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THEOREMS ON INTERPOLATION OF BESOV TYPE SPACES

Abstract

In the paper, two theorems of Riesz-Torin type are proved for the spaces $\bigcap_{i\in Q} \Lambda^{<m^i,N^i>}_{p_i,\theta_i}(G,\varphi^i)$, *i.e. some properties of functions belonging to intersection of spaces*

$$\bigcap_{i\in Q} \Lambda^{<m^i,N^i>}_{p_i,\theta_i}(G,\varphi^i)$$

are studied from the point of view of imbedding theory.

Introduction. In the paper, two theorems of Riesz-Torin type are proved for the spaces

$$\bigcap_{i\in Q} \Lambda^{<m^i,N^i>}_{p_i,\theta_i}(G,\varphi^i),$$

i.e. some properties of functions belonging to intersection of spaces

$$\bigcap_{i\in Q} \Lambda^{<m^i,N^i>}_{p_i,\theta_i}(G,\varphi^i)(\lambda = 1,2,...,M)$$

are studied from the point of view of imbedding theory.

Note that imbedding theorems for the spaces $\bigcap_{i\in Q} \Lambda^{<m^i,N^i>}_{p_i,\theta_i}(G,\varphi^i)$ (in the case $\lambda = 1$) were studied in the paper [8], while the spaces $\bigcap_{i\in Q} \Lambda^{<m,N>}_{p,\theta}(G)$ were determined and studied in the papers of A.J. Jabrailov and M.Aliyev. The proof of the theorem in the considered paper is carried out by the integral representations method obtained in the paper [6]. The similar theorems for another functional spaces were proved in the papers [3], [6], [9], [10], [11].

Let Q be a set of all possible vectors $i = (i_1,...,i_s)$, with the coordinates $i_k \in \{0,1,2,...,n_k\}$ $(k = 1,2,...,s)$, and notice that the number of all possible vectors $i = (i_1,...,i_s) \in Q$ equals $|Q| = \prod_{k=1}^{s}(1+n_k)$,therefore $(n+1) \le |Q| \le 2^n$, $(n = n_1,...,n_s)$ and let the vectors $m^i = (m_1^{i_1},...,m_s^{i_s})$ and $N^i = (N_1^{i_1},...,N_s^{i_s})$ with integer non-negative coordinate-vectors $m_k^{i_k} = (m_{k,1}^{i_k},...,m_{k,n_k}^{i_k})$. $N_k^{i_k} = (N_{k,1}^{i_k},...,N_{k,n_k}^{i_k})$ $(k = 1,2,...,s)$, i.e. $m_k^{i_k} \ge 0$, $N_k^{i_k} \ge 0$ $(k = 1,2,...,s)$; $|m^i| = |m_1^{i_1}| + \cdots + |m_s^{i_s}|$, $D^{m_i} f(x) = D_1^{m_1^{i_1}} \cdots D_s^{m_s^{i_s}} f(x_1,...,x_s)$, and the mixed difference

$$\Delta^{N^i}(t;G)g(x) = \Delta^{N^i}(t)g(x) = \Delta_1^{N^{i_1}}(t_1)\cdots\Delta_s^{N^{i_s}}(t_s)g(x)$$

be constructed on the vertices of a polyhedron entirely lying in the domain $G \subset E_n$, otherwise we assume $\Delta^{N^i}(t;G)g(x) = 0$.

Introduce a semi-norm in the space $\bigcap\limits_{i\in Q}\Lambda^{<m^i,N^i>}_{p_i,\theta_i}(G,\varphi^i)$ (in the special case a norm) by the equality

$$||f||_{Q\Lambda^{<m^i,N^i>}_{p_i,\theta_i}}=\sum_{i\in Q}||f||_{\Lambda^{<m^i,N^i>}_{p_i,\theta_i}(G,\phi^i)}<\infty$$

where the sum is taken over all possible vectors $i=(i_1,...,i_s)\in Q$. Notice that for any $i=(i_1,...,i_s)\in Q$

$$||f||_{\Lambda^{<m^i,N^i>}_{p_i,\theta_i}(G,\phi^i)}=\left\{\int_{E_{|\varepsilon_{N^i}|}}\left\|\frac{\Delta^{N^i}\left(\frac{z}{N^i};G\right)D^{m^i}f(\cdot)}{\prod_{k\in\varepsilon_{N^i}}\prod_{j\in\varepsilon_{N_k^{i_k}}}\phi^{i_k}_{k,j}(z_{k,j})}\right\|^{\theta_i}_{L_{p_i(G)}}\frac{dz}{z}\right\}^{\frac{1}{\theta_i}}<\infty$$

if $1\le\theta_i<\infty\,(i\in Q)$, and

$$\frac{dz}{z}=\prod_{k\in\varepsilon_{N^i}}\prod_{j\in\varepsilon_{N_k^{i_k}}}\frac{dz_{k,j}}{z_{k,j}},$$

in the case $\theta_i=\infty\,(i\in Q)$ it is assumed that

$$||f||_{\Lambda^{<m^i,N^i>}_{p_i,\theta_i}(G,\phi^i)}=\underset{z\in E_{|\varepsilon_{N^i}|}}{vrai\ \sup}\left\|\frac{\Delta^{N^i}\left(\frac{z}{N^i};G\right)D^{m^i}f(\cdot)}{\prod_{k\in\varepsilon_{N^i}}\prod_{j\in\varepsilon_{N_k^{i_k}}}\phi^{i_k}_{k,j}(z_{k,j})}\right\|_{L_{p_i(G)}}.$$

Notice that (for any $i=(i_1,...,i_s)\in Q$) the vector function $\phi^i(t)=(\phi_1^{i_1}(t_1),...,\phi_s^{i_s}(t_s))$, with coordinate vector-functions $\phi_k^{i_k}(t_k)=(\phi^{i_k}_{k,1}(t_{k,1}),...,\phi^{i_k}_{k,n_k}(t_{k,n_k}))$ $(k=1,2,...,s)$, is such that $\phi^{i_k}_{k,j}(t_{k,j})>0$ for $t_{k,j}\ne 0$, and $\phi^{i_k}_{k,j}(t_{k,j})\downarrow 0$ for $t_{k,j}\downarrow 0$ $(j=1,2,...,n_k)$ $(k=1,2,...,s)$, the set $E_{|\varepsilon_{N^i}|}=\prod_{k\in\varepsilon_{N^i}}E_{|\varepsilon_{N_k^{i_k}}|}$ for any

$$E_{|\varepsilon_{N_k^{i_k}}|}=\left\{z_k\in E_{n_k};\ z_{k,j}=0\,(j=\{1,2,...,n_k\}\backslash\varepsilon_{N_k^{i_k}})\right\},$$

therewith the set $\varepsilon_{N_k^{i_k}}=supp\,N_k^{i_k}$ is a support of the coordinate-vector $N_k^{i_k}=(N^{i_k}_{k,1},...,N^{i_k}_{k,n_k})$, i.e. this is a set of second indices of the coordinates of a vector for which the corresponding coordinate $N^{i_k}_{k,j}\ne 0$, consequently

$$\varepsilon_{N^i}=\left\{k\in e_s=\{1,2,...,s\};\ \varepsilon_{N_k^{i_k}}\ne\emptyset\right\}.$$

Let $H=(H_1,...,H_s)$, $H_k>0\,(k\in e_s)$ be positive vector $\sigma=(\sigma_1,\sigma_2,...,\sigma_s)$ with coordinate-vectors $\sigma_k=(\sigma_{k,1},...,\sigma_{k,n_k})\,(k\in e_s)$, i.e. $\sigma_{k,j}>0\,(j=1,2,...,n_k,$ $k\in e_s)$, and the vector $H^\sigma=(H_1^{\sigma_1},...,H_s^{\sigma_s})$, where $H_k^{\sigma_k}=(H_k^{\sigma_{k,1}},...,H_k^{\sigma_{k,n_k}})$ for all $k\in e_s$.

The subdomain $\Omega \subset G$ is said to be a subdomain satisfying the σ-semi-horm condition if there exists a vector $\delta = (\delta_1, ..., \delta_s)$ with cordinate vectors $\delta_k = (\delta_{k,1}, ..., \delta_{k,n_k})$ $(k \in e_s)$, therewith $\delta_{k,j} = 1$ or $\delta_{k,j} = -1$ $(j = 1, 2, ..., n_k, k \in e_s)$, for which

$$x + R_\delta(H^\sigma) = x + \bigcup_{0<\upsilon_k \le h_k} \{y \in R^n;\ c_{k,j}\nu_k^{\sigma_{k,1}} \le$$

$$\le y_{k,j}\delta_{k,j} \le c^*_{k,j}\ (j = 1, 2, ..., n\,;\ k \in e_s)\} \subset G$$

for all $x \in \Omega$.

It is said that the domain $G \in C_\varepsilon(\sigma, H) = C_\varepsilon(H^\sigma)$ satisfies the "strong σ-horn condition" if $G \subset \bigcup\limits_{\mu=1}^{N} G_\mu$ and it holds the condition $G \subset \bigcup\limits_{\mu=1}^{N} G_{\mu,\varepsilon}$, where $G_{\mu,\varepsilon} = \{y : y \in G_\mu, \rho(y, G\backslash G_\mu) > \varepsilon\}$, i.e.the set of points $y \in G_\mu$, distant from $G\backslash G\mu$ by $\rho(y, G\backslash G_\mu) > \varepsilon > 0$.

Main results. Let's prove two theorems of Riesz-Torin type for the functions from the spaces $\bigcap\limits_{i\in Q} \Lambda^{<m^i,N^i>}_{p_i^\lambda,\theta_i}(G, \varphi^i)$.

Let $\beta_\lambda \ge 0$, $\sum\limits_{\lambda=1}^{M} \beta_\lambda = 1$, $\frac{1}{p_i} = \sum\limits_{\lambda=1}^{M} \frac{\beta_\lambda}{p_i^\lambda}$, $\frac{1}{\theta_i} = \sum\limits_{\lambda=1}^{M} \frac{\beta_\lambda}{\theta_i^\lambda}$, $m^{i_k}_{k,j} = \sum\limits_{\lambda=1}^{M} \beta_\lambda m^{i_k^\lambda}_{k,j}$.

Theorem 1. *Let $G \in C_\varepsilon(a(h))$, $1 \le p_i^\lambda \le q_i^\lambda \le \infty$, $1 \le p_i^\lambda \le \theta_i^\lambda \le \infty$ $(\lambda = 1, 2, ..., M,\ i \in Q)$; $\nu = (\nu_1, ..., \nu_s)$ be an integer non-negative vector with coordinate-vectors $\nu_k = (\nu_{k,1}, ..., \nu_{k,n_k})$ $(k = 1, 2, ..., s)$, satisfy the condition in the case $i_k = 0$, $\nu_{k,j} \ge m^0_{k,j} + N^0_{k,j}$ $(j = 1, 2, ..., n_k)$; in the case $i_k \ne 0$ $\nu_{k,j} \ge m^{i_k}_{k,j} + N^{i_k}_{k,j}$ $(j \ne i_k)$,$\nu_{k,j} < m^{i_k}_{k,j} + N^{i_k}_{k,j}$ $(j = i_k)$ for all $k = 1, 2, ..., s$ and for any $i = (i_1, ..., i_s) \in Q$ and $f \in \bigcap\limits_{\lambda=1}^{M} \bigcap\limits_{i\in Q} \Lambda^{<m^{i^\lambda},N^{i^\lambda}>}_{p_i^\lambda,\theta_i^\lambda}(G, \varphi^i)$.*

Further, for each fixed $i \in Q$ it is assumed the finiteness of the integral expression

$$H_{k,i_k}(h_k) =$$

$$= \int_0^{h_k} \left\{ \prod_{j=1}^{n_k} (a_{k,j}(h_k))^{\sum\limits_{\lambda=1}^{M} m^{i_k,\lambda}_{k,j}\beta_\lambda - \nu_{k,j} - \frac{1}{p_i} + \frac{1}{q_i}} \prod_{j\in E_{N_k^{i_k}}} \varphi^{i_k}_{k,j}(a_{k,j}(\upsilon_k)) \right\} \frac{da_{k,i_k}(\upsilon_k)}{a_{k,i_k}(\upsilon_k)} \quad (1)$$

as soon as $i_k \ne 0$ *at corresponding* k.

Then in the domain $G \subset E_n$ there exist generalized derivatives $D^\nu f$, and the following inequalities are valid

$$\|D^\nu f\|_{L_q(G)} \le C \sum_{i\in Q} \left(\prod_{k=1}^{s} H_{k,i_k}(h_k) \right) \prod_{\lambda=1}^{M} \|f\|^{\beta_\lambda}_{\Lambda^{<m^{i,\lambda},N^{i,\lambda}>}_{p_i^\lambda,\theta_i^\lambda}(G,\varphi^i)} \quad (2)$$

where the sum is taken over all possible vectors $i \in Q$ and for some $h_0 = (h_{0,1}, ..., h_{0,s})$, $0 < h_k < h_{0,k}$ $(k = 1, 2, ..., s)$, C is independent of f and h_0, and in the case

$$H_{k,0}(h_k) = \prod_{j=1}^{n_k} (a_{k,j}(h_k))^{\sum_{\lambda=1}^{M} m^{0,\lambda}_{k,j} - \nu_{k,j} + \frac{1}{p_\lambda} + \frac{1}{q_\lambda}} \prod_{j\in E_{N_k^0}} \varphi^0_{k,j}(a_{k,j}(h_k)) \quad (3)$$

Proof. Let$f \in \bigcap_{\lambda=1}^{M}\bigcap_{i\in Q}\Lambda^{<m^{i,\lambda},N^{i,\lambda}>}_{p_i^\lambda,\theta_i^\lambda}(G,\varphi^i) \to \bigcap_{i\in Q}\Lambda^{<m^{i,\lambda},N^{i,\lambda}>}_{p_i^\lambda,\theta_i^\lambda}(G,\varphi^i) \to \Lambda^{<m^{i,\lambda},N^{i,\lambda}>}_{p_i^\lambda,\theta_i^\lambda}(G,\varphi^i)$ $(\lambda=1,2,...,M)$, then the existence of the generalized derivative follows from theorem 1 [8], and for almost each point $x\in G$ it is valid the integral representation of differentiable functions $f=f(x)$ with respect to k-th bundle of variables $x=(x_1,...,x_s)\in E_n$ in the form

$$D^\nu f(x)=\sum_{i\in Q}B_{i,\delta}f(x), \tag{4}$$

where the sum is taken over all possible vectors $i=(i_1,...,i_s)\in Q$, with the coordinate $i_k\in\{0,1,2,...,n_k\}$ $(k=1,2,...,s)$, and

$$B_{i,\delta}f(x)=B_{1,i_1,\delta_1}\cdots B_{s,i_s,\delta_s}f(x_1,...,x_s)\ (i=(i_1,...,i_s)\in Q).$$

The integral standing at the right side of equality (4) has the form

$$B_{i,\delta}f(x)=A_{e_s\backslash e^i}(\vec{h})\int_{\vec{0}}^{\vec{h}}A_{e^i}(\vec{\upsilon})\prod_{k\in e^i}\frac{da_k(\upsilon_k)}{a_k(\upsilon_k)}\times$$

$$\times\int_{E_{|\varepsilon_{N^i}|}}\frac{dz}{mes\left(R_\delta\cdot E_{|\varepsilon_{N^i}|}\right)}\int_{E_n}\left\{\Delta^{N^i}\left(\frac{z}{N^i}\right)D^{m^i}f(x+y)\right\}\times$$

$$\times\Phi_{i,\delta}\left(\frac{y}{a(\cdot)},\frac{z}{a(\cdot)}\right)\frac{dy}{mes(R_\delta\cdot E_n)}, \tag{5}$$

where $e^i=supp\,i$ is the support of the vector $i=(i_1,...,i_s)\in Q$, therefore $e^i\subset e_s=\{1,2,...,s\}$.

Note that in the integral expressions determined by equalities (5), the following notation are used:

1) $A_{e^i}(\vec{\upsilon})=\prod_{k\in e^i}\left\{(-1)^{|m_k^{i_k}-\nu_k|}c_{k,i_k}\prod_{j=1}^{n_k}(a_k(\upsilon_k))^{m_{k,j}^{i_k}-\nu_{k,j}}\right\}$

2) $A_{e_s\backslash e^i}(\vec{h})=\prod_{k\in e_s\backslash e}\left\{(-1)^{|m_k^0-\nu_k|}c_{k,0}\prod_{j=1}^{n_k}(a_{k,j}(h_k))^{m_{k,j}^0-\nu_{k,j}}\right\}$

3) $mes\left(R_\delta\cdot E_{|\varepsilon_{N^i}|}\right)=\prod_{k\in\varepsilon_{N^i}}mes\left(R_{\delta_k}\cdot E_{|\varepsilon_{N_k^{i_k}}|}\right)=$

$=\prod_{k\in\varepsilon_{N^i}}\begin{cases} mes\left(R_{\delta_k}\cdot E_{|\varepsilon_{N_k^0}|}\right)_{h_k} & \text{from } i_k=0,\\ mes\left(R_{\delta_k}\cdot E_{|\varepsilon_{N_k^{i_k}}|}\right)_{\upsilon_k} & \text{from } i_k\neq 0,\end{cases}$ where $i=(i_1,...,i_s)\in Q$, and

$$\varepsilon_{N^i}=\{k\in e^i;\ \varepsilon_{N_k^{i_k}}=supp\,N_k^{i_k}\neq\emptyset\}.$$

Notice that the support of the integral representation of differentiable functions $f=f(x)$, given by equalities (4), (5) is "$a(h)$-semi-horn"

$$x+R_\delta(a(h))\subset G,$$

with a vertex at the point $x\in E_n$.

The kernel of the integral operator (5) is determined by the equality

$$\Phi_{i,\delta}\left(\frac{y}{a(\cdot)},\frac{z}{a(\cdot)}\right)=\left\{\prod_{k\in e^i}\Phi_{k,i_k,\delta_k}\left(\frac{y_k}{a_k(\upsilon_k)},\frac{z_k}{a_k(\upsilon_k)}\right)\right\}\times$$

$$\times\left\{\prod_{k\in e_s\backslash e^i}\Phi_{k,0,\delta_k}\left(\frac{y_k}{a_k(h_k)},\frac{z_k}{a_k(h_k)}\right)\right\},$$

for each $i=(i_1,...,i_s)\in Q$, where

$$\frac{y_k}{a_k(\upsilon_k)}=\left(\frac{y_{k,1}}{a_{k,1}(\upsilon_k)},...,\frac{y_{k,n_k}}{a_{k,n_k}(h_k)}\right),\ \frac{z_k}{a_k(\upsilon_k)}=\left(\frac{z_{k,1}}{a_{k,1}(\upsilon_k)},...,\frac{z_{k,n_k}}{a_{k,n_k}(\upsilon_k)}\right)$$

for $\upsilon_k\in(0,h_k]$ $(k=1,2,...,s)$.

Then by means of integral identities (4), (5), we contsruct a set of auxiliary functions

$$f_{\nu,G_\mu+R_{\delta^\mu}}(x)=\sum_{i=(i_1,...,i_s)\in Q}B_{i,\delta^\mu,G_\mu+R_{\delta^\mu}}f(x), \tag{6}$$

determined on all E_n and coinciding on corresponding $G_\mu+R_{\delta^\mu}(a(h))\subset G$ $(\mu=1,2,...,N)$ with the function $D^\nu f(x)$.

In (6) the integral operators standing at the right side of these equalities are determined by the formulae;

$$B_{i,\delta^\mu,G_\mu+R_{\delta^\mu}}f(x)=A_{e_s\backslash e^i}(\vec{h})\int_{\vec{0}}^{\vec{h}}A_{e^i}(\vec{\upsilon})\prod_{k\in e^i}\frac{da_{k,i_k}(\upsilon_k)}{a_{k,i_k}(\upsilon_k)}\times$$

$$\times\int_{E_{|\varepsilon_{N^i}|}}\frac{dz}{mes\left(R_{\delta^\mu}\cdot E_{|\varepsilon_{N^i}|}\right)}\int_{E_n}\left\{\Delta^{N^i}\left(\frac{z}{N^i},G_\mu+R_{\delta^\mu}\right)D^{m^i}f(x+y)\right\}\times$$

$$\times\Phi_{i,\delta^\mu}\left(\frac{y}{a(\cdot)},\frac{z}{a(\cdot)}\right)\frac{dy}{mes(R_{\delta^\mu}\cdot E_n)}, \tag{7}$$

for each $i=(i_1,...,i_s)\in Q$ and for $\mu=1,2,...,N$.

Notice that when the vector N^i is a zero vector, for some $i\in Q$, instead of the function

$$\Delta^{N^i}\left(\frac{z}{N^i},G_\mu+R_{\delta^\mu}\right)D^{m^i}f(x+y)=D^{m^i}f(x+y)$$

under the integral we have the function

$$X(G_\mu+R_{\delta^\mu})D^{m^i}f(x+y),$$

where $X=X(G_\mu+R_{\delta^\mu})$ is a characteristic function of the set $G_\mu+R_{\delta^\mu}$.

The construction of auxiliary functions (6), (7) provides the following cycle of integral inequalities

$$||D^\nu f||_{L_q(G)}\le c\sum_{\mu=1}^{N}||f_{\nu,G_\mu+R_{\delta^\mu}}||_{L_q(G)}\le$$

$$\leq c\sum_{\mu=1}^{N}\sum_{i=(i_1,...,i_s)\in Q}||B_{i,\delta^{\mu};G_{\mu}+R_{\delta^{\mu}}}f||_{L_q(G)} \tag{8}$$

whence it follows that the proof of the theorem is reduced to estimation of integral operators (7). On the other hand,

$$||B_{i,\delta^{\mu};G_{\mu}+R_{\delta^{\mu}}}f||_{L_q(G)}\leq|A_{e_s\backslash e^i}|\int_{\vec{0}}^{\vec{h}}|A_{e^i}(\vec{\upsilon})|\,||F_i||_{L_q(G)}\frac{da_{k,i_k}(\upsilon_k)}{a_{k,i_k}(\upsilon_k)}, \tag{9}$$

$$F_i=\int_{E_{|\varepsilon_{N^i}|}}\frac{dz}{mes\left(R_{\delta^{\mu}}\cdot E_{|\varepsilon_{N^i}|}\right)}\int_{E_n}\left\{\Delta^{N^i}\left(\frac{z}{N^i},G_{\mu}+R_{\delta^{\mu}}\right)D^{m^i}f(x+y)\right\}\times$$

$$\times\Phi_{i,\delta^{\mu}}\left(\frac{y}{a(\cdot)},\frac{z}{a(\cdot)}\right)\frac{dy}{mes(R_{\delta^{\mu}}\cdot E_n)}. \tag{10}$$

Applying the Holder inequality for $|F(x,\upsilon)|$ by the exponents

$$\alpha_{\lambda}=\frac{q_{\lambda}}{q_{\beta_{\lambda}}}\ (\lambda=1,2,...,M)\ \left(\sum_{\lambda=1}^{M}\frac{1}{\alpha_{\lambda}}=q\sum_{\lambda=1}^{M}\frac{\beta_{\lambda}}{\alpha_{\lambda}}=1\right)$$

we get

$$||F_i(\cdot,\upsilon)||_{L_q(G)}\leq C\left(\int_G\prod_{\lambda=1}^{M}\{|F_i(x,\upsilon)|\}^{q\beta_{\lambda}}dx\right)^{\frac{1}{q}}\leq$$

$$\leq C_1\prod_{\lambda=1}^{M}\left\{\left(\int_G|F_i(x,\upsilon)|^{q_{\lambda}}dx\right)^{\frac{1}{q_{\lambda}}}\right\}^{\beta_{\lambda}}=C_1\prod_{\lambda=1}^{M}\{||F_i||_{q_{\lambda},G}\}^{\beta_{\lambda}}. \tag{11}$$

Represent the subintegrand function (10) in the form

$$|\Delta^{N^i}D^{m^i}f\Phi_{i,\delta}|=$$

$$=\left(|\Delta^{N^i}D^{m^i}f|^{p_i^{\lambda}}|\Phi_{i,\delta}|^{s_{\lambda}}\right)^{q_{\lambda}}\left(|\Delta^{N^i}D^{m^i}f|^{p_i^{\lambda}}\chi\left(\frac{y}{a(\upsilon)}\right)\right)^{\frac{1}{p_i^{\lambda}}-\frac{1}{q^{\lambda}}}\left(|\Phi_{i,\delta}|^{s_{\lambda}}\right)^{\frac{1}{s^{\lambda}}-\frac{1}{q^{\lambda}}},$$

where $\frac{1}{s^{\lambda}}=1-\frac{1}{p_i^{\lambda}}+\frac{1}{q^{\lambda}}$, apply again for $|F_i|$ the Holder inequality with the following exponents

$$\frac{1}{q^{\lambda}}+\left(\frac{1}{p_i^{\lambda}}-\frac{1}{q^{\lambda}}\right)+\left(\frac{1}{s^{\lambda}}-\frac{1}{q^{\lambda}}\right)=1,$$

χ is a characteristic function of the set $S(\Phi_{i,\delta})$. Then taking into account (11) in relation (9), we get

$$||B_{i,\delta^{\mu};G_{\mu}+R_{\delta^{\mu}}}f||_{L_q(G)}\leq C\left\{\prod_{k=1}^{s}H_{k,i_k}(h_k)\right\}\prod_{\lambda=1}^{M}||f||^{\beta_{\lambda}}_{\Lambda^{<m^{i,\lambda};N^{i,\lambda}>}_{p_i^{\lambda},\theta_i^{\lambda}}(G_{\mu}+R_{\delta^{\mu}};\varphi^i)}. \tag{12}$$

Hence by means of inequalities (8) and (12) we get the required inequality (2).

Theorem 2. *Let all the conditions of theorem 1 be satisfied, and further more* $l = (l_1, ..., l_s)$, $r = (r_1, ..., r_s)$*-be an integer non-negative vector with coordinate vectors* $l_k = (l_{k,1}, ..., l_{k,n_k})$, $r_k = (r_{k,1}, ..., r_{k,n_k})$, $l_k \geq 0$, $r_k \geq 0$ *and* $supp\,(l_k + r_k) \supset \{i_k\}$ *for* $i_k \neq 0$ $(k = 1, 2, ..., s)$, $\theta_i \leq \theta_1$*; and also* $\nu_{k,j} + l_{k,j} \geq m^0_{k,j} + N^0_{k,j}$ $(j = 1, 2, ..., n_k$, $i_k = 0)$, $\nu_{k,j} + l_{k,j} \geq m^{i_k}_{k,j} + N^{i_k}_{k,j}$ *for all* $i_k \neq 0$ $(j \neq i_k)$*;* $\nu_{k,j} + l_{k,j} < m^{i_k}_{k,j} + N^{i_k}_{k,j}$ $(k = 1, 2, ..., s)$ $i \in Q$, *and the infiniteness of the inetgral expression*

$$H_{k,i_k}(h_k, l_k) =$$

$$= \int_0^{h_k} \left\{ \prod_{j=1}^{n_k} (a_{k,j}(\upsilon_k))^{\sum_{\lambda=1}^{M} m^{i_k,\lambda}_{k,j}\beta_\lambda - \nu_{k,j} - \frac{1}{p_i} + \frac{1}{q} - |l_k|} \prod_{j \in E_{N^{i_k}_k}} \varphi^{i_k}_{k,j}\,(a_{k,j}(\upsilon_k)) \right\} \frac{da_{k,i_k}(\upsilon_k)}{a_{k,i_k}(\upsilon_k)},$$

for $i_k \neq 0$ *be assumed.*

Then in the domain G *there exist generalized derivatives* $D^\nu f$*, for which the inequalities*

$$\|D^\nu f\|_{\Lambda^{<l,r>}_{q,\theta}(G,\psi)} \leq C \sum_{i \in Q} \left(\prod_{k=1}^{s} H_{k,i_k}(h_k, l_k) \right) \prod_{\lambda=1}^{M} \|f\|^{\beta_\lambda}_{\Lambda^{<m^{i,\lambda}, N^{i,\lambda}>}_{p^\lambda_i, \theta^\lambda_i}(G,\varphi^i)}. \quad (13)$$

are valid.

Proof. For obtaining estimation (13), it suffices to estimate the norm

$$||\Delta^r\left(\frac{z}{r}, G\right) D^{\nu+l} f||_{L_q(G)}.$$

Similar to inequality (8), we have

$$||\Delta^r\left(\frac{z}{r}, G\right) D^{\nu+l} f||_{L_q(G)} \leq C \sum_{\mu=1}^{N} \sum_{i \in Q} ||\Delta^r\left(\frac{z}{r}, G\right) \widetilde{B}_{i,\delta^\mu, G_\mu + R_{\delta^\mu}} f||_{L_q(G)}, \quad (14)$$

where $\widetilde{B}_{i,\delta^\mu, G_\mu + R_{\delta^\mu}} f$ are obtained from $B_{i,\delta^\mu, G_\mu + R_{\delta^\mu}} f$ if instead of ν we take $\nu + l$.

As in the proof of theorem 1 we get

$$||\Delta^r\left(\frac{z}{r}, G\right) \widetilde{B}_{i,\delta^\mu, G_\mu + R_{\delta^\mu}} f||_{L_q(G)} \leq C \prod_{\lambda=1}^{M} ||\Delta^r\left(\frac{z}{r}, G\right) \widetilde{B}_{i,\delta^\mu, G_\mu + R_{\delta^\mu}} f||^{\beta_\lambda}_{L_{q^\lambda}(G)}. \quad (15)$$

Applying the Holder inequality with the exponents $\alpha_1 = \frac{q^\lambda p^\lambda_i}{q^\lambda - p^\lambda_i}$, $\alpha_2 = q^\lambda$, $\alpha_3 = \frac{p^\lambda_i}{p^\lambda_i - 1}$, we get

$$||\Delta^r\left(\frac{z}{r}, G\right) \widetilde{B}_{i,\delta^\mu, G_\mu + R_{\delta^\mu}} f||_{L_{q^\lambda}(G)} \leq$$

$$\leq C_1 \prod_{k=1}^{s} H_{k,i_k}(h_k, l_k) ||\Delta^{N^i}\left(\frac{z}{N^i}, G_\mu + R_{\delta^\mu}\right) D^{m^i} f||_{p^\lambda_i, G_\mu + R_{\delta^\mu}}.$$

Substituting this inequality in inequalities (14) and (15), for $\theta_i \leq \theta$ $(i \in Q)$, we get the required inequality.

The theorem is proved.

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