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ON THE ASYMPTOTICS OF THE MEAN VALUE OF THE MOMENT OF THE FIRST LEVEL-CROSSING BY THE FIRST ORDER AUTOREGRESSION PROCESS $(AR(1))$

Abstract

In the paper we fine the asymptotics for the mean value of the first level-crossing time by the trajectory of random walk described by the first order autoregression process $(AR(1))$.

In the paper we study a linear boundary value problem for a first order autoregression process $(AR(1))$ by mean of which one can construct mathematical models of some problems from applied fields of science and engineering.

Let on some probability space (Ω, F, P) be given a sequence of independent identically distributed random variables $\xi_n,\ n \geq 1$.

It is well known that ([1], [2] the first order autoregression process $(AR(1))$ is the solution of the equation of the following form

$$X_{n+1} = \beta X_n + \xi_n \quad n \geq 0, \quad X_0 = x, \tag{1}$$

where x and β are non-random variables, i.e. x and β are constants. Here it is assumed that $x \geq 0$ and $|\beta| < 1$.

Let's consider a random process of the form

$$T_n = \sum_{k=1}^{n} X_{k-1} X_k, \quad n \geq 1 \tag{2}$$

and a family of stopping times

$$\tau_a = \inf \{n \geq 1 : T_n \geq a\} \tag{3}$$

the first reaching time of the level $a \geq 0$ by the process T_n of the form (2).

Note that the autoregression scheme (1), the process of the form (2) and the family of stopping times of the form (3) arise when studying different probability models from applied fields of theory of random processes ([1]-[6]).

Let's give an example of a probability model described by an autoregression scheme in studying the following problem of hydrology ([4]-[5]). A number of problems of hydrology as is known are connected with some water basins as for example the Caspian Sea. In these problems the water level deviation in the considered basin is studied as a random process of its own mean value (mathematical expectation).

The water level deviations in the basin are stipulated by vibrations in discharge and water surface evaporation.

Take for a unit of measurement of time a year and by J_n denote the level in the basin in the n-th year.

Write the balance equation

$$J_{n+1} - J_n = \prod\nolimits_{n+1} -\gamma R(J_n),$$

where $\prod_{n+1}$ is the quantity of discharge in the $(n+1)$-th year, $R(J_n)$ is the quantity of the surface of the water basin on the level J_n, and γ is evaporation rate.

Denote by J the mean value of water level and suppose that it is fulfilled the equality

$$R(J_n) = R(J) + c(J_n - J),$$

where c is some number.

Then from the balance equation we get the $AR(1)$ sequence of the form

$$X_{n+1} = \beta X_n + \xi_{n+1},$$

where

$$X_n = J_n - J, \quad \beta = 1 - c\gamma, \; \xi_n = \prod\nolimits_n -\gamma R(J).$$

This equality permits to predict the water level deviation for the next year by the results of observations for the present and previous years. Note that the mean value J is found from the result of long-term observations.

In the paper [3] it is proved the integral limit theorem for a family of the first crossing time τ_a by means of which one can get approximate values of the probability $P(\tau_a \leq n)$.

As it was noted in the papers [4] and [6], the finding of the approximate mean value $E\tau_a$ is of great theoretical and practical interest in theory of boundary value problems for random processes.

We consider the case $0 < \beta < 1$, $E\xi_n = 0$ and $D\xi_n = 1$. For this case in the paper [3] the following relations were proved

$$E\ X_n X_{n-1} \to \lambda = \frac{\beta}{1-\beta^2} \quad \text{as} \quad n \to \infty$$

as

$$\frac{\tau_a}{a} \overset{a.s.}{\to} \frac{1}{\lambda} \quad \text{as} \quad a \to \infty. \tag{4}$$

By means of these relations, in the present paper we prove the following theorem on the asymptotics of $E\tau_a$ as $a \to \infty$ by means of which one can calculate approximate values of $E\tau_a$.

Theorem. *Let the following condition be fulfilled*

$$\sum_{n=1}^{\infty} P\left(T_n - \lambda n \leq 0\right) < \infty. \tag{5}$$

Then

$$\frac{E\tau_a}{a} \to \frac{1}{\lambda} \quad \text{as} \;\; a \to \infty.$$

Note that in practice for calculating $E\tau_a$ we can use the approximate equality $E\tau_a \approx \dfrac{a}{\lambda}$ for $a \geq a_0$, where $a_0 > 0$ is chosen from the results of the carried out observations. Note that condition (5) is trivially fulfilled for a random walk described by the sum of independent identically distributed random variables with final mathematical expectation ([6]).

For proving the theorem we need the following results formulated in the form of lemmas from what the affirmation of the theorem follows.

Lemma 1. *Let the family of random variables Y_a, $a \geq 0$ be uniformly integrable and Y_a converge as $a \to \infty$ in distribution to the random variable Y. Then it holds*

$$E\left|Y\right| < \infty \quad \text{and} \quad EY_a \to EY \quad \text{as} \quad a \to \infty.$$

This lemma is one of the variants of the known theorem on limit passage under the sign of mathematical expectation whose proof may be found for example in [6] (see also [5]).

Lemma 2. *Let condition (5) be fulfilled. Then the family $\frac{\tau_a}{a}$, $a > 0$ is uniformly integrable, i.e. it holds*

$$\sup_a E\left[\frac{\tau_a}{a} J\left(\frac{\tau_a}{a} > c\right)\right] \to 0 \quad \text{as} \quad c \to \infty,$$

where $J(B)$ is the indicator of the event B.

Proof. For some $\varepsilon \in (0, \lambda)$ we assume

$$N_a = \left[\frac{a}{\lambda - \varepsilon}\right] + 1.$$

Taking into account

$$\frac{a}{\lambda - \varepsilon} < \left[\frac{a}{\lambda - \varepsilon}\right] + 1 = N_a,$$

we get $a < n\left(\lambda - \varepsilon\right)$ for $n > N_a$. Then by the definition of the first exit time τ_a we have for $n > N_a$

$$P\left(\tau_a > n\right) < P\left(T_n \leq a\right) \leq P\left(T_n - n\lambda \leq -n\varepsilon\right) \leq$$
$$\leq P\left(T_n - n\lambda \leq 0\right)$$

or

$$\sum_{n=N_a+1}^{\infty} P\left(\tau_a > n\right) \leq \sum_{n=N_a+1}^{\infty} P\left(T_n - n\lambda \leq 0\right).$$

Therefore from condition (5) of the theorem we get

$$\sum_{n=N_a+1}^{\infty} P\left(\tau_a > n\right) = 0(1) \quad \text{as} \quad a \to \infty.$$

Hence it follows that

$$E\left[\tau_a J\left(\tau_a > N_a\right)\right] \to 0 \quad \text{as} \quad a \to \infty.$$

The affirmation of lemma 2 follows from the last relation.

Now the affirmation of the theorem follows from the convergence (4) and lemmas 1,2.

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Received November 07, 2013; Revised January 16, 2014.