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TO THEORY OF SOBAILITY OF FOURTH ORDER OPERATOR - DIFFERENTIAL EQUATIONS

Abstract

In the paper, sufficient conditions providing regular solvability of a boundary value problem containing an operator coefficient, for an elliptic type operator-differential operator of fourth order are found. These coefficient are expressed by the coefficient of the boundary value problem.

Let A be a positive-definite self-adjoint separable operator in separable Hilbert space H, and H_γ be a scale of Hilbert spaces generated by the operator A i.e. $H_\gamma = D(A^\gamma), (x,y)_\gamma = (A^\gamma x, A^\gamma y), \ x,y \in H_\gamma \quad (\gamma \geq 0)$ .For $\gamma = 0$, we assume that $H_0 = H, \ (x,y)_0 = (x,y)$.

Denote by $L_2(R_+;H)$ a Hilbert space of functions $f(t)$ determined in $R_+ = (0,\infty)$ almost everywhere , with the values in H , for which

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^\infty \|f(t)\|^2\right)^{1/2} < \infty.$$

Let

$$W_2^4(R_+;H) = \left\{u : u^{(4)} \in L_2(R_+;H), \ A^4 u \in L_2(R_+;H)\right\}$$

be Hilbert space with the norm [1]

$$\|u\|_{W_2^4(R_+;H)} = \left(\left\|u^{(4)}\right\|^2_{L_2(R_+;H)} + \left\|A^4 u\right\|^2_{L_2(R_+;H)}\right)^{1/2}$$

Here and in the sequel , the derivatives are understood in the sense of distributions theory. Denote by $L(X,Y)$ a space of linear bounded operators acting from the space X to the space Y, and suppose that the operator $S \in L(W_2^4(R_+;H), H_{5/2})$. Then from the theorem on traces it follows that the sub-space

$$W_{2,S}^4(R_+;H) = \left\{u : u \in W_2^4(R_+;H), \ u(0) = 0, \ u'(0) = Su\right\}$$

was well- defined.

Consider in H the boundary value problem

$$P(d/dt)\,u(t) = u^{(4)}(t) + A^4 u(t) + \sum_{j=0}^{4} A_{4-j} u^{(j)}(t) = f(t), \ t \in R_+, \qquad (1)$$

$$u(0) = 0, \ u'(0) = Su, \qquad (2)$$

where $f(t), u(t) \in H$, for $t \in R_+$ almost everywhere, and the operator coefficients satisfy the conditions:

1) A is a positive-definite self –adjoint operator;

2)The operators A_j are linear, $B_j = A_j A^{-j}$ $\left(j = \overline{0,4}\right)$ are bounded in H;

3)The operators $S \in L(W_2^4(R_+;H), H_{5/2})$, and $\|S\|_{W_2^4(R_+;H)\to H_{5/2}} = \chi$.

Definition 1. *In for any $f \in W_2^4(R_+;H)$ there exists the function $(u)t \in W_2^4(R_+;H)$ that satisfies equation (1) almost everywhere in R_+, boundary conditions (2) in the sense of convergence*

$$\lim_{t\to+0} \|u(t)\|_{7/2} = 0, \ \lim_{t\to+0} \left\|u'(t) - Su\right\|_{5/2} = 0,$$

and it holds the estimation

$$\|u\|_{W_2^4(R_+;H)} \le const \, \|f\|_{L_2(R_+;H)},$$

problem (1),(2) is said to be regularly solvable.

In the present paper, we'll find sufficient conditions on the coefficients of boundary value (1),(2) that provide regular solvability of problem (1),(2). Note that for $S = 0$ this problem was studied in the paper [2]. When $Su = T_1 u(0)$, where $T_1 \in L\left(H_{7/2}, H_{5/2}\right)$, the problem was investigated in [3] normal solvability of problem (1),(2) was studied in [4]. For operator-differential equations of second and third order, such non-local problems were studied for example, in the papers [6,7].

Denote by

$$P_0 u = P_0(d/dt)\, u = u^{(4)} + A^4 u, \quad P_1 u = \sum_{j=0}^{4} A_{4-j} u^{(j)},$$

$$Pu = P_0 u + P_1 u, \quad u \in W_{2,S}^4(R_+;H).$$

It is obvious that subject to the conditions 1)-3), the operators P_0, P_1 and P are linear bounded operators acting from $W_{2,S}^4(R_+;H)$ to $L_2(R_+;H)$.

Note also that problem (1) (2) for $A_j = 0$ $(j = 0, 4)$ was studied in [5] and the following fact were proved.

Lemma 1[5]. *Let* $\omega_1 = -\frac{1}{\sqrt{2}}(1+i)$, $\omega_2 = -\frac{1}{\sqrt{2}}(1-i)$. *Then for $x \in H_{7/2}$ it holds the following equalties*

$$\left\|A^4 e^{\omega_i t A} x\right\|_{L_2(R_+;H)} = \frac{1}{\sqrt{2}} \|x\|_{7/2}, \ i = 1,2 \tag{3}$$

$$\left\|e^{\omega_1 t A} x - e^{\omega_2 t A} x\right\|_{W_2^4(R_+;H)} = \sqrt[4]{2}\, \|x\|_{7/2}. \tag{4}$$

Theorem 1[5]. *Let conditions 1) and 3) be fulfilled, and $\chi < \sqrt[4]{2}$ .Then the operator P_0 realizes isomorphism between the spaces $W_{2,S}^4(R_+;H)$ and $L_2(R_+;H)$.*

In this paper it was shown that the operator Q acting in $H_{7/2}$ in the following way

$$Qx = \frac{i}{\sqrt{2}} A^{-1} S \left(e^{\omega_1 t A} x - e^{\omega_2 t A} x \right) \tag{5}$$

is bounded also subject to the condition of theorem 1

$$\|Q\|_{H_{7/2} \to H_{7/2}} \leq \frac{\chi}{\sqrt[4]{2}} = q < 1. \tag{6}$$

From this theorem it follows

Corollary 1. *Let conditions 1) be fulfilled. Then problem*

$$P_0 (d/dt) u(t) = u^{(4)}(t) + A^4 u(t) = f(t), \tag{7}$$

$$u(0) = u'(0) = 0 \tag{8}$$

is regularly solvable.

Now using the results of the paper [8], prove the following

Lemma 2. *Let $\xi(t)$ be a regular solution of problem (7) (8), and*

$$|||\xi||| = \left(\|\xi\|_{L_2(R_+;H)} + 2 \left\| A^2 \xi'' \right\|^2_{L_2(R_+;H)} + \left\| A^4 \xi \right\|^2_{L_2(R_+;H)} \right)^{1/2}.$$

Then it holds the following inequality

$$\left\| A^{4-j} \xi^{(j)} \right\|_{L_2(R_+;H)} \leq c_j \, |||\xi||| \, , \; j = \overline{0,4}, \tag{9}$$

where $c_0 = c_4 = 1, \; c_1 = \frac{3^{3/4}}{4}, \; c_2 = \frac{1}{2}, c_3 = \frac{1}{\sqrt[4]{3}}.$

Proof. By the theorem on intermediate derivatives [1], the norms $\|\xi\|_{W_2^4(R_+:H)}$ and $|||\xi|||$ are equivalent. Then by integration by parts we see that for $\xi(t)$ it holds the equality

$$\|P_0(d/dt)\xi\|_{L_2(R_+;H)} = \left\| \xi^{(4)} + A^4 \xi \right\|_{L_2(R_+;H)} =$$

$$= \left\| \xi^{(4)} \right\|^2_{L_2(R_+;H)} + \left\| A^4 \xi \right\|^2_{L_2(R_+;H)} + 2 \operatorname{Re} \left(\xi^{(4)}, A^4 \xi \right)_{L_2(R_+;H)} =$$

$$= \left\| \xi^{(4)} \right\|^2_{L_2(R_+;H)} + 2 \left\| A^2 \xi'' \right\|^2_{L_2(R_+;H)} + \left\| A^4 \xi \right\|^2_{L_2(R_+;H)} = |||\xi|||^2. \tag{10}$$

Here we used that $\xi(0) = \xi'(0) = 0$.

From equality (10) it follows that inequality (9) is valid for $j = 0$ and $j = 4$. Following [8], for proving inequality (9), for $j = 1, j = 2$ and $j = 3$ we consider the polynomial bundles of operators

$$P_j(\lambda; \beta; A) = \left(\lambda^8 E + 2\lambda^4 A^4 + A^4 \right) - \beta (i\lambda)^{2j} A^{8-2j}, \; j = 1, 2, 3.$$

The operator bundles $P_j(\lambda; \beta)$ for $\beta \in \lfloor 0, d_{4,j})$, where $d_{4,j} = \dfrac{16}{27^{1/2}}$ for $j = 1$, $j = 3$ and $d_{4,j} = 4$ for $j = 2$ are represented in the form

$$P_j(\lambda; \beta; A) = \phi_j(\lambda; \beta; A) \cdot \phi_j(-\lambda; \beta; A) - \beta (i\lambda)^{2j} A^{8-2j}, \; j = 1, 2, 3, \tag{11}$$

$$\phi_j(\lambda;\beta;A)=\prod_{k=1}^{4}(\lambda_k E-\eta_k(\beta)A)=\lambda^4E+A^4+\sum_{k=1}^{3}\alpha_{k,j}(\beta)\lambda^4A^{4-k},$$

where $\operatorname{Re}\eta_k(\beta)<0,\alpha_k(\beta)>0$. Then from (11) it follows that the coefficients $\alpha_{kj}(\beta)$ satisfy the conditions for $j=1$

$$\alpha_{11}^2-2\alpha_{21}=-\beta,\ \alpha_{21}^2=2\alpha_{11}\alpha_{31},\ \ \alpha_{31}^2=2\alpha_{21}; \tag{12}$$

for $j=2$

$$\alpha_{12}^2=2\alpha_{22},\ \alpha_{22}^2-2\alpha_{12}=-\beta,\ \ \ \alpha_{32}^2=2\alpha_{22}; \tag{13}$$

for $j=3$

$$\alpha_{13}^2=2\alpha_{23},\ \alpha_{23}^2=2\alpha_{13}\alpha_{33},\ \ \ \alpha_{33}^2-2\alpha_{23}=-\beta. \tag{14}$$

From the results of [8] it follows that for finding the exact values of the number c_j in the inequality (9),it is necessary to solve the system of equations (13) –(14) together with the equation

$$\det\begin{pmatrix}\alpha_{3j}\alpha_{2j}-\alpha_{1j} & \alpha_{2j}\\ \alpha_{2j} & \alpha_{3j}\end{pmatrix}=0.$$

If this system has no solution from the interval $(0,d_{4,j})$, then $c_j=d_{4,j}^{-1/2}$, and if the system has no solution from the interval $(0,d_{4,j})$,and $\beta_{0,j}$ is the least of these , then $c_j=\beta_{0,j}^{-1/2}$.

There fore $j=1$ we get the following system of equations $\alpha_{31}^2\alpha_{21}-\alpha_{11}\alpha_{31}=\alpha_{21}^2,\ \alpha_{11}^2-2\alpha_{21}=-\beta,\ \alpha_{21}^2=2\alpha_{11}\alpha_{31},\ \alpha_{31}^2=2\alpha_{21}$. Taking into account the fourth equation in the first one, we get $\alpha_{11}^2-\alpha_{11}\alpha_{31}=\alpha_{21}^2$, or $\alpha_{21}^2=\alpha_{11}\alpha_{31}$. Then from the third equation it follows $2a_{11}a_{31}=a_{11}a_{31}$. Since $a_{11}a_{31}>0$, the system has no solution, i.e.$c_1=\frac{3^{3/4}}{4}$.

For $j=2$ we have the system of equalities

$$\alpha_{31}^2\alpha_{22}-\alpha_{12}\alpha_{32}=\alpha_{22}^2,\ \alpha_{12}^2=2\alpha_{22},\ \alpha_{22}^2-2\alpha_{12}\alpha_{32}=-\beta,\ \alpha_{32}^2=2\alpha_{22}.$$

Hence we have $a_{32}=a_{12},\alpha_{22}^2=\alpha_{12}\alpha_{32}=\alpha_{12}^2=2\alpha_{22}$, i.e.$\alpha_{22}=2$. Then $\beta=2\ \alpha_{12}^2-\ \alpha_{22}^2=\ \alpha_{22}^2=4\notin(0,4)$, i.e. $c_2=\frac{1}{2}$.

For $j=3$ we have

$$\alpha_{33}^2\alpha_{23}-\alpha_{13}\alpha_{33}=\alpha_{23}^2,\ \alpha_{13}^2=2\alpha_{23},\ \alpha_{23}^2=2\alpha_{13}\alpha_{33},,\ \alpha_{33}^2-2\alpha_{23}=-\beta.$$

From the first equation, allowing for the fourth equation, it follows that

$$(2\alpha_{23}-\beta)\alpha_{23}-\frac{\alpha_{23}^2}{2}=\alpha_{23}^2,\ \text{i.e. }\alpha_{23}=2\beta,\ \text{then }\ \alpha_{13}=2\beta^{1/2}\ ,\alpha_{33}=\beta^{3/2}.$$

Then from the fourth equation it follows $\beta^2=3$, i.e. $\beta_{0,3}=\sqrt{3},c_3=\frac{1}{\sqrt[4]{3}}$. The lemma is proved.

Now prove the following

Theorem 2. *Let conditions 1) and 2) be fulfilled,* $\chi < \sqrt[4]{2}$. *Then for any* $u \in W_{2,S}^4(R_+;H)$ *it holds the inequality*

$$\left\| A^{4-j}u^{(j)} \right\|_{L_2(R_+;H)} \leq d_j \left\| P_0 u \right\|_{L_2(R_+;H)}, \ j = \overline{0,4}, \qquad (15)$$

where

$$d_0 = d_4 = 1 + \frac{\chi}{\sqrt[4]{2} - \chi}, \ d_1 = \frac{3^{3/4}}{4} + \frac{\chi}{\sqrt[4]{2} - \chi},$$

$$d_2 = \frac{1}{2} + \frac{\sqrt{3}\chi}{\sqrt[4]{2} - \chi}, \ d_3 = \frac{1}{\sqrt[4]{3}} + \frac{\sqrt{3}\chi}{\sqrt[4]{2} - \chi}.$$

Proof. Since $P_0 u = P_0(d/dt)\xi = f$, then $u \in W_{2,S}^4(R_+;H)$ is represented in the form

$$u(t) = \xi(t) + e^{\omega_1 tA}x_1 + e^{\omega_2 tA}x_2,$$

where $\omega_1 = -\frac{1}{\sqrt{2}}(1+i)$, $\omega_2 = -\frac{1}{\sqrt{2}}(1-i)$, $x_1, x_2 \in H_{7/2}$ are unknown vectors to be defined. Obviously $\left(\xi(0) = 0, \xi'(0) = 0\right)$, and $u(0) = \xi(0) + x_1 + x_2 = x_1 + x_2$ and $u'(0) = \omega_1 A x_1 + \omega_2 A x_2$. Then from condition (2) it follows that $x_2 = -x_1$ $(\omega_1 - \omega_2)x_1 = A^{-1}S\left(e^{\omega_1 tA}x_1 - e^{\omega_2 tA}x_2\right) + A^{-1}S(\xi(t))$. Consequently for x_1 we get the equation $(E - Q)x_1 = \psi$, where $\psi = A^{-1}S(\xi(u)) \in H_{7/2}$. Then $x_1 = (E+Q)^{-1}\psi$, $x_2 = -(E+Q)^{-1}\psi$ .Obviously $x_1 x_2 \in H_{7/2}$ .Therefore $u(t) \in W_{2,S}^4(R_+ : H)$. Then from the representation $u(t)$ it follows that

$$\left\| A^{4-j}u^{(j)} \right\|_{L_2(R_+;H)} \leq \left\| A^{4-j}\xi^{(j)} \right\|_{L_2(R_+;H)} + \left\| A^4(\omega_1^j e^{\omega_1 tA}x_1 - \omega_2^j e^{\omega_2 tA}x_1) \right\|_{L_2(R_+;H)}. \qquad (16)$$

On the other hand, from lemma 1

$$\left\| A^4(\omega_1^j e^{\omega_1 tA}x_1 - \omega_2^j e^{\omega_2 tA}x_1) \right\|_{L_2(R_+;H)} = \left\| A^4 e^{\omega_1 tA}x_1 \right\|_{L_2(R_+;H)}^2 +$$

$$+ \left\| A^4 e^{\omega_2 tA}x_1 \right\|_{L_2(R_+;H)}^2 - 2\operatorname{Re}\omega_1^{2j}\left(A^4 e^{\omega_1 tA}x_1, A^4 e^{\omega_2 tA}x_1\right)_{L_2(R_+;H)} \leq$$

$$\leq \sqrt{2}\left\| x_1 \right\|_{7/2}^2 - 2\operatorname{Re}\omega_1^{2j}\left(A^4 e^{\omega_1 tA}x_1, A^4 e^{\omega_2 tA}x_1\right)_{L_2(R_+;H)}. \qquad (17)$$

Further, assuming $A^{7/2}x_1 = y \in H$ and from the spectral expansion of the operator A after simple calculations we get

$$-2\operatorname{Re}\omega_1^{2j}\left(A^4 e^{\omega_1 tA}x_1, A^4 e^{\omega_2 tA}x_1\right) = -2\operatorname{Re}\omega_1^{2j}\int_0^\infty \left(A^{1/2}e^{\omega_1 tA}y, A^{1/2}e^{\omega_2 tA}y\right)dt =$$

$$= -2\operatorname{Re}\omega_1^{2j}\left(\int_0^\infty \left(Ae^{2\omega_1 tA}y, y\right)dt\right) = -2\operatorname{Re}\omega_1^{2j}\int_0^\infty\left(\int_{\mu_0}^\infty \mu e^{2\omega_1 t\mu}\left(dE_\mu y, y\right)dt\right) =$$

$$= - - 2\,\mathrm{Re}\,\omega_1^{2j}\int\limits_{\mu_0}^{\infty}\mu\Big(\int\limits_0^{\infty}e^{2\omega_1 t\mu}dt\,(dE_\mu y,y)\Big)=k_j\,\|x\|_{7/2}^2\,, \tag{18}$$

where $k_0=k_1=k_4=-\frac{1}{\sqrt{2}}, k_2=k_3=\frac{1}{\sqrt{2}}$.

Then, taking into account (18) in (17), we get

$$\left\|A^4\left(\omega_1^j e^{\omega_1 tA}x-\omega_2^j e^{\omega_2 tA}x\right)\right\|_{L_2(R_+;H)}^2\le\left(\sqrt{2}+k_j\right)\|x\|_{7/2}^2\,,\ j=\overline{0,4}. \tag{19}$$

Consequently, using inequality (19) lemma 2, allowing for

$$\left\|(E+Q)^{-1}\right\|\le\frac{1}{1-q},\ q=\frac{\chi}{\sqrt{2}}<1,$$

we get

$$\left\|A^{4-j}u^{(j)}\right\|_{L_2(R_+;H)}\le c_j\,|||\xi|||+\left(\sqrt{2}+k_j\right)^{1/2}\|x_1\|_{7/2}\le c_j\,|||\xi|||+$$

$$+\left(\sqrt{2}+k_j\right)^{1/2}\left\|(E+Q)^{-1}\psi\right\|_{7/2}\le c_j\,|||\xi|||+\frac{\left(\sqrt{2}+k_j\right)^{1/2}}{1-q}\,\|\psi\|_{7/2}=$$

$$\le c_j\,|||\xi|||+\frac{\left(\sqrt{2}+k_j\right)^{1/2}}{1-q}\left\|A^1 S\left(\xi\left(t\right)\right)\right\|_{7/2}\le$$

$$\le c_j\,|||\xi|||+\frac{\left(\sqrt{2}+k_j\right)^{1/2}}{1-q}\chi\,\|\psi\|_{W_2^4(R_+;H)}\le\left(c_j+\frac{\chi\left(\sqrt{2}+k_j\right)^{1/2}\cdot 2^{1/4}}{\sqrt[4]{2}-\chi}\right)|||\xi|||$$

$$=d_j\,|||\xi|||=d_j\,\|P_0(d/dt)\xi\|_{L_2(R_+;H)}=d_j\,\|P_0u\|_{L_2(R_+;H)}\,.$$

The theorem is proved.

Now prove the main theorem.

Theorem 3. *Let conditions 1) -3), $\chi<\sqrt[4]{2}$ be fulfilled, and it hold the inequality*

$$d=\sum_{j=0}^{4}d_j\,\|B_{4-j}\|<1,$$

where the numbers d_j are determined from theorem 2. Then problem (1),(2) is regularly solvable.

Proof. Write problem in(1),(2) the form of the equation $Pu=P_0u+P_1u=f$, where $f\in L_2\left(R_+;H\right),\ u\in W_{2,S}^4\left(R_+;H\right)$. Since P_0^{-1} exists and is bounded (theorem 1), then after substitution of $\omega=P_0u$ we get the equation $\omega+P_1P_0^{-1}\omega=f$ in $L_2\left(R_+;H\right)$. Then allowing for theorem 2 we get

$$\left\|P_1P_0^{-1}\omega\right\|_{L_2(R_+;H)}=\|P_1u\|_{L_2(R_+;H)}\le\sum_{j=0}^{4}\left\|A_{4-j}u^{(j)}\right\|_{L_2(R_+;H)}\le$$

$$\leq \sum_{j=0}^{4} \|B_{4-j}\| \left\| A^{4-j} u^{(j)} \right\|_{L_2(R_+;H)} \leq \left(\sum_{j=0}^{4} d_j \|B_{4-j}\| \right) \|P_0 u\|_{L_2(R_+;H)} =$$

$$= d \|P_0 u\|_{L_2(R_+;H)} = d \|\omega\|_{L_2(R_+;H)} .$$

According to the theorem condition $d < 1$, therefore $\omega = \left(E + P_1 P_0^{-1}\right) f$ and $u = P_0^{-1} \left(E + P_1 P_0^{-1}\right)^{-1} f$ $\|u\|_{W_2^4(R_+;H)} \leq const \|f\|_{L_2(R_+;H)}$. The theorem is proved.

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