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REDUCTION OF THE SPECTRAL PROBLEM FOR AN ELLIPTIC TYPE EQUATION OF FIRST ORDER ON A CURVILINEAR STRIP TO FREDHOLM TYPE SECOND ORDER INTEGRAL EQUATION

Abstract

The paper deals with homogeneous boundary value problem with a parameter for the Cauchy–Riemann equation with non local conditions on a plane strip. The solution is sought in the form dictated by Green's second formula. The indeterminacy is eliminated by means of the obtained necessary conditions.

Introduction. As is known, in the boundary value problem for an ordinary linear differential equation the number of boundary conditions coincides with the order of the equation under consideration [1]. The boundary value problem for partial equations on the whole is considered for an elliptic type equation. In the course of mathematical physics equations the Laplace equation is a physical model of an elliptic equation. The Dirichlet, Neumann problem or the third boundary value problem [2] is considered for the Laplace equation. The boundary conditions of these problems have local form and the boundary is the support for the boundary conditions. Thus, unlike boundary value problems for a linear ordinary differential equation, in the boundary value problem for partial equations the number of boundary conditions cofncides with the half of the highest order derivative of the equation under consideration.

We consider an elliptic type equation of first order. Therefore there arises a question on how to give the boundary condition for the problem to be accessible. Furthermore, we have to observe that the boundary was the support of the reduced condition. In the paper [2] while investigating the process in the nuclear reactor the mathematical model is obtained in the form of a first order linear integro- differential equation and boundary conditions are given on a part of the boundary. This in not a boundary value problem but the Cauchy generalized problem. The while boundary is not a support for the boundary condition. Proceeding from this fact we give a non –local boundary condition.

Giving the boundary condition in non-local form we have opportunity to investigate the boundary value problem for partial equation of any (both even and add) order, and for each condition the boundary is a support. Note that in our boundary conditions the Carleman conditions are observed [3].

So , consider the following boundary value problem:

$$\frac{\partial u(x)}{\partial x_2}+i\frac{\partial u(x)}{\partial x_1}=\lambda u(x)\,,\ x\in D\subset R^2, \tag{1}$$

$$u(x_1,\gamma(x_1))=\alpha(x_1)\,u(x_1,0)\,,\ x_1\in R, \tag{2}$$

where

$$D=\{x=(x_1,x_2):x_2\in(0,\gamma(x_1))\,,\ x_1\in R\}\,,$$

$i=\sqrt{-1},\ \lambda\in\mathbf{C}$ is a parameter $\alpha(x_1)$ is a complex-valued continuous function, $u(x)$ is a desired function, $\gamma(x_1)>0,\ x_1\in R$. It is known that [2]

$$U(x-\xi)=\frac{1}{2\pi}\frac{1}{x_2-\xi_2+i(x_1-\xi_1)}, \tag{3}$$

is the fundamental solution of the Cauchy- Riemann equation (1). Proceeding from (3) and from equation (1), we obtain the following main relation [4]:

$$\int_D \frac{\partial u}{\partial x_2} U dx + i \int_D \frac{\partial u}{\partial x_1} U dx = \lambda \int_D uU dx,$$

or

$$\int_{\overline{D}\backslash D} uU \left(\cos\left(\nu, x_2\right) + i \cos\left(\nu, x_1\right)\right) dx - \int_D u \left(\frac{\partial U}{\partial x_2} + i \frac{\partial U}{\partial x_1}\right) dx = \lambda \int_D uU dx,$$

or

$$\int_{\overline{D}\backslash D} uU \left(\cos\left(\nu, x_2\right) + i \cos\left(\nu, x_1\right)\right) dx - \lambda \int_D uU dx = \begin{cases} u\left(\xi\right), & \xi \in D, \\ \frac{1}{2} u\left(\xi\right), & \xi \in \overline{D}\backslash D \end{cases}, \tag{4}$$

where ν is an external normal to the boundary of domain D.

Thus, we established

Theorem 1. *Let $D \subset R^2$ be a strip in the upper half-plane with a curvilinear upper boundary of Lyapunov type. Then each solution of equation (1) determined in D satisfies main relations (4).*

Consider the necessary conditions [4], [5]:

$$\frac{1}{2} u\left(\xi_1, 0\right) = -\int_R u\left(x_1, 0\right) U\left(x_1 - \xi_1, 0\right) dx_1 + \int_R u\left(x_1, \gamma\left(x_1\right)\right) \times$$

$$\times U\left(x_1 - \xi_1, \gamma\left(x_1\right) - 0\right) \left[1 - i\gamma'\left(x_1\right)\right] dx_1 - \lambda \int_D u\left(x\right) U\left(x - \xi\right) \Big|_{\xi_2 = 0} dx,$$

$$\frac{1}{2} u\left(\xi_1, \gamma\left(\xi_1\right)\right) = -\int_R u\left(x_1, 0\right) U\left(x_1 - \xi_1, -\gamma\left(\xi_1\right)\right) dx_1 +$$

$$+ \int_R u\left(x_1, \gamma\left(x_1\right)\right) U\left(x_1 - \xi_{1,}\left(x_1\right) - \gamma\left(\xi_1\right)\right) \times$$

$$\times \left[1 - i\gamma'\left(x_1\right)\right] dx_1 - \lambda \int_D u\left(x\right) U\left(x - \xi\right) \Big|_{\xi_2 = \gamma\left(\xi_1\right)} dx,$$

Substituting the fundamental solutions from (3), we have:

$$u\left(\xi_1, 0\right) = -\frac{1}{\pi i} \int_R \frac{u\left(x_1, 0\right)}{x_1 - \xi_1} dx_1 + \frac{1}{\pi} \int_R \frac{u\left(x_1, \gamma\left(x_1\right)\right)}{\gamma\left(x_1\right) + i\left(x_1 - \xi_1\right)} \times$$

$$\times \left[1 - i\gamma'\left(x_1\right)\right] dx_1 - \frac{\lambda}{\pi} \int_D \frac{u\left(x\right)}{x_2 + i\left(x_1 - \xi_1\right)} dx, \tag{5}$$

$$u(\xi_1,\gamma(\xi_1)) = -\frac{1}{\pi}\int\limits_R \frac{u(x_1,0)}{-\gamma(\xi_1)+i(x_1-\xi_1)}dx_1 - \frac{i}{\pi}\int\limits_R \frac{u(x_1,\gamma(x_1))}{x_1-\xi_1}dx_1 +$$
$$+\frac{i}{\pi}\int_R \frac{\gamma(x_1)-\gamma(\xi_1)-\gamma'(x_1)(x_1-\xi_1)}{[\gamma(x_1)-\gamma(\xi_1)+i(x_1-\xi_1)](x_1-\xi_1)}\times$$
$$\times u(x_1,\gamma(x_1))\,dx_1 - \frac{\lambda}{\pi}\int_D \frac{u(x)}{x_2-\gamma(\xi_1)+i(x_1-\xi_1)}dx. \tag{6}$$

Proceeding from boundary conditions (2), by means of necessary conditions (5) and (6), we construct the following linear combination [6],[7]:

$$u(\xi_1,\gamma(\xi_1)) + \alpha(\xi_1)u(\xi_1,0) = -\frac{i}{\pi}\int_R \frac{u(x_1,\gamma(x_1))-\alpha(x_1)u(x_1,0)}{x_1-\xi_1}dx_1 +$$
$$+\frac{i}{\pi}\int_R \frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}u(x_1,0)\,dx_1 + \frac{1}{\pi}\int_R \frac{u(x_1,0)}{\gamma(\xi_1)-i(x_1-\xi_1)}dx_1 +$$
$$+\frac{i}{\pi}\int_R \frac{\gamma(x_1)-\gamma(\xi_1)-\gamma'(x_1)(x_1-\xi_1)}{[\gamma(x_1)-\gamma(\xi_1)+i(x_1-\xi_1)](x_1-\xi_1)}u(x_1,\gamma(x_1))\,dx_1 -$$
$$-\frac{\lambda}{\pi}\int_D \frac{u(x)}{x_2-\gamma(\xi_1)+i(x_1-\xi_1)}dx +$$
$$+\frac{\alpha(\xi_1)}{\pi}\int_R \frac{u(x_1,\gamma(x_1))}{\gamma(x_1)+i(x_1-\xi_1)}\left[1-i\gamma'(x_1)\right]dx_1 - \frac{\lambda\alpha(\xi_1)}{\pi}\int_D \frac{u(x)}{x_2+i(x_1-\xi_1)}dx, \tag{7}$$

not containing a singular integral.

This proves the following theorem.

Theorem 2. *Under conditions of theorem 1, if $\alpha(x_1)$ belongs to the Holder class, then regular relations (7) are valid for the solution of problem (1)-(2).*

Now, combining the obtained regular relation (7) with the given boundary conditions (2), assuming

$$\alpha(x_1) \neq 0, \quad x_1 \in R, \tag{8}$$

we have:

$$u(\xi_1,\gamma(\xi_1)) = \frac{i}{2\pi}\int\limits_R \frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}u(x_1,0)\,dx_1 + \frac{1}{2\pi}\int\limits_R \frac{u(x_1,0)}{\gamma(\xi_1)-i(x_1-\xi_1)}dx_1 +$$
$$+\frac{i}{2\pi}\int\limits_R \frac{\gamma(x_1)-\gamma(\xi_1)-\gamma'(x_1)(x_1-\xi_1)}{[\gamma(x_1)-\gamma(\xi_1)+i(x_1-\xi_1)](x_1-\xi_1)}u(x_1,\gamma(x_1))\,dx_1 -$$
$$-\frac{\lambda}{2\pi}\int\limits_D \frac{u(x)}{x_2-\gamma(\xi_1)+i(x_1-\xi_1)}dx +$$
$$+\frac{\alpha(\xi_1)}{2\pi}\int\limits_R \frac{u(x_1,\gamma(x_1))}{\gamma(x_1)+i(x_1-\xi_1)}\left[1-i\gamma'(x_1)\right]dx_1 - \frac{\lambda\alpha(\xi_1)}{2\pi}\int\limits_D \frac{u(x)}{x_2+i(x_1-\xi_1)}dx,$$
$$u(\xi_1,0) = \frac{i}{2\pi\alpha(\xi_1)}\int\limits_R \frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}u(x_1,0)\,dx_1 +$$

$$+\frac{1}{2\pi\alpha(\xi_1)}\int_R \frac{u(x_1,0)}{\gamma(\xi_1)-i(x_1-\xi_1)}dx_1+$$

$$+\frac{i}{2\pi\alpha(\xi_1)}\int_R \frac{\gamma(x_1)-\gamma(\xi_1)-\gamma'(x_1)(x_1-\xi_1)}{[\gamma(x_1)-\gamma(\xi_1)+i(x_1-\xi_1)](x_1-\xi_1)}u(x_1,\gamma(x_1))\,dx_1-$$

$$-\frac{\lambda}{2\pi\alpha(\xi_1)}\int_D \frac{u(x)}{x_2-\gamma(\xi_1)+i(x_1-\xi_1)}dx+$$

$$+\frac{1}{2\pi}\int_R \frac{u(x_1,\gamma(x_1))}{\gamma(x_1)+i(x_1-\xi_1)}\left[1-i\gamma'(x_1)\right]dx_1-\frac{\lambda}{2\pi}\int_D \frac{u(x)}{x_2+i(x_1-\xi_1)}dx,$$

We represent the obtained system of Fredholm integral equations of second kind with regular kernels in the form [4]-[7]:

$$u(\xi_1,0)=\int_R [K_{11}(\xi_1,x_1)\,u(x_1,0)+K_{12}(\xi_1,x_1)\,u(x_1,\gamma(x_1))]\,dx_1-$$

$$-\lambda\int_D K_1(\xi_1,x)\,u(x)\,dx,$$

$$u(\xi_1,\gamma(\xi_1))=\int_R [K_{21}(\xi_1,x_1)\,u(x_1,0)+K_{22}(\xi_1,x_1)\,u(x_1,\gamma(x_1))]\,dx_1-$$

$$-\lambda\int_D K_2(\xi_1,x)\,u(x)\,dx, \tag{9}$$

where

$$K_{11}(\xi_1,x_1)=\frac{\lambda}{2\pi\alpha(\xi_1)}\left[i\frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}+\frac{1}{\gamma(\xi_1)-i(x_1-\xi_1)}\right], \tag{9$_1$}$$

$$K_{12}(\xi_1,x_1)=\frac{1}{2\pi\alpha(\xi_1)}\left[i\frac{\gamma(x_1)-\gamma(\xi_1)-\gamma'(x_1)(x_1-\xi_1)}{\gamma(x_1)-\gamma(\xi_1)+i(x_1-\xi_1)}+\right.$$

$$\left.+\alpha(\xi_1)\frac{1-i\gamma'(x)}{\gamma(x_1)+i(x_1-\xi_1)}\right], \tag{9$_2$}$$

$$K_1(\xi_1,x)=\frac{1}{2\pi\alpha(\xi_1)}\left[\frac{1}{x_2-\gamma(\xi_1)+i(x_1-\xi_1)}+\frac{\alpha(\xi_1)}{x_2+i(x_1-\xi_1)}\right], \tag{9$_3$}$$

$$K_{21}(\xi_1,x_1)=\frac{1}{2\pi}\left[i\frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}+\frac{1}{\gamma(\xi_1)-i(x_1-\xi_1)}\right], \tag{9$_4$}$$

$$K_{22}(\xi_1,x_1)=$$

$$=\frac{1}{2\pi}\left[i\frac{\gamma(x_1)-\gamma(\xi_1)-\gamma'(x_1)(x_1-\xi_1)}{\gamma(x_1)-\gamma(\xi_1)+i(x_1-\xi_1)}+\alpha(\xi_1)\frac{1-i\gamma'(x_1)}{\gamma(x_1)+i(x_1-\xi_1)}\right], \tag{9$_5$}$$

$$K_2(\xi_1,x)=\frac{1}{2\pi}\left[\frac{1}{x_2-\gamma(\xi_1)+i(x_1-\xi_1)}+\alpha(\xi_1)\frac{x_1}{x_2+i(x_1-\xi_1)}\right]. \tag{9$_6$}$$

Let system (9) be solvable, and its resolvent be of the from:

$$R(\xi_1,x_1)=\begin{pmatrix} R_{11}(\xi_1,x_1) & R_{12}(\xi_1,x_1)\\ R_{21}(\xi_1,x_1) & R_{22}(\xi_1,x_1)\end{pmatrix}. \qquad (10)$$

Then the solution of system (9) is representable in the form:

$$u(\xi_1,0)=-\lambda\int_D\Big\{K_1(\xi_1,\eta)+\int_R[R_{11}(\xi_1,x_1)K_2(\xi_1,\eta)+$$

$$+R_{12}(\xi_1,x_1)K_2(\xi_1,\eta)]\,dx_1\Big\}u(\eta)\,d\eta, \qquad (11_1)$$

$$u(\xi_1,\gamma(\xi_1))=-\lambda\int_D\Big\{K_2(\xi_1,\eta)+\int_R[R_{21}(\xi_1,x_1)K_1(\xi_1,\eta)+$$

$$+R_{22}(\xi_1,x_1)K_2(\xi_1,\eta)]\,dx_1\Big\}u(\eta)\,d\eta. \qquad (11_2)$$

Thus, from (4) for the solution of problem (1) (2) we get the following representation:

$$u(\xi)=-\int_R u(y_1,0)\,U(y_1-\xi_1,-\xi_2)\,dy_1+$$

$$+\int_R u(y_1,\gamma(y_1))\,U(y_1-\xi_1,\gamma(y_1)-\xi_2)[1-i\gamma'(y_1)]dy_1-$$

$$-\lambda\int_D u(y)U(y-\xi)dy,\quad \xi\in D. \qquad (*)$$

Substituting (11_1) and (11_2), in (*) we get:

$$u(\xi)=\frac{\lambda}{2\pi}\int\limits_R\frac{dy_1}{-\xi_2+i(y_1-\xi_1)}\int\limits_D\Big\{K_1(y_1,\eta)+\int\limits_R[R_{11}(y_1,x_1)K_1(\xi_1,\eta)+$$

$$+R_{12}(y_1,x_1)K_2(x_1,\eta)]\,dx_1\Big\}u(\eta)\,d\eta-$$

$$-\frac{\lambda}{2\pi}\int\limits_R\frac{[1-i\gamma'(y_1)]\,dy_1}{\gamma(y_1)-\xi_2+i(y_1-\xi_1)}\int\limits_D\Big\{K_2(y_1,\eta)+\int\limits_R[R_{21}(y_1,x_1)K_1(x_1,\eta)+$$

$$+R_{22}(y_1,x_1)K_2(x_1,\eta)]\,dx_1\Big\}u(\eta)\,d\eta-\frac{\lambda}{2\pi}\int\limits_D\frac{u(\eta)\,d\eta}{\eta_2-\xi_2+i(\eta_1-\xi_1)},\ \xi\in D.$$

Thus, we obtained the following problem. Find the eigen values and eigen functions of Fredholm homogeneous integral equation of second kind [8],[9]:

$$u(\xi)=\lambda\int_D K(\xi,\eta)\,u(\eta)\,d\eta, \qquad (12)$$

where

$$K(\xi,\eta)=\frac{-1}{2\pi}\int_R\frac{dy_1}{\xi_2-i(y_1-\xi_1)}\times$$

$$\times\Big\{K_1(y_1,\eta)+\int\limits_R[R_{11}(y_1,x_1)K_1(x_1,\eta)+R_{12}(y_1,x_1)K_2(x_1,\eta)]\,dx_1\Big\}-$$

$$-\frac{-1}{2\pi}\int\limits_{R}\frac{\left[1-i\gamma'\left(y_1\right)\right]dy_1}{\gamma\left(y_1\right)-\xi_2+i\left(y_1-\xi_1\right)}\times$$

$$\times\left\{K_2\left(y_1,\eta\right)+\int\limits_{R}\left[R_{21}\left(y_1,x_1\right)K_1\left(x_1,\eta\right)+R_{12}\left(y_1,x_1\right)K_2\left(x_1,\eta\right)\right]dx_1\right\}-$$

$$-\frac{1}{2\pi}\frac{1}{\eta_2-\xi_2+i\left(\eta_1-\xi_1\right)}, \tag{13}$$

is a regular kernel of equation (12) .We proved the following statement:

Theorem 3. *If $\alpha\left(x_1\right)\neq 0$ and the conditions of theorem 2 are fulfilled, boundary value problem (1)- (2) is Fredholm.*

The Fredholm property of problems for the Laplace equation [6],[7] and integro-differential loaded equation was investigated by the similar scheme.

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