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**Ziyadkhan S. ALIYEV, Sevinj B. GULIYEVA**

# OSCILLATION PROPERTIES OF EIGEN FUNCTIONS OF A VIBRATIONAL BOUNDARY VALUE PROBLEM

**Abstract**

*A spectral problem for an ordinary differential opertor of fourth order with self-adjoint boundary conditions is considered. The structure of the root subspaces and oscillation properties of eigenfunctions of this problem is studied completely.*

Let's consider the following boundary value problem

$$l_r(y) \equiv (p(x)y'')'' - (q(x)y') + r(x)y(x) = \lambda\tau(x)y,\ 0 < x < l, \tag{1}$$

$$y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \tag{2a}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{2b}$$

$$y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \tag{2c}$$

$$y(l)\cos\delta - Ty(l)\sin\delta = 0, \tag{2d}$$

where $\lambda \in C$ is a spectral parameter, $Ty \equiv (py'') - qy',\ p(x),\ \tau(x) > 0,\ q(x) \geq 0$ for $x \in [0,l],\ p' \in AC[0,l],\ r,\ \tau \in C[0,l],\ \alpha,\beta,\gamma,\delta$ are real constants, and $\alpha,\beta,$ $\gamma \in [0,\pi/2],\ \delta \in [0,\pi)$.

Problem (1), (2) arises by separating variables in the dynamic boundary value problem describing small lateral oscillations of a non-homogeneous bar subjected to axial forces.

Under rather wide class of boundary conditions, equation (1) was studied in [1,2]. In these papers, the classes of regular and completely regular Sturmian systems were introduced and studied. For completely regular Sturmian systems it is established that eigenvalues of these systems are real and form an infinitely monotonically increasing subsequence, and in the case $r \equiv 0$ all of them are positive and simple, and the corresponding eigenfunctions have Sturm oscillation properties (see also [3,4]). In the case when $r(x)$ doesn't vanishes identically on any interval constituting the part of $[0,l]$ it is shown that the eigenvalues are simple, except may be the first $m$ ones, and the corresponding eigenfunctions with ordinary numbers greater that $m$ possess Sturm oscillation properties (see definition of number $m$ in the context).

Note that problem (1), (2) for $\delta \in [0,\pi/2]$, except the case $\beta = \delta = \pi/2$, is a completely regular Sturmian system, and in the case $\delta \in [\pi/2,\pi]$ is a regular Strumian system.

Oscillation properties of eigenvalues and their derivatives of problem (1), (2) for $r \equiv 0,\ \delta \in [0,\pi/2]$ were studied in detail in [4], for $\delta \in [\pi/2,\pi]$ in [5] and [6].

Oscillation properties of eigenfunctions corresponding to the first $m$ eigenvalues of completely regular Strumian systems were studied in the papers [7, 8].

The present paper is devoted to studying the structure of root subspaces and oscillation properties of eigenfunctions corresponding to the first $m$ eigenvalues of problem (1), (2).

It is known that for each fixed $\lambda \in \mathbb{C}$ there exists a unique nontrivial solution $y(x,\lambda)$ to within constant factor of problem (1), (2a)-(2c) for $r \equiv 0$, i.e. of differential equation

$$l_0(y)(x) \equiv (p(x)y''(x))'' - (q(x)y'(x))' = \lambda r(x)y(x),\ 0 < x < l, \tag{3}$$

satisfying the boundary conditions (2a)-(2c). For any fixed $x \in [0,l]$ the function $y(x,\lambda)$ is an entire function of $\lambda$.

Obviously, the eigenvalues $\mu_n(0)$ and $\mu_n(\pi/2),\ n \in \mathbb{N}$, of boundary value problem (3), (2) for $\delta = 0$ and $\delta = \pi/2$ are the zeros of entire functions $y(l,\lambda)$ and $Ty(l,\lambda)$, respectively. Notice that the function

$$F_0(\lambda) = Ty(l,\lambda),/y(l,\lambda)$$

was determined for the values

$$\lambda \in A \equiv \left( \bigcup_{n=1}^{\infty} A_n \right) \bigcup (\mathbb{C} \backslash \mathbb{R}),$$

where $A_n = \left(\mu_{n-1}(0),\ \mu_n(0)\right),\ n \in \mathbb{N},\ \ \mu_0(0) = -\infty$ and is a meromophic function of finite order, $\mu_n(\pi/2)$ and $\mu_n(0),\ n \in \mathbb{N}$ are the zeros and poles of these function, respectively.

Denote: $\delta_0 = \begin{cases} \pi/2, & \text{if } \beta \in [0,\pi/2), \\ arctg F_0(0), & \text{if } \beta = \pi/2. \end{cases}$

Recall that the problem (3), (2) was investigated in the papers [4, 5], where in particular, the following theorem was proved.

**Theorem** *A. For the fixed $\alpha,\ \beta,\ \gamma$, the eigenvalues of the problem (3), (2) for $\delta \in [0,\pi)$ are real, simple and form an infinitely increasing sequence $\{\mu_k(\delta)\}_{k=1}^{\infty}$ such that $\mu_1(\delta) < \mu_2(\delta) < ... < \mu_k(\delta) < ...$, and $\mu_k(\delta) > 0$ for $k \geq 2,\ \mu_1(\delta) > 0$ in the case $\delta \in [0,\delta_0),\ \mu_1(\delta) = 0$ in the case $\delta = \delta_0,\ \mu_1(\delta) < 0$, in the case $\delta \in (\delta_0,\pi)$. Furthermore, the eigenfunction $y_k^{(\delta)}(x)$ corresponding to the eigenvalue $\lambda_k(\delta)$ for $k \geq 2$ has exactly $k-1$ simple zeros in the interval $(0,l)$, eigenfunction $y_1^{(\delta)}(x)$ in the case $\delta \in [0,\delta_0)$ has no zeros, and in the case $\delta \in [\delta_0,\pi)$ may have arbitrary number of zeros in the interval $(0,l)$ which are also simple.*

For studying oscillation properties of eigenfunctions of problem (3), (2), in the papers [4, 5] the following Prufer type transformation was used

$$\begin{cases} y(x) = r(x)\sin\psi(x)\cos\theta(x), \\ y'(x) = r(x)\cos\psi(x)\sin\varphi(x), \\ (py)''(x) = r(x)\cos\psi(x)\cos\varphi(x), \\ Ty(x) = r(x)\sin\psi(x)\sin\theta(x). \end{cases} \tag{4}$$

Following the corresponding reasonings carried out in the course of proof of theorem $A$, we are convinced that for any $\delta_1,\ \delta_2 \in (0,\pi)$ such that $\delta_1 < \delta_2$ the following relations are fulfilled

$$\mu_1(\delta_2) < \mu_1(\delta_1) < \mu_1(0) < \mu_2(\delta_2) < \mu_2(\delta_1) < \mu_2(0) < ... . \tag{5}$$

Based on max-min properties of eigenvalues [9, p.343], the eigenvalues of problem (1), (2) are determined from the relation:

$$\lambda_k = \max_{V^{(k-1)}} \min_{y\in B.C.} \left\{ R[y] \middle| \int_0^l \tau(x)y(x)\varphi(x)dx = 0,\ \varphi \in V^{(k-1)} \right\}, \qquad (6)$$

where $R[y]$ is the Reley ratio

$$R[y] = \left\{ \int_0^l (py''^2 + qy'^2 + ry^2)dx + N[y] \right\} \Big/ \int_0^l \tau y^2 dx,$$
$$N[y] = y'^2(0)ctg\alpha + y^2(0)ctg\beta + y'^2(l)ctg\gamma + y^2(l)ctg\delta, \qquad (7)$$

$V^{(k-1)}$ is an arbitrary set of linearly-independent functions $\varphi_j \in B.C.$, $1 \le j \le k-1$, $B.C.$ is the set of functions satisfying the boundary conditions (2).

Assume:

$$r_0 = \min_{x\in[0,l]} r(x),\ r_1 = \max_{x\in[0,l]} r(x),\ \tau_0 = \min_{x\in[0,l]} \tau(x),\ \tau_1 = \max_{x\in[0,l]} \tau(x).$$

Denote by $(\Psi_0)$ a comlepetely regular Sturmian system that is obtained from the system (1), (2) by substituting $r_0$ for $r(x)$ and $\tau_1$ for $\tau(x)$. By substituting $\lambda' = \lambda\tau_1 - r_0$ the system $(\Psi_0)$ goes into the equaivalent system $(\Psi_1)$ for which the statement of theorm $A$ is valid.

Let $\lambda_{k,1}$, $k \in \mathbb{N}$ be the $k$-th eigenvalue of the system $(\Psi_1)$ which is positive by theorem $A$, and $\lambda_{k,0} = (\lambda_{k,1} + r_0)/\tau_1$, $k \in \mathbb{N}$ be the $k$-th eigenvalue of the system $(\Psi_0)$. Then by theorem $A$, the eigenfunction $y_{k,0}(x)$ corresponding to the eigenvalue $\lambda_{k,0}, k \in \mathbb{N}$ has exactly $k-1$ simple zeros in the interval $(0,l)$.

Now pass from the system $(\Psi_0)$ to the system (1), (2) using the "$\mu$-process" (see [1.2]) by means of deformation

$$r(x,\mu) \equiv (1-\mu')r_0 + \mu' r(x),$$

$$\tau(x,\mu) = (1-\mu'')\tau_1 + \mu''\tau(x),\ x \in [0,l],\ \mu',\ \mu'' \in [0,1].$$

Since $r(x,\mu)$ increases, and $\tau(x,\mu)$ decreases, then by [2, lemma 4] the positive eigenvalues don't decrease.

Define the positive integer $m_0$ from the relations

$$\lambda_{m_0+1,0} > (r_1\tau_1 - r_0\tau_0)/r_0 \ge \lambda_{m_0,0} \quad \text{and} \quad \lambda_{m_0+1,0} > 0.$$

It is known [2] that if $k > m = \max\{2, m_0\}$, then the following inequality is fulfilled

$$r(x,\mu) - \lambda_k(\mu)\tau(x,\mu) < 0,\quad x \in [0,l],\ \mu \in [0,1].$$

where $\lambda_k(\mu)$ is the $k$-th eigenvalue of the Sturmian system that is obtained from the system (1), (2) by substituting $r(x,\mu)$ for $r(x)$ and $\tau(x,\mu)$ for $\tau(x)$. Consequently, by Corollary 1, Lemma 7 and Remark 1 from [2], the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_m$, $\lambda_{m+1}, ...$, of the completely regular Sturmian system (1), (2), except may be the first $m$ ones, are positive and simple, and the eigenfunction $\vartheta_k(x)$ corresponding to the eigenvalue $\lambda_k$ for $k > m$ has $k-1$ simple zeros in the interval $(0,l)$.

Obviously, these statements are valid also for the regular Sturmian system

$$\left.\begin{array}{c} l_0\left(y\right)+\mu r(x)y=\lambda\tau(x)y, \quad x\in(0,l), \\ y(x)\in B.C., \mu\in[0,1]. \end{array}\right\} \tag{8}$$

**Lemma 1.** *The following relation is valid*

$$\nu_k(\mu)\in\left[\mu_k+\mu r_0/\tau_0, \mu_k+\mu r_1/\tau_0\right], \tag{9}$$

*where* $\nu_k(\mu)$ *is the* $k$-th *eigenvalue of problem (8),* $\mu_k=\mu_k(\delta)$.

**Proof.** From (6) we have

$$\lambda_k(\mu)=\max_{V^{(k-1)}}\min_{y(x)\in B.C.}\left\{R_\mu[y]\left|\int_0^l\tau(x)y(x)\varphi(x)dx=0,\ \varphi(x)\in V^{(k-1)}\right.\right\}, \tag{10}$$

where

$$R_\mu[y]=\left(\int_0^l\left(py''^2+qy'^2+\mu ry^2\right)dx+N[y]\right)\Big/\int_0^l\tau y^2dx. \tag{11}$$

For an arbitrary choice of $V^{(k-1)}$ from (11) we get

$$R_\mu[y]=R_0[y]+\mu\left(\int_0^l ry^2dx\Big/\int_0^l\tau y^2dx\right), \tag{12}$$

where

$$R_0[y]=\left(\int_0^l\left(py''^2+qy'^2\right)dx+N[y]\Big/\int_0^l\tau y^2dx\right). \tag{13}$$

From (12), (13) it follows that

$$R_0[y]+\mu r_0/\tau_0\le R_\mu[y]\le R_0[y]+\mu r_1/\tau_0. \tag{14}$$

Taking into account (14), from (10) we get the relation (9). The lemma is proved.

Let $E=C^3[0,l]\cap B.C.$ Banach space endowed with the norm $\|y\|_j=\sum\limits_{i=0}^{j}\left|y^{(i)}\right|_0$, where $|\cdot|_0$ is an ordinary sup.-norm in $C[0,l]$.

Let $S=\{y\in E\}\left|y^{(i)}(x)\neq0,\ x\in(0,l),\ i=\overline{0,3}\right\}\cup\{y\in E|$ if $y(\xi)=0$ or $y''(\xi)=0$ for $\xi\in(0,l)$, then $y'(\xi)Ty(\xi)<0$; if $y'(\eta)=0$ or $Ty(\eta)=$ $=0$ for $\eta\in(0,l)$ then $y(\eta)y''(\eta)<0\}$.

Denote by $S_k^\nu$, $k\in\mathbb{N}$, $\nu=+$ or $-$, the set of functions $y\in S$ satisfying the following conditions:

1) $y(x)$ has exactly $k-1$ zeros in the interval $(0,l)$:

2) $\lim\limits_{x\to0}\nu sgny(x)=1$;

3) the angular function $\psi$ satisfies either the condition $\psi(x)\in(0,\pi/2)$ or $\psi(x)\in(\pi/2,\pi)$;

4) the boundary values of angular functions $\theta$ and $\varphi$ from (4) are determined as follows:

$$\theta(0)=\pi/2-\beta,\quad \theta(l)=k\pi-\pi/2-\delta;$$

$$\varphi(0)=\alpha,\quad \varphi(l)=n_k\pi-\gamma,\quad k\in N,$$

where $\alpha = 0$ in the case $\psi(0) = \pi/2$, $\gamma = 0$ in the case $\psi(l) = \pi/2$, $n_k = k$ or $k+1$ in the case $\psi(0) \in (0, \pi/2)$, $n_1 = 1$ and $n_k = k$ or $k-1$, $k \in \mathbb{N}\backslash\{1\}$, in the case $\psi(0) \in [\pi/2, \pi)$; $w(0) = ctg\psi(0)$ determined at least by one of the following equalities:

$$a)\ \ w(0) = \frac{y'(0)\sin\beta}{y(0)\sin\alpha}, \qquad b)\ \ w(0) = -\frac{(py'')(0)\cos\beta}{Ty(0)\cos\alpha},$$

$$c)\ \ w(0) = \frac{(py'')(0)\sin\beta}{y(0)\cos\alpha}, \quad b)\ \ w(0) = -\frac{y'(0)\cos\beta}{Ty(0)\sin\alpha};$$

5) the graphs of the functions $\theta(x)$ and $\varphi(x)$, $x \in [0, l]$, intersect the lines

$$\theta = (2m-1)\,\pi/2,\ \theta = m\pi \ \text{ and } \ \varphi = m\pi, \quad m = 0, 1, 2, ...,$$

strongly increasing;

6) if $(i)$ $y(0)y'(0) > 0$, $(ii)$ $y(0) = 0$ or $(iii)$ $y'(0) = 0$ and $y(0)y''(0) > 0$, then $\psi(x) \in (0, \pi/2)$ for $x \in (0, l)$, and if $(i\nu) y(0)y'(0) < 0$, $(\nu) y'(0) = 0$ and $y(0)y'(0) < 0$ or $(\nu i)$ $y'(0) = 0$, then $\psi(x) \in (\pi/2, \pi)$ for $x \in (0, l)$.

Denote: $S_k = S_k^+ \cup S_k^-$. By lemma 2.2, corollary of theorem 3.1, theorems 3.3, 3.4, 5.1, 5.5, 6.1, 6.3 from [4], and theorem $A$, the eigenfunction $\vartheta_k(x) = \vartheta_k^{(\delta)}(x)$ corresponding to the eigenvalue $\mu_k = \mu_k(\delta)$ of problem (3), (2) is contained in the set $S_k$ for $k \in \mathbb{N}$ in the case $\delta \in [0, \delta_0)$, for $k \in \mathbb{N}\backslash\{1\}$ in the case $\delta \in (\delta_0, \pi)$. Consequently, the sets $S_k^\nu$, $k \in \mathbb{N}$, $\nu = +$ or $-$, are nonempty. The sets $S_k^\nu$, $k \in \mathbb{N}$, $\nu = +$ or $-$, are open subsets in $E$ [10].

Denote: $\mathbb{N}_0 = \begin{cases} \mathbb{N}, & \text{if } \ \delta \in [0, \delta_0), \\ \mathbb{N}\backslash\{1\}, & \text{if } \ \delta \in [\delta_0, \pi), \end{cases}$

Alongside with problem (1), (2) we consider the following nonlinear "approximation" problem

$$\left.\begin{array}{c} (l_0 y)(x) + r(x)\left\|y(x)\right\|_3^\varepsilon y(x) = \lambda\tau(x)y(x), \quad x \in (0, l), \\ y(x) \in B.C., \end{array}\right\} \tag{15}$$

where $\varepsilon \in (0, 1]$.

Let's determine the function $g(y) \in C[0, l]$, $y \in E$, as follows:

$$g(y)(x) = -r(x)y(x), \quad x \in [0, l]\,. \tag{16}$$

Since $r(x) \in C[0, l]$, then map $g : E \to C[0, l]$ is continuous. We can rewrite problem (15) in the following equivalent form

$$\left.\begin{array}{c} l_0(y) = \lambda r y + g\left(\left\|y\right\|_3^\varepsilon y\right), \ x \in (0, l)\,, \\ y \in B.C. \end{array}\right\} \tag{17}$$

By (16) for any fixed $\varepsilon \in (0, 1]$ the following relation

$$g\left(\left\|y\right\|_3^\varepsilon y\right) = 0\left(\left\|y\right\|_3\right) \ \text{ from } \ \left\|y\right\|_3 \to 0,$$

is valid. Consequently, for problem (15) the statement of theorem 1 from [10] is true. Then for any $k \in \mathbb{N}_0$ and $\nu = +$ or $-$, there exists an unbounded continuum of the set of solutions of problem (15) $C_{k,\varepsilon}^\nu$ such that

$$(\mu_k, 0) \in C_{k,\varepsilon}^\nu \subset (\mathbb{R} \times S_k^\nu) \cup \{(\mu_k, 0)\}\,.$$

**Lemma 2.** *There exists $\rho_0 \in (0,1)$ such that for any $\varepsilon \in (0,\rho_0)$ and $\gamma > 0$ there doesn't exist the non-trivial solution $(\lambda, w)$ of problem (15) satisfying the conditions*

$$w \in S_k^{\nu},\ k \le m_0,\ k \in \mathbb{N}_0,\ \nu = +or-,\ \|w\|_3 \le p_0 \text{ and } dist\,(\lambda, I_0) = \gamma,$$

*where* $I_0 = [\mu_1 + r_0/\tau_0, \mu_m + r_1/\tau_0]$.

The proof is carried out by the scheme of the proof of lemma 2 from [7], using lemma 1.

Assume that $k \le m,\ k \in \mathbb{N}_0$. Since $C_{k,\varepsilon}^{\nu}$ is a connected set, then for any $\varepsilon \in (0,\rho_0)$ there exists the solution $(\lambda_\varepsilon, y_\varepsilon)$ of problem (15) such that $\lambda_\varepsilon \in I_0$ and $\|y_\varepsilon\|_3 = \rho_0$. Following the above reasonings conducted in proving lemma 2, we can find such a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,\rho_0)$ and $\lim\limits_{n\to\infty} \varepsilon_n = 0$ that the sequence $\{(\lambda_{\varepsilon_n}, y_{\varepsilon_n})\}_{n=1}^{\infty}$ converges to the solution $\left(\widehat{\lambda}, \widehat{y}\right)$ of problem (1), (2), where $\widehat{\lambda} \in I_0,\ \widehat{y} \in S_k$. So, for any $k \in \{1,2,...,m\} \cap \mathbb{N}_0$ there exists an eigenfunction $y_k = \widehat{y} \in S_k$ of problem (1), (2) corresponding to the eigenvalue $\lambda_s = \widehat{\lambda},\ s \in \{1,2,...,m\} \cap \mathbb{N}_0$. Using system (8) by applying the "$\mu$-process", we pass from the regular Sturmian system (3), (2) to the regular Sturmian system (1), (2). Since the eigenvalues $\nu_k(\mu)$ for the "$\mu$-process" are displaced from the initial value (origin) $\mu_k$ to which corresponds the eigenfunction $\vartheta_k \in S_k$, we can assume $s = k$.

Thus, the eigenfunction $y_k(x)$ of problem (1), (2) corresponding to the eigenvalue $\lambda_k,\ k \in \{1,2,...,m\} \cap \mathbb{N}_0$ has exactly $k-1$ simple zeros in the interval $(0,l)$, and

$$\lambda_1 \le \lambda_2 \le ... \le \lambda_m < \lambda_{m+1}.$$

Determine the numbers $d_0 > 0$ and $d_1 \ge 0$ from the following relations:

$$d_0 = \min_{k=\overline{1,m}} \left\{\mu_{k+1} - \mu_k\right\},$$

$$d_1 = \inf\left\{z \in R_+ \left| r(x) + z\tau(x) > 0, x \in [0,l]\right.\right\}.$$

By substituting $\xi = \lambda + d$, the system (1), (2) goes into the equivalent system

$$\left.\begin{array}{l} \ell_{\overline{r}}(y) \equiv \ell_0(y) + \overline{r}y = \xi\tau y \\ y \in B.C., \end{array}\right\} \tag{18}$$

where $\overline{r} = r + d_1\tau$. Now pass from the system (3), (2) to the system (18) using the "$\mu$-process":

$$\left.\begin{array}{l} \ell_0(y) + \overline{\mu}\overline{r}y = \xi\tau y \\ y \in B.C.. \end{array}\right\} \tag{19}$$

Since the coefficient $r(x,\mu) = \overline{\mu}r(x)$ increases, then the eigenvalues don't decrease. Notice that if the condition $r_1/\tau_0 + d_1 < d_0$ is fulfilled, then the eigenvalues $\xi_1(\mu), \xi_2(\mu), ..., \xi_m(\mu)$ of problem (19) don't coincide for the "$\mu$-process" and consequently, all of them are simple.

**Remark 1.** From the above reasonings it is seen that for $\delta \in [\delta_0, \pi)$ the first eigenvalue $\xi_1(\mu)$ is also simple.

Note that if the condition $r_1/\tau_0 + d_1 < d_0$ is not fulfilled, then we can choose such $\overline{\mu} \in (0,1)$ that the inequality $\overline{\mu}(r_1/\tau_0 + d_1) < d_0$ is valid. Then obviously, the eigenvalues $\xi_1(\mu), \xi_2(\mu), ..., \xi_m(\mu)$ of problem (19) for $\mu \in (0,\overline{\mu})$ are also simple.

Now show that then eigenvalues $\xi_1(\mu), \xi_2(\mu), ..., \xi_m(\mu)$ of problem (19) remain simple for $\mu_0 \in (\overline{\mu}, 1]$ as well. Indeed, if it is not so, then there exist $\mu_0 \in (\overline{\mu}, 1]$ closest to $\overline{\mu}$, $k \in \{1, 2, ..., m-1\}$ and $\delta_1 \in [0, \pi)$ such that $\xi_k(\mu_0, \delta_1) = \xi_{k+1}(\mu_0, \delta_1)$. Take rather small $\varepsilon > 0\,(\varepsilon < \overline{\mu} - \mu_0)$ and consider the eigenvalues $\xi_k(\mu_0 - \varepsilon)$ and $\xi_{k+1}(\mu_0 - \varepsilon)$. Obviously, $\xi_k(\mu_0 - \varepsilon) < \xi_{k+1}(\mu_0 - \varepsilon)$.

Let $\delta_2 \in (0, \pi/2)$ be such that if $\delta_1 \in (0, \pi)$, then $0 < \delta_2 < \delta_1$. By property 1 from [4] and relations (5) we have

$$\xi_k(\mu_0 - \varepsilon, \delta_1) < \xi_{k+1}(\mu_0 - \varepsilon, \delta_2) < \xi_{k+1}(\mu_0 - \varepsilon, \delta_1), \text{ if } \delta_1 = 0,$$

$$\xi_k(\mu_0 - \varepsilon, \delta_1) < \xi_k(\mu_0 - \varepsilon, \delta_2) < \xi_{k+1}(\mu_0 - \varepsilon, \delta_1), \text{ if } \delta_1 \in (0, \pi).$$

Further, passing in the latter two inequalities to limit as $\varepsilon \to 0$ and taking into account $\xi_k(\mu_0, \delta_1) = \xi_{k+1}(\mu_0, \delta_1)$, we get

$$\xi_{k+1}(\mu_0, \delta_2) = \xi_{k+1}(\mu_0, \delta_1) \text{ for } \delta_1 = 0,$$

$$\xi_k(\mu_0, \delta_1) = \xi_k(\mu_0, \delta_2) \text{ for } \delta \in (0, \pi),$$

that contradict relations (5).

By substituting the variable $\lambda = \xi - d_1$, the system (18) goes into the equivalent system (1), (2).

Consequently, the following theorem is valid.

**Theorem 1.** *For the fixed $\alpha, \beta, \gamma$ the eigenvalues of problem (1), (2) for $\delta \in [0, \pi)$ are real, simple and form an infinitely increasing sequence such that $\{\lambda_k(\delta)\}_{k=1}^{\infty}, \lambda_1(\delta) < \lambda_2(\delta) < ... < \lambda_k(\delta) < ....$ Furthermore, the eigenfunction $y_k^{(\delta)}(x)$ corresponding to the eigenvalue $\lambda_k(\delta)$ for $k \in \mathbb{N}_0$ has exactly $k-1$ simple zeros in the interval $(0, l)$, more exactly $y_k^{(\delta)}(x) \in S_k$.*

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**Ziyadkhan S. Aliyev**
Baku State University,
23, Z.Khalilov str, AZ 1148, Baku, Azerbaijan
Tel.: (99412) 538-05-82 (off.).

Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, B.Vahabzade str., AZ1141, Baku Azerbaijan.
Tel.: (99412) 539 47 20 (off.)

**Sevinj B. Guliyeva**
Ganja State University,
187, Sh.I.Khatai ave., AZ2000, Ganja, Azerbaijan.
Tel.: (99412) 539 47 20 (off.).

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