Transactions of NAS of Azerbaijan, 2014, vol. XXXIV, No 4, pp. 143-148. ________ 143

Elnur H. KHALILOV

ON DIRECT VALUE OF THE DERIVATIVE OF AN ACOUSTIC SINGLE LAYER POTENTIAL

Abstract

In the paper, the existence of the direct value of the acoustic single layer potential is proved.

It is known that some problems of physics and mechanics are reduced to the system of singular integral equations dependent on direct values of the derivative of the acoustic single layer potential (see [1])

$$\upsilon(x) = \int_S \operatorname{grad}_x \Phi_k(x,y) \cdot \rho(y)\, dS_y, \quad x \in S, \tag{1}$$

where $S \subset R^3$ is the Lyapunov surface with the exponent α, $\vec{n}(y)$ is the external unit normal at the point $y \in S$, $\Phi_k(x,y) = \frac{\exp(ik|x-y|)}{4\pi|x-y|}$ is the fundamental solution of the Helmholts equation, k is a wave number, moreover $\operatorname{Im} k \geq 0$, and $\rho(y)$ is a continuous function on the surface S.

The counterexamples constructed by Gunther show that for a single layer potential with continuous density, the derivatives, generally speaking, don't exist. However, it is proved in the paper [1] that if S is a twice differentiable surface, and $\rho \in C^{\beta}$ (C^{β} is a space of functions satisfying the Holder condition with the exponent $\beta \in (0,1]$), then integral (1) exists in the sense of the Cauchy principal value. But as is seen the conditions imposed on the surface S and on the density $\rho(x)$ are rather hard, therefore there appears interest to the proof of the convergence of integral (1) in the sense of the Cauchy principal value under weakened conditions, and our paper is devoted to this issue.

For the function $\varphi(x)$ continuous on the surface S we introduce a modulus of continuity in the form $\omega(\varphi,\delta) = \delta \cdot \sup\limits_{\tau \geq \delta} \frac{\overline{\omega}(\varphi,\tau)}{\tau}$, $\delta > 0$, where

$$\overline{\omega}(\varphi,\tau) = \max_{\substack{|x-y| \leq \tau \\ x,y \in S}} |\varphi(x) - \varphi(y)|.$$

Theorem. *Let S be a Lyapunov surface with the exponent $0 < \alpha \leq 1$ and $\int\limits_0^{diamS} \frac{\omega(\rho,t)}{t} dt < +\infty$. Then integral (1) exists in the sense of the Cauchy principal value, and*

$$\sup_{x \in S} |\upsilon(x)| \leq M^{*)} \cdot \left(\|\rho\|_{\infty} + \int_0^{diamS} \frac{\omega(\rho,t)}{t} dt \right). \tag{2}$$

Proof. Let $\upsilon(x) = (\upsilon_1(x), \upsilon_2(x), \upsilon_3(x))$, where

$$\upsilon_m(x) = \int_S \frac{\partial \Phi_k(x,y)}{\partial x_m} \cdot \rho(y)\, dS_y, \quad x \in S \quad (m = 1,2,3).$$

It is easy to calculate

$$\upsilon_m(x) = (1 - ik) \cdot \int_S \frac{(\exp(ik|x-y|) - 1) \cdot (y_m - x_m)}{|x-y|^3} \cdot \rho(y)\, dS_y +$$

$$+ (1 - ik) \cdot \int_S \frac{y_m - x_m}{|x-y|^3} \rho(y)\, dS_y, \quad x \in S \quad (m = 1,2,3). \tag{3}$$

Since

$$\left| \frac{(\exp(ik|x-y|) - 1) \cdot (y_m - x_m)}{|x-y|^3} \right| \le \frac{M}{|x-y|},$$

the integral

$$\int_S \frac{(\exp(ik|x-y|) - 1) \cdot (y_m - x_m)}{|x-y|^3} \cdot \rho(y)\, dS_y$$

converges as improper one, and

$$\left| \int_S \frac{(\exp(ik|x-y|) - 1) \cdot (y_m - x_m)}{|x-y|^3} \cdot \rho(y)\, dS_y \right| \le$$

$$\le M \cdot \|\rho\|_\infty, \quad \forall x \in S \quad (m = 1,2,3). \tag{4}$$

Denote by $d > 0$ the radius of a standard sphere for S (see [3]). Then for any point $x \in S$ the vicinity $S_d(x) = \{y \in S \,|\, |y - x| < d\}$ intersects the straight line parallel to the normal $\vec{n}(x)$, at a unique point, or don't intersect at all, i.e. the set $S_d(x)$ is uniquely projected onto the set $\Omega_d(x)$ lying in the circle of radius d centered at the point x in the tangential plane $\Gamma(x)$ to S at the point x. On the piece $S_d(x)$ we choose a local rectangular system of coordinates (u, v, w) with the origin at the point x where we direct the axis w along the normal $\vec{n}(x)$, and the axes u and v lie in the tangential plane $\Gamma(x)$. Then in these coordinates the vicinity $S_d(x)$ may be given by the equation $w = f(u, v)$, $(u, v) \in \Omega_d(x)$, moreover

$$f \in C^{1,\alpha}(\Omega_d(x)) \quad \text{and} \quad f(0,0) = 0, \quad \frac{\partial f(0,0)}{\partial u} = 0, \quad \frac{\partial f(0,0)}{\partial v} = 0.$$

Furthermore, if $\widetilde{y} \in \Gamma(x)$ is the projection of the point $y \in S$, then (see [4])

$$|x - \widetilde{y}| \le |x - y| \le C_1 \cdot |x - \widetilde{y}|,$$

where C_1 is a positive constant dependent only on S (for the sphere $C_1 = \sqrt{2}$).

Obviously,

$$\int_S \frac{y_m - x_m}{|x-y|^3} \cdot \rho(y)\, dS_y = \int_{S \backslash S_d(x)} \frac{y_m - x_m}{|x-y|^3} \cdot \rho(y)\, dS_y +$$

$$+\int\limits_{S_d(x)} \frac{y_m - x_m}{|x-y|^3} \cdot (\rho(y) - \rho(x))\, dS_y +$$

$$+\rho(x) \cdot \int\limits_{S_d(x)} \frac{y_m - x_m}{|x-y|^3} dS_y, \quad x \in S \quad (m = 1,2,3).$$

The integral $\int\limits_{S \backslash S_d(x)} \frac{y_m - x_m}{|x-y|^3} \cdot \rho(y)\, dS_y$ exists as proper one, and

$$\left| \int\limits_{S \backslash S_d(x)} \frac{y_m - x_m}{|x-y|^3} \cdot \rho(y)\, dS_y \right| \leq M \cdot \|\rho\|_\infty, \quad \forall x \in S \quad (m = 1,2,3). \tag{5}$$

Furthermore, passing to the double integral, we have:

$$\left| \int\limits_{S \backslash S_d(x)} \frac{y_m - x_m}{|x-y|^3} \cdot (\rho(y) - \rho(x))\, dS_y \right| \leq \int\limits_{S \backslash S_d(x)} \frac{|\rho(y) - \rho(x)|}{|x-y|^2} dS_y \leq$$

$$\leq M \cdot \int\limits_0^{diam S} \frac{\omega(\rho, t)}{t} dt < +\infty, \quad \forall x \in S \quad (m = 1,2,3). \tag{6}$$

It remains to prove that the integral $\int\limits_{S_d(x)} \frac{y_m - x_m}{|x-y|^3} dS_y \quad (m = 1,2,3)$ exists in the sense of the Cauchy principal value. Let $d_0 = d/C_1$ and $O_{d_0} = \left\{u^2 + v^2 < d_0\right\} \subset \Gamma(x)$ (obviously, $O_{d_0} \subset \Omega_d(x)$). Then according to the formula of reduction of the surface integral to the repated one, we get:

$$\int\limits_{S_d(x)} \frac{y_1 - x_1}{|x-y|^3} dS_y = \int\limits_{\Omega_d(x)} \frac{u}{\left(\sqrt{u^2 + v^2 + f^2(u,v)}\right)^3} \cdot \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2} dudv =$$

$$= \int\limits_{\Omega_d(x) \backslash O_{d_0}} \frac{u}{\left(\sqrt{u^2 + v^2 + f^2(u,v)}\right)^3} \cdot \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2} dudv +$$

$$+ \int\limits_{O_{d_0}} \frac{u}{\left(\sqrt{u^2 + v^2 + f^2(u,v)}\right)^3} \cdot \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2} dudv =$$

$$= \int\limits_{\Omega_d(x) \backslash O_{d_0}} \frac{u}{\left(\sqrt{u^2 + v^2 + f^2(u,v)}\right)^3} \cdot \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2} dudv +$$

$$+ \int\limits_{O_{d_0}} \frac{u}{\left(\sqrt{u^2 + v^2}\right)^3} dudv +$$

$$+\int\limits_{O_{d_0}} \frac{u}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3}\cdot\left(\sqrt{1+\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2}-1\right)dudv+$$

$$+\int\limits_{O_{d_0}} u\cdot\left(\frac{1}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3}-\frac{1}{\left(\sqrt{u^2+v^2}\right)^3}\right)dudv.$$

Denote the additive integrals in the right side of this equality by A_1, A_2, A_3 and A_4, respectively.

The integral A_1 exists as proper one.

The integral A_2 exists in the sense of the Cauchy principal value and equals zero. Indeed, let $O_\varepsilon=\left\{u^2+v^2<\varepsilon, w=0\right\}$, where $\varepsilon>0$. Then

$$\int\limits_{O_{d_0}} \frac{u}{\left(\sqrt{u^2+v^2}\right)^3}dudv=\lim_{\varepsilon\to 0}\int\limits_{O_{d_0}\backslash O_\varepsilon} \frac{u}{\left(\sqrt{u^2+v^2}\right)^3}dudv=$$

$$=\lim_{\varepsilon\to 0}\int\limits_{\varepsilon}^{d_0}\int\limits_{0}^{2\pi}\frac{r\cdot\cos\varphi}{r^3}\cdot r\cdot d\varphi\cdot dr=\lim_{\varepsilon\to 0}\int\limits_{\varepsilon}^{d_0}\int\limits_{0}^{2\pi}\frac{1}{r}\cdot\cos\varphi d\varphi dr=0.$$

Furthermore, taking into account (see [3])

$$\left|\frac{\partial f}{\partial u}\right|\le M\cdot\left(\sqrt{u^2+v^2}\right)^\alpha \quad\text{and}\quad \left|\frac{\partial f}{\partial v}\right|\le M\cdot\left(\sqrt{u^2+v^2}\right)^\alpha,$$

we get

$$|A_3|=\left|\int\limits_{O_{d_0}}\frac{u\cdot\left(\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2\right)}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3\cdot\left(1+\sqrt{\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2}\right)}dudv\right|\le$$

$$\le M\cdot\int\limits_{O_{d_0}}\left|\frac{u\cdot\left(\sqrt{u^2+v^2}\right)^{2\alpha}}{\left(\sqrt{u^2+v^2}\right)^3}\right|dudv=M\cdot\int\limits_0^{d_0}\int\limits_0^{2\pi}\frac{|\cos\varphi|}{r^{1-2\alpha}}d\varphi dr\le\frac{\pi\cdot M\cdot d_0^{2\alpha}}{\alpha}.$$

For the integral A_4 we have:

$$|A_4|=\left|\int\limits_{O_{d_0}}\frac{uf^2(u,v)\left(2\left(u^2+v^2\right)+f^2(u,v)+\sqrt{(u^2+v^2)\cdot(u^2+v^2+f^2(u,v))}\right)dudv}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3\cdot\left(\sqrt{u^2+v^2}\right)^3\left(\sqrt{u^2+v^2+f^2(u,v)}+\sqrt{u^2+v^2}\right)}\right|\le$$

$$\le M\cdot\int\limits_{O_{d_0}}\frac{|u|\left(u^2+v^2\right)^{1+\alpha}\left(u^2+v^2\right)\left(2+\left(u^2+v^2\right)^\alpha+\sqrt{1+(u^2+v^2)^\alpha}\right)}{2\cdot\left(\sqrt{u^2+v^2}\right)^7}dudv=$$

$$= M \cdot \int_0^{d_0}\int_0^{2\pi} \frac{|\cos\varphi| \cdot \left(2 + r^{2\alpha} + \sqrt{1+r^{2\alpha}}\right)}{2 \cdot r^{2-2\alpha}} r d\varphi dr \leq$$

$$\leq M\pi \left(2 + d_0^{2\alpha} + \sqrt{1 + d_0^{2\alpha}}\right) \cdot \frac{d_0^{2\alpha}}{2\alpha}.$$

Behaving in the same way, we can prove that the integral $\int\limits_{S_d(x)} \frac{y_2 - x_2}{|x - \mathrm{y}|} dS_y$ exists in the sense of the Cauchy principal value.

Since, $|f(u,v)| \leq M \cdot \left(\sqrt{u^2+v^2}\right)^{1+\alpha}$ (see [3]), we have:

$$\left| \int\limits_{S_d(x)} \frac{y_3 - x_3}{|x - \mathrm{y}|^3} dS_y \right| =$$

$$= \left| \int\limits_{\Omega_d(x)} \frac{f(u,v)}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3} \cdot \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2} dudv \right| \leq$$

$$\leq M \cdot \int\limits_{\Omega_d(x)} \frac{1}{\left(\sqrt{u^2+v^2}\right)^{2-\alpha}} dudv \leq M.$$

As a result we get

$$\left| \rho(x) \cdot \int\limits_{S_d(x)} \frac{y_m - x_m}{|x - \mathrm{y}|^3} \cdot \rho(y)\, dS_y \right| \leq M \cdot \|\rho\|_\infty, \qquad \forall x \in S \quad (m = 1,2,3). \tag{7}$$

Taking into account inequalities (4)-(7) in equality (3), we get the proof of the theorem.

References

[1]. Kolton D., Kress R. *Methods of integral equations in scattering theory.* M.Mir, 1987, 311 p. (Russian).

[2]. Gunther N.M. *Potential theory and its application to main problems of mathematical physics.* M. Gostekhizdat, 1953, 415 p. (Russian).

[3]. Vladinmirov V.S. *Equations of mathematical physics.* M. Nauka, 1976, 527 p. (Russian).

[4]. Kustov Yu.A., Musayev B.I. *Cubic formula for two-dimensional singular integral and its applications.* Dep. In VINITI, No 4281-81, 60 p. (Russian).

[5]. Mikhlin S.G. *Many-dimensional singular integrals and integral equations.* M. Fizmatgiz. 1962, 254 p. (Russian).

Elnur H. Khalilov
Azerbaijan State Oil Academy
20, Azadlig av. AZ 1601, Baku, Azerbaijan
Tel.: (99412) 539-47-20 (off.).

Received February 20, 2014; Revised April 30, 2014.