to, satisfies under $\lambda > 1$ conditions (2.3), and describes the regime with exacerbation.

When $\alpha \neq 0$ the invariant solution is written as

$$u = t^{1/2\sigma/\alpha}\varphi(\xi), \;\; \xi = rt^{-1/2\sigma/\alpha}$$

where φ satisfies the corresponding differential equation. For $\alpha = -\lambda - 1$, it is integrated in quadratures, and the solution satisfying conditions (2.3) (for $\lambda < 0$) is written as

$$w = \int\limits_{rt^{-1/(1+\lambda)}}^{t^{-1/(1+\lambda)}} \xi^{-1/\lambda}\left(\frac{1-\lambda}{(1+\lambda)(3\lambda+1)}\xi^{(3\lambda-1)/\lambda} + \xi_0\right)d\xi$$

and the corresponding mode

$$f(t) = \rho t^{1/[\lambda(\lambda+1)]}\left(\frac{1-\lambda}{(1+\lambda)(3\lambda-1)}t^{(1-3\lambda)/[\lambda(\lambda+1)]}\right)^{1/(\lambda-1)}$$

When $\Phi(u_r) = u_r^{-1}$ equation (2.2) is replaced $x = r^2$ by the equation

$$u_t = (1/u_x)_x \tag{2.4}$$

which $x_1 = u, \;\; u_1 = x$ is converted by substitution to the linear heat conduction equation [1, 12]. However, from the solutions of this equation, it is difficult to obtain and study a solution of equation (2.1) that satisfies the conditions (2.3).

If we introduce a function $v = u_x$, then by differentiating (2.2) with respect to x, we obtain an equation that has a self-similar solution (the function ϕ has a parametric representation):

$$v = 1/2\sqrt{2\xi}\phi(\ln\xi), \; \xi = xt^{-1/2} \tag{2.5}$$

$$\varphi = \frac{1}{s - F(s)}, \; \xi = exp\int\limits_{s_2}^{s}\frac{\left(F'(s) - 1\right)ds}{(s - F(s))\left(sF(s) - F^2(s) - 1\right)}$$

$$F(s) = exp\left(\frac{s}{2}\right)^2\left(\int\limits_{s_0}^{s} exp\left(\frac{\tau^2}{2}\right)d\tau + s_1\right)^{-1}$$

Then the solution of equation (2.1) satisfying conditions (2.3) can be represented by (2.5) as

$$w = \sqrt{t} \int\limits_{\ln r^2 t^{-1/2}} \frac{dz}{\varphi(z)} \tag{2.6}$$

The solution of equation (2.2) corresponding to (2.6) is no longer invariant under point transformations, but will be invariant under some tangent transformation [12].

We present a method for constructing some tangent symmetries for equation (2.2), considered in [2].

Differentiating equation (2.2) and introducing a new function $v = u_r$, we obtain the equation:

$$v_t = \left(r^{-1}\left(r\Phi(v)\right)_r\right)_r \tag{2.7}$$

The group classification of equation (2.7) with respect to point transformations leads to the following result. If $\Phi(v)$ is an arbitrary function, then equation (2.7) admits a two-dimensional algebra with basis $Y_1 = \partial/\partial t$, $Y_2 = 2t\partial/\partial t + r\partial/\partial r$. The extension of this algebra occurs under the following specializations $\Phi(v)$ (cases $\Phi' \equiv 0,\ \Phi' \equiv 1$ are excluded):

1 $\Phi(v) = e^v$; additional operator

$$Y_3 = r\partial/\partial r + 2\partial/\partial v$$

2 $\Phi(v) = v^\lambda$; additional operator

$$Y_3 = (\lambda - 1)\, r\partial/\partial r + 2v\partial/\partial v$$

3 $\Phi(v) = v^{-1/5}$; additional operators

$$Y_4,\ Y_5 = r^3\partial/\partial r - 5r^2 v\partial/\partial v$$

4 $\Phi(v) = v^{-1}$; additional operators

$$Y_4,\ Y_6 = r^{-1}\partial/\partial r + vr^{-2}\partial/\partial v$$

The operators $Y_3, \ldots, Y_6$ associate tangent symmetry operators for equation (2.2) and allow us to construct invariant solutions corresponding to them.

In particular, when $\Phi(u_r) = u_r^{-1/5}$ the invariant solution corresponding to the associated operator $Y_2 + 5/6Y_4 + Y_5$ can be written as

$$w = \int\limits_r^1 r\, exp\left(\frac{1}{6r^2}\right) F\left(t^{1/2} exp\left(\frac{1}{6r^2}\right)\right)^{-5} dr$$

where $F(z)$ satisfies the ordinary differential equation

$$2z^2 F'' + 6zF' + 15F^{-6}z^{-5}F' + 18F = 0 \tag{2.8}$$

and the operator $Y_2 + Y_5$ corresponds to the solution

$$w = \int\limits_r^1 \left(r^2+1\right)^{-5/2}\, G^{-5}\left(\frac{r^2}{(r^2+1)\,t}\right) dr$$

where $G(z)$ satisfies the equation

$$4z^2G'' + 4zG' + 5z^2G^{-6}G' - G = 0 \tag{2.9}$$

In equations (2.8), (2.9) by introducing new functions

$$F(z) = z^{-1}\varphi(\ln z), \; G(z) = z^{1/6}\psi(\ln z)$$

you can lower the order and investigate them using the methods of the qualitative theory of differential equations.

It should be noted that equation (2.2) is used to describe the plane-radial filtration of non-Newtonian media [5]. Therefore, the results obtained can also have a filtering interpretation.

3 Invariant solutions of equations of nonisothermal stationary flow of a viscous fluid in pipes

A stationary nonisothermal flow of a viscous Newtonian fluid in a flat channel filled with a porous material is investigated. The Brinkman equation is used as the motion equation. The viscosity is considered temperature-dependent. When writing the energy equation, a single-temperature model is used. Dissipative heat releases are taken into account. The problem is solved for temperature boundary conditions of the first kind. For the first time, the problem of nonisothermal fluid flow in a flat channel completely filled with a porous material is posed and solved by an approximate method. The temperature at the channel inlet can be very different from the temperature of the channel wall, which immediately makes the non-isothermal factor decisive in this process. The mathematical model takes into account the temperature dependence of viscosity and energy dissipation. The results obtained showed that the neglect of any of these factors can lead to significant distortions of the actual flow pattern and heat exchange in the channel [4].

A laminar stationary nonisothermal flow of a power-law fluid in a cylindrical channel with sudden narrowing is studied. A mathematical model of the flow is formulated, which includes the hydrodynamic equations written in the variables current function-vortex, and the energy equation. The rheological properties of a liquid are described by the Ostwald-de-Waale power law, in a modified form of which the dependence of the effective viscosity on temperature is taken into account. To solve the problem, we use the establishment method followed by the implementation of a finite-difference method based on the variable direction scheme. The effect of viscous dissipation on the flow structure of pseudoplastic, Newtonian, and dilatant fluids is estimated. The temperature and effective viscosity fields are demonstrated. The results of a parametric study of the local hydraulic resistance coefficient are presented. Mathematical modeling of the flow of a power-law fluid in a channel with a narrowing cross-section jump under nonisothermal conditions is performed. An algorithm for numerical solution of the problem is developed. The calculation results made it possible to estimate the effect of viscous dissipation and the temperature dependence of viscosity on the introduced dimensionless geometric characteristics of the flow structure. In the course of parametric studies, it was found that when the main parameters of the problem (the non-linearity indicator of the fluid, the Peclet number, and the Reynolds number) vary, the length of the two-dimensional flow zone changes significantly behind the cross-section jump. Of particular interest is the fact that an increase in the nonlinearity index for the isothermal and non-isothermal cases has a different effect on the length of this zone: in the first case, it decreases, and in the second, it increases. Characteristic features of the effective viscosity and temperature fields for pseudoplastic, Newtonian, and dilatant fluids are shown. Based on the obtained data, the dependences of the local hydraulic resistance coefficient on the dimensionless criteria of the problem and the non-linearity index of the fluid are constructed [23].

The influence of the nature of changes in the viscosity-temperature dependence on the flow stability of a 45% aqueous solution of propylene glycol in a flat channel with a linear temperature distribution is considered. The study of hydrodynamic stability was reduced to solving the generalized Orr-Sommerfeld equation by the spectral method. The eigenvalue spectra and eigenfunctions for both sections of the viscosity-temperature dependence are constructed, and the corresponding critical Reynolds numbers are determined. The results of the study indicate that the laminar-turbulent transition depends not only on the presence of a temperature dependence of viscosity, but also on the intensity of its change. The results of numerical studies made it possible to establish a significant influence of the inhomogeneous temperature distribution on the flow regimes of a thermos-viscous liquid. It is found that taking into account the dependence of viscosity on temperature reduces the critical Reynolds number and increases the range of unstable flow regimes. Comparison of two different sections of the viscosity function as a function of temperature for a propylene glycol solution, which differ in the values of the derivative, showed that the interface between stable and unstable flow regimes also depends on the intensity of the viscosity change: the higher it is, the smaller the region of stable laminar flows. It should be added that averaging the viscosity over the range of its variation leads to the classical Orr-Sommerfeld equation and its corresponding results [20].

We study the group properties [21] of a system of equations describing the flow in pipes of a liquid whose viscosity depends on temperature at high Peclet numbers. It is shown that for exponential and power-law dependences, the main transformation group expands. For these cases, we consider invariant solutions that have a physical meaning.

Equations describing the motion of a viscous fluid in a cylindrical tube can be written in dimensionless form as follows for $\delta \ll 1$, $Pe \gg 1$[25]:

$$\frac{\partial p}{\partial R} = 0, \ \ \frac{\partial p}{\partial z} = \delta Pe \frac{1}{R}\frac{\partial}{\partial R}\left(\mu R u\right) \tag{3.1}$$

$$\frac{\partial v}{\partial R} + R\frac{\partial u}{\partial z} = 0, \ \ \frac{\partial^2 T}{\partial R^2} + \frac{1-v}{R}\frac{\partial T}{\partial R} = u\frac{\partial T}{\partial z} \tag{3.2}$$

Here

$$z = \frac{x}{Per_0}, \ \ \mu = \frac{\eta}{\eta_0}, \ \ T = \frac{t}{t_0}, \ \ R = \frac{r}{r_0}, \ \ \delta = \frac{r_0}{l}, \ \ p = \frac{P}{P_0}$$

$$v = R\frac{V_r Pe}{2V_0}, \ \ u = \frac{V_x}{2V_0}, \ \ P_0 = \frac{2\eta_0 l V_0}{r_0^2}, \ \ Pe = \frac{2V_0 r_0}{a}$$

where x is the longitudinal coordinate $r-$ is the distance from the pipe axis r_0, and is the pipe radius,

$t-$ temperature $V_x, \ V_r-$ - longitudinal and radial velocity components $\eta-$ - fluid viscosity $l-$ - pipe length $P-$ - pressure, $t_0, \ \eta_0, \ V_0$- characteristic values of temperature, viscosity and velocity $Pe-$ - Peclet number.

From the first equation (3.1) it follows that $\partial p/\partial z-$ is a certain function z, we denote it $g\left(z\right)$. Therefore, the second equation (3.1) can be integrated once with respect to R under the natural symmetry condition $\partial u/\partial R|_{R=0} = 0$. Introducing the notation $f\left(T\right) = \delta \mu Pe/2$ equation (3.1), we replace it with the following:

$$\partial u/\partial R = Rf\left(T\right) g\left(z\right) \tag{3.3}$$

We carry out a group classification [21] of the system (3.2), (3.3).

For an arbitrary function type f, the system allows infinitesimal operators:

$$X_1 = R\left(1 - z\frac{g'}{g}\right)\frac{\partial}{\partial R} + 4z\frac{\partial}{\partial z} + 2u\left(1 + z\frac{g'}{g}\right)\frac{\partial}{\partial u} - R^2 u\left(z\frac{g'}{g}\right)\frac{\partial}{\partial v}$$

$$X_2 = -R\frac{g'}{g}\frac{\partial}{\partial R} + 4\frac{\partial}{\partial z} + 2u\frac{g'}{g}\frac{\partial}{\partial u} - R^2 u\left(\frac{g'}{g}\right)'\frac{\partial}{\partial v}$$

Extensions of this algebra are obtained under the following specifications $f\left(T\right)$ up to equivalence transformations [21]:

1) $f\left(T\right) \equiv const$; additional basis operators

$$X_3 = \partial/\partial T,\ \ X_4 = T\partial/\partial T$$

2) $f\left(T\right) = T^{\gamma}$; additional operator

$$X_5 = \gamma R\partial/\partial R - 4T\partial/\partial T + 2\gamma u\partial/\partial u$$

3) $f\left(T\right) = e^{T}$; additional operator

$$X_6 = R\partial/\partial R - 4\partial/\partial T + 2u\partial/\partial u$$

Let us consider some invariant solutions corresponding to these operators that allow physical interpretation.

By

$$f\left(T\right) = e^{\varepsilon T},\ \ g\left(z\right) = -2p_0 e^{-\varepsilon z},\ \ p_0 = Const$$

the invariant solution of the operator $X_2 - X_6$ has the form:

$$v = \varphi_1\left(R\right),\ \ u = \varphi_2\left(R\right),\ \ T = z + \varphi_3\left(R\right)$$

$\varphi_i\left(i = 1,\ 2,\ 3\right)$ satisfy a system of ordinary differential equations whose solution under boundary conditions is:

$$v|_{R=1} = u|_{R=1} = 0,$$

it can be written as:

$$\varphi_1 = 0,\ \ \varphi_k = \varphi_k^{(0)} + \varepsilon\varphi_k^{(1)} + O\left(\varepsilon^2\right),\ \ k = 2,\ 3;$$

$$\varphi_2^{(0)} = p_0\left(1 - R^2\right),\ \ \varphi_2^{(1)} = -2p_0\int_R^1 R\varphi_3^{(0)}dR;$$

$$\varphi_3^{(i)} = \int_0^R\left(\int_0^R R\varphi_2^{(i)}dR\right)\frac{dR}{R} + \alpha,\ \ \alpha = const,\ \ t = 0,1.$$

By

$$f\left(T\right) = e^{\varepsilon T},\ g\left(z\right) = -2p_0 z^{1-\varepsilon},\ p_0 = const,$$

the invariant solution of the operator $X_1 - X_6$ has the form

$$v = \varphi_1\left(R\right),\ \ u = z\varphi_2\left(R\right),\ \ T = \ln z + \varphi_3\left(R\right),$$

where the functions $\varphi_i\left(i = 1,\ 2,\ 3\right)$ satisfy a system of ordinary differential equations

$$\varphi_2' = -2p_0 Re^{\varepsilon\varphi_3},\ \ R\varphi_2 + \varphi_1' = 0;$$

$$\left(1 - \varphi_1\right)\varphi_3' + R\varphi_3'' - R\varphi_2 = 0,$$

the solution of which is

$$\varphi_i = \varphi_i^{(0)} + \varepsilon\varphi_i^{(1)} + O\left(\varepsilon^2\right),\ i = 1,\ 2,\ 3,$$

$$\varphi_1^{(0)} = p_0\left(R^4/4 - R^2/2\right), \ \ \varphi_2^{(0)} = p_0\left(1 - R^2\right),$$

$$\varphi_3^{(0)} = p_0\left(\int\limits_R^1\left(\int\limits_0^R\left(1 - R^2\right)RF\left(R\right)dR\right)\frac{dR}{RF\left(R\right)} + \alpha\right),$$

$$\varphi_1^{(1)} = -\int\limits_R^1 R\varphi_2^{(1)}dR,$$

$$\varphi_2^{(1)} = -2p_0\int\limits_R^1 R\varphi_3^{(0)}dR, \quad \varphi_3^{(1)} = \int\limits_R^1\left(\int\limits_0^R\left(R\varphi_2^{(1)} + \varphi_1^{(1)}\varphi_3^{(0)'}\right)RF\left(R\right)dR\right)\frac{dR}{RF\left(R\right)},$$

$$F\left(R\right) = \exp\left(\left(p_0/4\right)\left(R^2 - R^2/4\right)\right),$$

describes the flow in a pipe with permeable walls at a constant injection (suction) rate $v|_{R=1} = -p_0/4$.

For arbitrary functions f and g, the invariant solution of the operator X_2can be written as

$$v = -\frac{1}{4}\frac{g^{'}}{g}\varphi\left(\xi\right), \ \ u = \frac{\varphi\left(\xi\right)}{R^2}, \ \ T = A_1\ln\xi + A_2,$$

$$\varphi = \xi^{1/2}\left(\frac{1}{4}\int\xi^{-1/2}f(T)d\xi + A_3\right), \ \ \xi = R^4g(z),$$

where $A_i \ \ (i = 1, \ 2, \ 3)$ are arbitrary constants, they can be chosen in such a way that

$$\varphi\left(\beta_i\right) = 0, \ \ \beta_i > 0, \ \ i = 1, \ 2 \ .$$

This solution corresponds to the flow in an annular channel, the radii of the walls of which vary according to the law $R_i = \left(\beta_i/g\right)^{1/4}$. Since $g\left(z\right)$– is an arbitrary function, and the initial system of equations is invariant with respect to the shift by z, then you can choose the function $g\left(z\right)$ and the range of changes z so that $R_i$they are practically constant.

4 Conclusions

The group analysis method has shown high efficiency for the study of equations describing the unsteady laminar flow of non-Newtonian fluids in pipes. The group classification made it possible to identify cases of extending the symmetry algebra and obtain invariant solutions that have a physical meaning and satisfy natural boundary conditions.

Invariant and tangent solutions have been constructed for various types of rheological functions that characterize the law of friction of a non-Newtonian fluid, which describe various modes of motion, including modes with sharpening, modes with permeable walls, and flows in channels of variable radius. The obtained solutions have not only hydrodynamic, but also filtration interpretation, which expands the scope of the method.

Within the framework of non-isothermal flows, it is shown that the temperature dependence of viscosity significantly affects the flow structure, regime stability, and hydraulic resistance. Group analysis of the system of equations for large Peclet numbers revealed the presence of additional symmetries, which allowed us to obtain new invariant solutions describing physically significant non-isothermal processes, including the influence of energy dissipation and temperature inhomogeneity.

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