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ASYMPTOTICS OF THE SOLUTION IN A RECTANGLE OF A BOUNDARY VALUE PROBLEM FOR ONE-CHARACTERISTIC DIFFERENTIAL EQUATION DEGENERATING INTO A PARABOLIC EQUATION

Abstract

In a rectangular domain we consider a boundary value problem for a third order, non-classical type equation degenerating into a second order parabolic equation. The total asymptotic expansion of the solution of the problem under consideration in a small parameter is constructed and the remainder term is estimated.

While studying some real phenomena with non-uniform passages from one physical characteristics to another ones, it is necessary to study singularly perturbed boundary value problems. Such problems attracted the attention of many mathematicians. Non-classical singularly perturbed equations were not studied enough.

M.I. Vishik and L.A. Lusternik in [1] have introduced the so-called one-characteristic equations. They called the equations of $2k+1$ odd order and of type

$$L_{2k+1} \equiv A_1(A_{2k}u) + B_{2k}u = f \tag{1}$$

one-characteristic if A_1 is a first order operator, A_{2k} is $2k$ order elliptic operator, and B_{2k} is any differential operator of at most $2k$ order. Obviously, the real characteristics of equation (1) will be only the characteristics of the first order operator A_1. In the same paper in $D = \{(x,y)\,|0 \le x \le 1, 0 \le y \le 1\}$ they considered the boundary value problem

$$\varepsilon^2 \frac{\partial}{\partial x}(\Delta u) - \varepsilon \Delta u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = f(x,y), \tag{2}$$

$$u\Big|_{\Gamma} = 0, \frac{\partial u}{\partial x}\Big|_{x=1} = 0, \tag{3}$$

where Γ is the boundary of the domain D. Assuming $f(x,y)$ for $x=y$ together with its own derivatives of corresponding order vanishes, the authors constructed only the first terms of the asymptotic of the solution of boundary value problem (2), (3).

M.G.Javadov and M.M.Sabzaliyev M.M. in the paper [2] refused the go to zero condition $f(x,y)$ for $x=y$ and constructed the first terms of the asymptotics of the solution of boundary value problem (2), (3) with regard to internal layers arising near $x=y$. In the same paper the asymptotics of the solution of this problem was constructed to within any degree of the small parameter.

The total asymptotics in a small parameter of the solution of a boundary value problem for a third order one-characteristic equation degenerating into an elliptic equation was constructed by M.M. Sabzaliev in [3].

The boundary value problems for non-classic equations not being $(2m+1)$-th order one-characteristic equations degenerating into hyperbolic and parabolic equations were studied in the papers [5] – [7].

In the present paper in the rectangle $D = \{(t,x)\,|0 \le t \le T,\ 0 \le x \le 1\}$ we consider the boundary value problem

$$L_\varepsilon u \equiv \varepsilon^2 \frac{\partial}{\partial t}(\Delta u) - \varepsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au = f(t,x), \tag{4}$$

$$u\,|_{t=0} = 0, \quad u\,|_{t=T} = 0, \ \frac{\partial u}{\partial t}\,|_{t=T} = 0, \tag{5}$$

$$u\,|_{x=0} = 0, \quad u\,|_{x=1} = 0, \tag{6}$$

where $\varepsilon > 0$ is a small parameter, $\Delta \equiv \dfrac{\partial^2}{\partial t^2} + \dfrac{\partial^2}{\partial x^2}$, $\ a > 0$ is a constant, $f(t,x)$ is the given smooth function.

Our goal is to construct the asymptotic expansion of the solution of boundary value problem (4)-(6) in a small parameter. By constructing the asymptotics we follow the M.I. Vishik-L.A. Lusternik method represented in [1] and [4].

For constructing the asymptotics we perform iterative processes. In the first iterative process we'll look for approximate solution of equation (4) in the form

$$W = W_0 + \varepsilon W_1 + ... + \varepsilon^n W_n, \tag{7}$$

and the functions $W_i(t,x);\ i = 0,1,...,n$ will be chosen so that

$$L_\varepsilon W = 0\left(\varepsilon^{n+1}\right). \tag{8}$$

Substituting (7) in (8) and equating the terms with the same powers of ε, for determining $W_i; i = 0,1,...,n$ we get the following recurrently connected equations

$$\frac{\partial W_0}{\partial t} - \frac{\partial^2 W_0}{\partial x^2} + aW_0 = f(t,x), \tag{9}$$

$$\frac{\partial W_1}{\partial t} - \frac{\partial^2 W_1}{\partial x^2} + aW_1 = \frac{\partial^2 W_0}{\partial t^2}, \tag{10}$$

$$\frac{\partial W_k}{\partial t} - \frac{\partial^2 W_k}{\partial x^2} + aW_k = \frac{\partial^2 W_{k-1}}{\partial t^2} - \frac{\partial}{\partial t}(\Delta W_{k-2});\ k = 2,3,...,n. \tag{11}$$

Equations (9), (10), (11) differ only by the right sides. Equation (9) is obtained from equation (4) for $\varepsilon = 0$ and is called a degenerate equation corresponding to equation (4).

Obviously it is impossible to use all boundary conditions (5), (6) for equations (9), (10), (11). For equations (9), (10), (11) with respect to t it should be used the first condition from (5) and with respect to x the both conditions from (6), i.e.

$$W_i\,|_{t=0} = 0, \tag{12}$$

$$W_i\,|_{x=0} = 0, \quad W_i\,|_{x=1} = 0; \quad i = 0,1,...,n. \tag{13}$$

Thus, from (9) and (12), (13) for $i = 0$ we get that the function $W_0(t,x)$ is the solution of equation (9) satisfying the conditions

$$W_0\,|_{t=0}\, = 0; \quad (0 \le x \le 1), \tag{14}$$

$$W_0\,|_{x=0}\, = W_0\,|_{x=1}\, = 0; \ (0 \le t \le T). \tag{15}$$

Problem (9), (14), (15) is called a degenerate problem corresponding to problem (4)-(6). Looking for the function $W_0(t,x)$ in the form

$$W_0(t,x) = \sum_{k=1}^{\infty} W_{0k}(t)\sin k\pi x, \tag{16}$$

we get that the functions $W_{0k}(t)$ are the solutions of the equations

$$\frac{dW_{0k}}{dt} + (1 + k^2\pi^2)W_{0k} = f_k(t); \quad k = 1, 2, \ ..., \tag{17}$$

where

$$f_k(t) = 2\int_0^1 f(t,\xi)\sin k\pi\xi d\xi. \tag{18}$$

(12) yields that the functions $W_{0k}(t)$ should satisfy the initial conditions

$$W_{0k}(0) = 0; \quad k = 1, 2, ... \ . \tag{19}$$

The solutions of equations (17) satisfying initial conditions (19), are of the form

$$W_{0k}(t) = \int_0^t e^{-(a+k^2\pi^2)(t-\tau)} f_k(\tau)d\tau; \quad k = 1, 2, ... \ . \tag{20}$$

Taking into account (20) in (16), we get

$$W_0(t,x) = \sum_{k=1}^{\infty}\left[\int_0^t e^{-(a+k^2\pi^2)(t-\tau)} f_k(\tau)d\tau\right]\sin k\pi x. \tag{21}$$

Finally, substituting the expression for $f_k(t)$ from (18) to (21), and denoting

$$G(x,\xi,t-\tau) = 2\sum_{k=1}^{\infty} e^{-(a+k^2\pi^2)(t-\tau)}\sin k\pi x \cdot \sin k\pi\xi, \tag{22}$$

we get the solution of problem, (9), (14), (15) that is represented by the formula

$$W_0(t,x) = \int_0^t\int_0^1 G(x,\xi,t-\tau)f(t,\xi)d\xi d\tau. \tag{23}$$

Obviously, when $f(t,x)$ is smooth and vanishes together with its second order derivative with respect to x for $x = 0$ and $x = 1$, then the function $W_0(t,x)$ determined by formula (23) satisfies equation (9), initial condition (14) and boundary conditions (15). But in order to complete the first iterative process, it is necessary to impose on the function $f(t,x)$ such conditions that the functional series standing in the right side of equality (16) so that one could differentiate required number of times with respect to t and x, and the obtained new functional series would be uniformly convergent in D.

Introduce the denotation

$$C^{\alpha,\beta}(D) = \left\{ u(t,x) \left| \frac{\partial^i u(t,x)}{\partial t^{i_1} \partial x^{i_2}} \in C(D); i = i_1 + i_2; i_1 = 0,1,...,\alpha; i_2 = 0,1,...,\beta \right. \right\}.$$

It is valid

Lemma 1. *Let $f(t,x) \in C^{p-1,2p+2}(D)$, and the following condition be fulfilled*

$$\frac{\partial^{2r} f(t,0)}{\partial x^{2r}} = \frac{\partial^{2r} f(t,1)}{\partial x^{2r}} = 0; r = 0,1,...,p. \tag{24}$$

Then the solution of problem (9), (14), (15) that is represented by formula (23) satisfies the relations

$$W_0(t,x) \in C^{p,2p}(D), \tag{25}$$

$$\frac{\partial^{i_1+2i_2} W_0(t,0)}{\partial t^{i_1} \partial x^{2i_2}} = \frac{\partial^{i_1+2i_2} W_0(t,1)}{\partial t^{i_1} \partial x^{2i_2}} = 0; i_1 + i_2 \le p, \tag{26}$$

where i_1, i_2 are non-negative integers, p is an arbitrary natural number.

Proof. Taking into account conditions (24) and applying to the right side of equality (18) the integration by parts formula, we have

$$f_k(t) = -\frac{2}{k\pi} \int_0^1 f(t,\xi) d(\cos k\pi\xi) = \frac{2}{k\pi} \int_0^1 \frac{\partial f(t,\xi)}{\partial \xi} \cos k\pi\xi d\xi =$$

$$= \frac{2}{k^2\pi^2} \int_0^1 \frac{\partial f(t,\xi)}{\partial \xi} d(\sin k\pi\xi) = -\frac{2}{k^2\pi^2} \int_0^1 \frac{\partial^2 f(t,\xi)}{\partial \xi^2} \sin k\pi\xi d\xi =$$

$$= ... = \frac{2(-1)^{p+1}}{k^{2p+2}\pi^{2p+2}} \int_0^1 \frac{\partial^{2p+2} f(t,\xi)}{\partial \xi^{2p+2}} \sin k\pi\xi. \tag{27}$$

(27) yields the following estimation

$$|f_k(t)| \le \frac{2M_{0,2p+2}}{k^{2p+2}\pi^{2p+2}}, \quad t \in [0,T], \tag{28}$$

where $M_{0,2p+2} = \max\limits_{(t,x)\in D} \left| \frac{\partial^{2p+2} f(t,x)}{\partial x^{2p+2}} \right|$ denotes a constant independents of k. Note that such an estimation is valid for the derivatives $f_k(t)$ as well:

$$\left| f_k^{(i)}(t) \right| \le \frac{2M_{i,2p+2}}{k^{2p+2}\pi^{2p+2}}; i = 0,1,...,p-1; t \in [0,T], \tag{29}$$

where

$$M_{i,2p+2} = \max_{(t,x)\in D} \left| \frac{\partial^{i+2p+2} f(t,x)}{\partial t^i \partial x^{2p+2}} \right|; i = 0,1,...,p-1.$$

If we denote the members of the functional series in the right side of equality (16) by $\overline{W}_{0k}(t,x) = W_{0k}(t)\sin k\pi x$, it is obvious that

$$\left|\frac{\partial^i \overline{W}_{0k}(t,x)}{\partial t^{i_1}\partial x^{i_2}}\right| \le \left|\frac{d^{i_1}W_{0k}(t)}{dt^{i_1}}\right| k^{i_2}\pi^{i_2}, \quad (i = i_1 + i_2). \tag{30}$$

Taking into account estimation (28) from (20), we have

$$|W_{0k}(t)| \le \int_0^1 |f_k(\tau)|\, d\tau \le \frac{2TM_{0,2p+2}}{k^{2p+2}\pi^{2p+2}},\ t \in [0,T]. \tag{31}$$

Following estimations (28) and (31), from the equality

$$\frac{dW_{0k}(t)}{dt} = f_k(t) + k^2\pi^2 W_{0k}(t) \tag{32}$$

we have

$$\left|\frac{dW_{0k}(t)}{dt}\right| \le \frac{c_1}{k^{2p}\pi^{2p}},\ t \in [0,T], \tag{33}$$

where $c_1 > 0$ is a constant independent of k. Differentiating sequentially the both sides of (32) and each time taking into account the estimation for the previous derivative $W_{0k}(t)$, we get the validity of the estimation

$$\left|\frac{d^{i_1}W_{0k}(t)}{dt^{k_1}}\right| \le \frac{c_{i_1}}{k^{2p+2-2i_1}\pi^{2p+2-2i_1}},\ t \in [0,T]. \tag{34}$$

Based on (34) from (30) it follows that

$$\left|\frac{d^{i_1}\overline{W}_{0k}(t,x)}{dt^{i_1}\partial x^{i_2}}\right| \le \frac{c_{i_1}}{k^{2p+2-2i_1-i_2}\pi^{2p+2-2i_1-i_2}};\ (t,x) \in D. \tag{35}$$

Denoting $r = 2p + 2 - 2i_1 - i_2$, from (35) we get that the number series

$$\frac{c_{i_1}}{\pi^r}\sum_{k=1}^{\infty}\frac{1}{k^r} \tag{36}$$

is majorant for the functional series

$$\sum_{k=1}^{\infty}\frac{\partial^i \overline{W}_{0k}(t,x)}{\partial t^{i_1}\partial x^{i_2}}. \tag{37}$$

As the number series (36) converges for $r \ge 2$, then the functional series (37) converges uniformly in D. This means that relations (25), (26) are valid for the function $W_0(t,x)$ determined by equality (23).

Lemma 1 is proved.

If we take the natural number p contained in the condition of lemma 1 $p = n + 4$ then the smoothness of the function W_0 will admit to construct the remaining functions $W_1, W_2, ..., W_n$.

By lemma 1, the right side of equation (10) satisfies condition (24) for $p = n+2$. Therefore, by the same lemma, the function W_1 that is the solution of problem (10) and (12), (13) for $i = 1$ will satisfy conditions (25), (26) for $p = n + 3$. Continuing the process, we find all the functions $W_i; i = 0, 1, ..., n$, contained in the right side of (7).

From (7), (12), (13) it follows that the constructed function W satisfies the following boundary conditions:

$$W|_{t=0} = 0, \ (0 \leq x \leq 1); W|_{x=0} = W|_{x=1} = 0, (0 \leq t \leq T). \tag{38}$$

This function doesn't satisfy, generally speaking, the second and third boundary conditions from (5) for $t = T$. Therefore it is necessary to construct the boundary layer type function V near the boundary $t = T$ so that the obtained sum $W + V$ could satisfy the boundary conditions

$$(W + V)|_{t=T} = 0, \ \frac{\partial}{\partial t}(W + V)|_{t=T} = 0. \tag{39}$$

The first iterative process is performed on the basis of decomposition (4) of the operator L_ε that is called the first decomposition of the operator L_ε. For performing the other iterative process by means of which we'll construct a boundary layer function near the boundary $t = T$, at first it is necessary to write new decomposition of the operator L_ε near this boundary. For that we make a change of variables: $T - t = \varepsilon\tau, \ x = x$. The new decomposition of the operator L_ε in the coordinates (τ, x) has the form

$$L_{\varepsilon,1} \equiv \varepsilon^{-1}\left[-\left(\frac{\partial^3}{\partial\tau^3} + \frac{\partial^2}{\partial\tau^2} + \frac{\partial}{\partial\tau}\right) + \varepsilon\left(-\frac{\partial^2}{\partial x^2} + a\right) - \varepsilon^2\frac{\partial^3}{\partial\tau\partial x^2}\right]. \tag{40}$$

We'll look for the boundary layer function V near the boundary $t = T$ in the form

$$V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + ... + \varepsilon^{n+1}V_{n+1}, \tag{41}$$

as an approximate solution of the equation

$$L_{\varepsilon,1}V = 0. \tag{42}$$

Subsituting the expression for V from (41) in (42), and associating the members at the same powers of ε, we have:

$$\frac{\partial^3 V_0}{\partial\tau^3} + \frac{\partial^2 V_0}{\partial\tau^2} + \frac{\partial V_0}{\partial\tau} = 0, \tag{43}$$

$$\frac{\partial^3 V_1}{\partial\tau^3} + \frac{\partial^2 V_1}{\partial\tau^2} + \frac{\partial V_1}{\partial\tau} = -\frac{\partial^2 V_0}{\partial x^2} + aV_0, \tag{44}$$

$$\frac{\partial^3 V_k}{\partial\tau^3} + \frac{\partial^2 V_k}{\partial\tau^2} + \frac{\partial V_k}{\partial\tau} = -\frac{\partial^2 V_{k-1}}{\partial x^2} + aV_{k-1} - \frac{\partial^3 V_{k-2}}{\partial\tau\partial x^2}; k = 2, 3, ..., n + 1. \tag{45}$$

In order to find the boundary conditions for equations (43),(44), (45) it is necessary of substitute expansions (7), (41) for W and V in (39) and associate the members at the same powers with respect to ε. Then we get:

$$V_i|_{\tau=0} = -W_i|_{t=T}; \ i = 0, 1, ..., n; \ V_{n+1}|_{\tau=0} = 0, \tag{46}$$

$$\frac{\partial V_0}{\partial \tau}\mid_{\tau=0}=0, \frac{\partial V_j}{\partial \tau}\mid_{\tau=0}=\frac{\partial W_{j-1}}{\partial t}\mid_{t=T}; \quad j=1,2,...,n+1. \tag{47}$$

Now construct the functions $V_0, V_1, ..., V_{n+1}$. From (43), (46) for $i=0$ and (47) and we have that the function V_0 is a boundary layer type solution of equation (43) satisfying the boundary conditions

$$V_0\mid_{\tau=0}=-W_0\mid_{t=T}, \ \frac{\partial V_0}{\partial \tau}\mid_{\tau=0}=0. \tag{48}$$

The characteristic equation corresponding to ordinary differential equation (43), in addition to the zero root has two non-zero roots $\lambda_{1,2}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$ with negative real parts. This fact provides the regularity of degeneration of problem (4)-(6) on the boundary $t=T$.

The boundary layer type solution of problem (43), (48) has the form

$$V_0(\tau,x)=\frac{W_0(T,x)}{\lambda_1-\lambda_2}(\lambda_2 e^{\lambda_1\tau}-\lambda_1 e^{\lambda_2\tau}). \tag{49}$$

By lemma 1, from (49) it follows that the function $V_0(\tau,x)$ and all its even derivatives with respect to x vanish for $x=0$ and $x=1$.

Knowing the function V_0, determine the function V_1, as a boundary layer type solution of equation (44) satisfying the boundary conditions:

$$V_1\mid_{\tau=0}=-W_1\mid_{t=T}, \ \frac{\partial V_1}{\partial \tau}\mid_{\tau=0}=\frac{\partial W_0}{\partial t}\mid_{t=T}. \tag{50}$$

From (49) it follows that the right side of equation (44) is of the form

$$f_1=m_1(x)e^{\lambda_1\tau}+m_2(x)e^{\lambda_2\tau}, \tag{51}$$

where $m_1(x)$, $m_2(x)$ are determined by the following equalities

$$\begin{aligned} m_1(x)&=\frac{\lambda_2}{\lambda_1-\lambda_2}\left[aW_0(T,x)-\frac{\partial^2 W_0(T,x)}{\partial x^2}\right], \\ m_2(x)&=\frac{\lambda_2}{\lambda_1-\lambda_2}\left[\frac{\partial^2 W_0(T,x)}{\partial x^2}-aW_0(T,x)\right]. \end{aligned} \tag{52}$$

Following (51), we get that equation (44) has a particular solution in the form

$$V_1^{(1)}=\left[c_{10}(x)+c_{11}(x)\tau\right]e^{\lambda_1\tau}+\left[c_{20}(x)+c_{21}(x)\tau\right]e^{\lambda_2\tau}, \tag{53}$$

and the functions $c_{10}(x)$, $c_{11}(x)$, $c_{20}(x)$, $c_{21}(x)$ are expressed by the functions $W_0(T,x)$ and $\frac{\partial^2 W_0(T,x)}{\partial x^2}$. They may be determined by the method of undetermined coefficients.

Represent V_1 in the form $V_1=V_1^{(1)}+V_1^{(2)}$. Then $V_1^{(2)}$ will be a boundary layer type solution of the following problem:

$$\frac{\partial^3 V_1^{(2)}}{\partial \tau^3}+\frac{\partial^2 V_1^{(2)}}{\partial \tau^2}+\frac{\partial V_1^{(2)}}{\partial \tau}=0;$$

$$V_1^{(2)}\mid_{\tau=0}=\varphi_1(x),\ \frac{\partial V_1^{(2)}}{\partial\tau}\mid_{\tau=0}=\psi_1(x), \tag{54}$$

where

$$\varphi_1(x)=-W_1(T,x)-c_{10}(x),$$

$$\psi_1(x)=\frac{\partial W_0(T,x)}{\partial t}-c_{11}(x)-\lambda_1 c_{10}(x)-c_{21}(x)-\lambda_2 c_{20}(x). \tag{55}$$

Obviously, the boundary layer type solution of problem (54) has the form

$$V_1^{(2)}(\tau,x)=\frac{1}{\lambda_1-\lambda_2}\left\{[\psi_1(x)-\lambda_2\varphi_1(x)]\,e^{\lambda_1\tau}+[\lambda_1\varphi_1(x)-\psi_1(x)]\,e^{\lambda_2\tau}\right\}. \tag{56}$$

(53) and (56) yield that the function V_1 being the sum of $V_1^{(1)}$ and $V_1^{(2)}$ is determined by the formula

$$V_1(\tau,x)=[a_{10}(x)+a_{11}(x)\tau]\,e^{\lambda_1\tau}+[b_{10}(x)+b_{11}(x)\tau]\,e^{\lambda_2\tau}, \tag{57}$$

and $a_{10}(x),a_{11}(x),a_{20}(x),a_{21}(x)$ denote the functions

$$a_{10}(x)=\frac{1}{\lambda_1-\lambda_2}\left[\psi_1(x)-\lambda_2\varphi_1(x)\right]+c_{10}(x),a_{11}(x)=c_{11}(x),$$

$$b_{10}(x)=\frac{1}{\lambda_1-\lambda_2}\left[\lambda_1\varphi_1(x)-\psi_1(x)\right]+c_{20}(x),b_{11}(x)=c_{21}(x). \tag{58}$$

According to (55), (57), (58) we have that the function $V_1(\tau,x)$ and all its even derivatives with respect to x vanish for $x=0$ and $x=1$.

By constructing the functions $V_k;k=2,3,...,n+1$ we use the following statement.

Lemma 2. *The functions V_k being the boundary layer type solutions of equations (45) form $k=2,3,...,n+1$ and satisfying the appropriate conditions from (46), (47) are determined by the formula*

$$V_k(\tau,x)=\left[\sum_{i=0}^{k}a_{ki}(x)\right]e^{\lambda_1\tau}+\left[\sum_{i=0}^{k}b_{ki}(x)\right]e^{\lambda_2\tau};\ k=2,3,...,n+1, \tag{59}$$

and the coefficients $a_{ki}(x)$, $b_{ki}(x)$ are expressed uniformly by the functions $W_0(T,x)$, $W_1(T,x),...,W_k(T,x)$ and their derivatives with respect to t of first order, while with respect to x only of even order.

Proof. The lemma is proved by the mathematical induction method. Above it is shown that the functions V_0,V_1 are determined by formula (59). Now assume that the lemma statement is true for $V_0,V_1,V_2,...,V_{k-1}$ and prove that it is true for V_k as well. Note that the right side of equation (45) for V_k contains the functions V_{k-1},V_{k-2} that by supposition are determined by formula (59).

Repeating the reasonings carried out while determining the function V_1, we can affirm that V_k is also determined by formula (59).

Lemma 2 is proved.

Multiply all the functions V_j by the smoothing function and denote the obtained new functions again by $V_j;j=0,1,...,n+1$. As all the functions $V_j(\tau,x)$; $j=0,1,...,n+1$ vanish for $x=0$ and $x=1$, it follows from (38) and (41) that the sum $\widetilde{U} = W + V$ constructed by us, in addition to (39), satisfies also the boundary conditions

$$(W + V)\,|_{t=0} = 0\ ,\ (W + V)\,|_{x=0} = (W + V)\,|_{x=1} = 0\ . \tag{60}$$

Denote the difference of the exact solution of problem (4)-(6) and $\widetilde{U}$ by

$$U - \widetilde{U} = \varepsilon^{n+1} z, \tag{61}$$

and call $\varepsilon^{n+1} z$ a remainder term.

It holds the following statement.

Lemma 3. *For the function z it is valid the estimation*

$$\varepsilon^2 \left\| \frac{\partial z}{\partial t}\,|_{t=0} \right\|^2_{L_2(0,1)} + \varepsilon \left\| \frac{\partial z}{\partial t} \right\|^2_{L_2(D)} + \left\| \frac{\partial z}{\partial x} \right\|^2_{L_2(D)} + c_1 \left\| z \right\|^2_{L_2(D)} \le c_2, \tag{62}$$

where $c_1 > 0, c_2 > 0$ are the constants independent of ε.

Proof. Acting on both sides of equality (61) by the appropriate decompositions of operator L_ε, and taking into account equations (4), (9)-(11) and (43)-(45) we have

$$L_\varepsilon z = F(\varepsilon, t, x), \tag{63}$$

where $F(\varepsilon, t, x) = F_1(\varepsilon, t, x) + F_2(\varepsilon, \tau, x)$ is a uniformly bounded function. The function $F_1(\varepsilon, t, x)$ has the form

$$F_1 = \frac{\partial^2 W_n}{\partial t^2} - \frac{\partial}{\partial t}\left(\Delta W_{n-2}\right) - \varepsilon \frac{\partial}{\partial t}\left(\Delta W_{n-1}\right),$$

and $F_2(\varepsilon, \tau, x)$ near the boundary $t = T$ is of the form

$$F_2 = \frac{\partial^2 V_n}{\partial x^2} - aV_n + \varepsilon \left(\frac{\partial V_{n+1}}{\partial x^2} - aV_{n+1} \right) + \frac{\partial^3 V_{n-1}}{\partial \tau \partial x^2} + \varepsilon \frac{\partial^3 V_n}{\partial \tau \partial x^2} + \varepsilon^2 \frac{\partial^3 V_{n+1}}{\partial \tau \partial x^2}.$$

Obviously, z will satisfy the boundary conditions

$$z\,|_{t=0} = z\,|_{t=T} = 0\ ,\ \frac{\partial z}{\partial t}\,|_{t=T} = 0\ ,\ z\,|_{x=0} = z\,|_{x=1} = 0\ . \tag{64}$$

Multiplying the both sides of equation (63) scalarly by z and integrating the left side of the obtained equality with regard to boundary conditions (64), after some transformations we get estimation (62).

Lemma 3 is proved.

The obtained results may be generalized in the form of the following statement.

Theorem. *Suppose that the function $f(t,x) \in C^{n+3,2n+10}(D)$ satisfies condition (24) for $p = n+4$. Then for the solution of problem (4)-(6) it is valid the asymptotic representation*

$$u = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{j=0}^{n+1} \varepsilon^j V_j + \varepsilon^{n+1} z,$$

where the functions W_i are determined by the first iterative process, V_j are the boundary layer type functions near the boundary $t = T$, and are determined by the second iterative process, $\varepsilon^{n+1} z$ is a remainder term, and estimation (62) is valid for the function z.

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