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INTEGRAL LIMIT THEOREM FOR THE FIRST PASSAGE TIME FOR THE LEVEL OF RANDOM WALK, DESCRIBED BY A NONLINEAR FUNCTION OF THE SEQUENCE AUTOREQRESSION $AR(1)$

Abstract

In the paper, an integral limit theorem is proved for the first passage time for the level of random walk described by a nonlinear function of the sequence of first order autoregression $AR(1)$.

1. Introduction. Let on some probability space (Ω, F, P) a sequence of independent identically distributed random variables $\xi_n; n \geq 1$ be given.

Consider the recurrent relation

$$X_n = \beta X_{n-1} + \xi_n, \quad n \geq 1$$

where $X_0 = x \geq 0$ and $|\beta| < 1$ are non-random constants.

The sequence $X_n, n \geq 0$ is called an autoregression sequence of first order $(AR(1))$.

Let $\Delta(x)$, $x \in R$ be some Borel function and assume for $n \geq 1$

$$H_n = n\Delta\left(\frac{T_n}{n}\right), \quad T_n = \sum_{k=1}^{n} X_{k-1}X_k.$$

Consider the family of the first passage times

$$\tau_a = \inf\{n \geq 1 : H_n > c\} \tag{1}$$

of the process H_n, $n \geq 1$ for the level $c \geq 0$.

The family of stopping times of the form (1) oftenly arises in many applied problems of theory of random processes ([1]-[9]).

A series of asymptotic properties of distribution of boundary functionals connected with the first passage time of the level by the first order autoregression process with discrete and continuous time was studied in the paper [3].

In the present paper, for a rather wide class $\Delta(x)$ we prove an integral limit theorem for τ_a under which one can understand any statement that under some conditions there exist the normalizing constants $A(c)$ and $B(c) > 0$ dependent on the parameter c, for which it is fulfilled the convergence in distribution

$$\frac{\tau_c - A(c)}{B(c)} \xrightarrow{d} \eta \quad as \quad c \to \infty,$$

where η is some non-degenerate random value ([8], [9]).

For the linear case, when $\Delta(x) = x$ an integral limit theorem for the family of first passage times of the form (1) was proved in the paper [1].

2. Formulation and proof of the main result

It holds

Theorem. *Let $E\xi_1 = 0$, $D\xi_1 = 1$, $\beta \in (0,1)$ and let the function $\Delta(x)$ be twice continuously-differentiable in $R = (-\infty, \infty)$, and at the point $x = \lambda = \frac{\beta}{1-\beta^2}$ satisfy the conditions $\Delta(\lambda) > 0$ and $\Delta'(\lambda) \neq 0$.*

Then

$$\lim_{c\to\infty} P\left(\frac{\tau_v - N_c}{\Delta'(\lambda)\sqrt{N_c}} \leq \frac{x}{\Delta(\lambda)}\right) = \Phi(x),$$

where

$$N_c = \frac{c}{\Delta(\lambda)} \quad and \quad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y2/2}dy.$$

In order to prove the theorem we need the following facts formulated in the form of lemmas.

Lemma 1. *Let the conditions of the theorem be fulfilled. Then*

$$\frac{H_n}{n} \overset{a.s}{\to} \Delta(\lambda) \quad as \quad n \to \infty$$

and

$$\lim_{c\to\infty} P\left(\frac{H_n - n\Delta(\lambda)}{\sigma\Delta'(\lambda)\sqrt{n}} \leq x\right) = \Phi(x),$$

where $\sigma^2 = \frac{\beta}{\lambda}$.

Proof. By the assumptions for the function $\Delta(x)$ we have

$$H_n = n\Delta(\lambda) + n\Delta'(\lambda)\left(\overline{T}_n - \lambda\right) + \frac{1}{2}n\Delta''(\nu_n)\left(\overline{T}_n - \lambda\right)^2, \tag{2}$$

where $\overline{T} = \frac{T_n}{n}$ and ν_n is an intermediate point between λ and $\overline{T}_n$.

From (2) we get

$$\frac{H_n - n\Delta(\lambda)}{\Delta'(\lambda)} = T_n - \lambda n + \frac{1}{2}\frac{\Delta''(\nu_n)}{\Delta'(\lambda)}\left(\frac{T_n - n}{\sqrt{n}}\right)$$

or

$$\frac{H_n - n\Delta(\lambda)}{\sigma\Delta'(\lambda)\sqrt{n}} = \frac{T_n - \lambda n}{\sigma\sqrt{n}} + \varepsilon_n, \tag{3}$$

where

$$\varepsilon_n = \frac{\sigma}{2}\frac{\Delta''(\nu_n)}{\sqrt{n}\Delta'(\lambda)}\left(\frac{T_n - n\lambda}{\sigma\sqrt{n}}\right)^2.$$

It is well known that (see [7]),

$$\frac{T_n}{n} \overset{a.s}{\to} \lambda \quad as \quad n \to \infty$$

and

$$\lim_{c\to\infty} P\left(\frac{T_n - \lambda n}{\sigma\sqrt{n}} \leq x\right) = \Phi(x), \quad x \in R. \tag{4}$$

Prove that $\varepsilon_n \overset{P}{\to} o$ as $n \to \infty$.

For this it suffices to show

$$\eta_n = \frac{1}{\sqrt{n}} \left(\frac{T_n - n\lambda}{\sigma\sqrt{n}} \right)^2 \xrightarrow{P} o \qquad as \quad n \to \infty \tag{5}$$

since $\Delta''(\nu_n) \overset{a.s}{\to} \Delta''(\lambda)$ as $n \to \infty$.

Indeed, for any $\varepsilon > 0$ we prove

$$P(|\eta_n| > 0) = P\left(\left| \frac{T_n - n\lambda}{\sigma\sqrt{n}} \right| > \sqrt{\varepsilon\sqrt{n}} \right) \to o \quad as \quad n \to \infty.$$

By the second relation in (4) we have

$$P\left(\left| \frac{T_n - n\lambda}{\sigma\sqrt{n}} \right| < \varepsilon_n \right) - [\Phi(\varepsilon_n) - \Phi(-\varepsilon_n)] \to 0 \tag{6}$$

as $n \to \infty$, where $\varepsilon_n = \sqrt{\varepsilon\sqrt{n}}$.

It is clear that

$$\Phi(\varepsilon_n) \to 1 \quad and \quad \Phi(-\varepsilon_n) \to 0$$

as $n \to \infty$ since $\varepsilon_n \to \infty$, $n \to \infty$. Therefore from (6) it follows (5). Then the second statement of lemma 1 follows from (3).

The first statement of lemma 1 follows from (2) since $\overline{T}_n \overset{a.s}{\to} \lambda$ as $n \to \infty$ [7].

Lemma 2. *Let the conditions of the theorem be fulfilled. Then it holds*

1) $P(\tau_c < \infty) = 1$ *for all* $c \geq 0$*;*

2) $\tau_c \overset{a.s}{\to} \infty$ *as* $n \to \infty$*;*

3) $\frac{\tau_c}{c} \overset{a.s}{\to} \frac{1}{\Delta(\lambda)}$ *as* $c \to \infty$*.*

Proof. It is clear that from the convergence

$$\frac{H_n}{n} \overset{a.s}{\to} \Delta(\lambda) > 0$$

it follows that

$$P\left(\sup_n H_n = \infty \right) = 1.$$

Then the first statement of lemma 2 follows from the equality

$$P(\tau_c < \infty) = P\left(\sup_n H_n > c \right) = 1$$

for all $c \geq 0$.

In order to prove statement 2) it suffices to note that the process τ_c, $c \geq 0$ as a function of c increases and therefore there is a limit

$$\tau_\infty = \lim_{c\to\infty} \tau_c \leq \infty.$$

Then statement 2) follows from the following equality

$$P(\tau_\infty \leq n) = \lim_{c\to\infty} P\left(\sup_{1\leq k\leq n} H_k > c \right) = 0$$

for all $n \geq 1$.

Prove statement 3). Note that by the definition of quantity τ_c it is fulfilled the bilateral inequality

$$\frac{H_{\tau_c - 1}}{\tau_c} \leq \frac{c}{\tau_c} < \frac{T_{\tau_c}}{\tau_c}. \tag{7}$$

On the other hand, it is easy to show that (see also []) by statement 2) of the lemma being proved from the convergence $\frac{H_n}{n} \xrightarrow{a.s} \Delta(\lambda)$ as $n \to \infty$ it follows the convergence $\frac{H_{\tau_c}}{\tau_c} \xrightarrow{a.s} \Delta(\lambda)$ as $c \to \infty$. From (7) we get statement 3) of lemma 2.

Lemma 3. *Let t_a, $a > 0$ be an arbitrary family of integer random variables such that $\frac{t_a}{a} \xrightarrow{P} \theta > 0$ as $a \to \infty$, and let Y_n, $n \geq 1$ be an arbitrary sequence of random variables such that it converges in distribution to some random variable $Y_n \xrightarrow{P} Y$ and the equality*

$$\lim_{\delta \to 0} \sup_{n \geq 1} P\left\{ \max_{0 \leq k \leq n\delta} |Y_{n+k} - Y_n| > \varepsilon \right\} = 0 \tag{8}$$

for any $\varepsilon > 0$ is fulfilled.

Then

$$Y_{t_a} \xrightarrow{d} Y \quad as \quad a \to \infty.$$

The statement of this lemma is a special case of the Anscombe theorem ([8], [9]).

Note that if condition (8) is fulfilled, then it is said that the sequence Y_n, $n \geq 1$ is uniformly continuous in probability ([9]).

Lemma 4. *The sequence*

$$H_n^* = \frac{H_n - n\Delta(\lambda)}{\sqrt{n}}, \quad n \geq 1$$

is uniformly continuous in probability.

Proof. From (2) we have

$$H_n^* = \frac{H_n - n\Delta(\lambda)}{\sqrt{n}} = \Delta'(\lambda) \frac{T_n - n\lambda}{\sqrt{n}} +$$

$$\frac{1}{2} \frac{\Delta''(\nu_n)}{\sqrt{n}} \left(\frac{T_n - n\lambda}{\sqrt{n}} \right)^2 =$$

$$= \Delta'(\nu) T_n^* + \frac{1}{2} \frac{\Delta''(\nu_n)}{\sqrt{n}} (T_n^*)^2, \tag{9}$$

where $T_n^* = \frac{T_n - n\lambda}{\sqrt{n}}$.

In [7] it is proved that the sequence T_n^*, $n \geq 1$ is uniformly continuous in probability.

It is clear that $\Delta''(\nu_n) \xrightarrow{a.s} \Delta''(\lambda)$ as $n \to \infty$, and consequently the sequence $\Delta''(\nu_n)$, $n \geq 1$ is uniformly continuous in probability (see [9]). Then the statement of lemma 4 follows from (9) and lemma 1.4 of the paper [9], since from (4) the sequence T_n^*, $n \geq 1$ is stochastically bounded.

Theorem's proof. We have

$$\frac{H_{\tau_c} - \tau_c \Delta(\lambda)}{\sqrt{\tau_c}} = \frac{c - \tau_c \Delta(\lambda)}{\sqrt{\tau_c}} + \frac{\chi_c}{\sqrt{\tau_c}},$$

where $\chi_c = H_{\tau_c} - c$.

Hence we have

$$\frac{H_{\tau_c} - \tau_c \Delta(\lambda)}{\sigma \Delta'(\lambda)\sqrt{\tau_c}} = \frac{\tau_c - c/\Delta(\lambda)}{\sigma \Delta'(\lambda)\sqrt{\tau_c}} \Delta(\lambda) + \frac{\chi_c}{\sigma \Delta'(\lambda)\sqrt{\tau_c}}. \tag{10}$$

By (4) from lemmas 3 and 4 we get

$$\lim_{c\to\infty} P\left(\frac{H_{\tau_c} - \tau_c \Delta(\nu)}{\sigma \Delta'(\lambda)\sqrt{\tau_c}} \le x\right) = \Phi(x). \tag{11}$$

Prove that the second term in (10) converges to zero in probability, i.e. the following relation is fulfilled:

$$\frac{\chi_c}{\sqrt{\tau_c}} \xrightarrow{P} 0 \quad as \quad c \to \infty. \tag{12}$$

Indeed, from (2) we have

$$H_n - H_{n-1} = \Delta(\lambda) + \Delta'(\lambda)(X_n X_{n-1} - \lambda) + \beta_n - \beta_{n-1}, \tag{13}$$

where

$$\beta_n = \frac{1}{2} n \Delta''(\nu_n)\left(\frac{T_n}{n} - \lambda\right)^2.$$

Show that

$$\frac{H_n - H_{n-1}}{\sqrt{n}} \xrightarrow{P} o \quad as \quad n \to \infty. \tag{14}$$

It is seen from (13) that to prove (14) it suffices to show that

$$\frac{X_n X_{n-1}}{\sqrt{n}} \xrightarrow{P} o \quad as \quad n \to \infty. \tag{15}$$

$$\frac{\beta_n - \beta_{n-1}}{\sqrt{n}} \xrightarrow{P} o. \tag{16}$$

Relation (15) was proved in the paper [1], furthermore, it was shown in this paper that the sequence $\frac{X_n X_{n-1}}{\sqrt{n}}$, $n \ge 1$ is uniformly continuous in probability.

From the stochastic boundedness of the sequence T_n^*, $n \ge 1$ it follows that

$$\frac{\beta_n}{\sqrt{n}} = \frac{1}{2}\left(\frac{T_n - n\lambda}{\sqrt{n}}\right)\left(\frac{T_n}{n} - \lambda\right) \xrightarrow{P} o \quad as \quad n \to \infty. \tag{17}$$

Since $\frac{T_n}{n} \xrightarrow{a.s} \lambda$. Thus, (14) follows from (16) and (17). It is clear that by the mentioned lemma 1.4 from [9] the sequence $\frac{\beta_n}{\sqrt{n}}$, $n \ge 1$ is uniformly continuous is probability.

Therefore, it follows from equality (13) that the sequence $\frac{H_n - H_{n-1}}{\sqrt{n}}$, $n \ge 1$ is also uniformly continuous in probability. Then from (13) and lemma 3 we have

$$\frac{H_{\tau_c} - H_{\tau_c}}{\sqrt{\tau_c}} \xrightarrow{P} o \quad as \ \ c \to \infty. \tag{18}$$

Thus, relation (12) follows from the following estimation

$$0 \leq \chi_c = H_{\tau_c} - c \leq H_{\tau_c} - H_{\tau_c - 1}$$

and relation (18).

Then the theorem statement follows from (10), (11) and (12).

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