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CUBIC FORMULA FOR THE NORMAL DERIVATIVE OF A DOUBLE LAYER ACOUSTIC POTENTIAL

Abstract

In the paper, a cubic formula is constructed for the normal derivative of a double layer acoustic potential.

It is known that the Dirichlet and Neumann external boundary value problems and the Helmholtz equation and others (see [1]) are reduced to a singular integral equation dependent on the normal derivative of a double layer acoustic potential:

$$T(x) = \frac{\partial}{\partial \overrightarrow{n}(x)} \left(\int_S \frac{\Phi_k(x,y)}{\partial \overrightarrow{n}(y)} \cdot \rho(y) dS_y \right), \ x \in S,$$

where $S \subset R^3$ is Lyapunov's surface with the exponent α, $\overrightarrow{n}(x)$ is the external unit normal at the point $x \in S$, $\Phi_k(x,y) = \dfrac{\exp(ik|x-y|)}{4\pi|x-y|}$, $x \neq y$, is the fundamental solution of the Helmholtz equation, k is a wave number, $\mathrm{Im}\, k \geq 0$, and $\rho(y)$ is a continuously differentiable function on S. As it is impossible to find the exact solution to these equations, , there arises an interest to ground the collocation method for such equations. To this end, at first it is necessary to construct the cubic formula for the normal derivative of a double layer acoustic potential.

Introduce the sequence $\{h\} \subset R$ of the values of discretization parameter h tending to zero, and partition S into the elementary domains $S = \bigcup_{l=1}^{N(h)} S_l^h$:

(1) for any $l \in \{1,2,...,N(h)\}$ S_l^h is closed, and its point sets S interior with respect to $\overset{0}{S}{}_l^h$ is not empty, and $mes\overset{0}{S}{}_l^h = mesS_l^h$, for $j \in \{1,2,...,N(h)\}$, $j \neq l$ $\overset{0}{S}{}_l^h \bigcap \overset{0}{S}{}_j^h = \varnothing$;

(2) for any $l \in \{1,2,...,N(h)\}$ S_l^h is a connected piece of the surface S with continuous boundary;

(3) for any $l \in \{1,2,...,N(h)\}$ S_l^h $diamS_l^h \leq h$;

(4) for any $l \in \{1,2,...,N(h)\}$ there exists the so-called support point $x_l \in S_l^h$ such that :

(4.1) $r_l(h) \sim R_l(h)$ $\left(r_l(h) \sim R_l(h) \iff C_1 \leq \frac{r_l(h)}{R_l(h)} \leq C_2\right.$, where C_1 and C_2 are positive constants independent of h), here $r_l(h) = \min\limits_{x \in \partial S_l^h} |x - x_l|$ and $R_l(h) = \max\limits_{x \in \partial S_l^h} |x - x_l|$

(4.2)$R_l(h) \leq \frac{d}{2}$, where d is a radius of a standard sphere (see[2]);

(4.3) for any $j \in \{1,2,...,N(h)\}$ $r_j(h) \sim r_l(h)$.

Obviously $r(h) \sim R(h)$, where $R(h) = \max\limits_{l=\overline{1,N(h)}} R_l(h)$, $r(h) = \min\limits_{l=\overline{1,N(h)}} r_l(h)$.

Let $S_d(x)$ and $\Gamma_d(x)$ be the parts of the surface S and the tangential plane $\Gamma(x)$ at the point $x \in S$ included into the sphere $B_d(x)$ of radius d centered at the point x. Furthermore, let $\widetilde{y} \in \Gamma(x)$ be the projection of the point $y \in S$.

Then

$$\left| x - \widetilde{y} \right| \leq |x - y| \leq C_1(S) \left| x - \widetilde{y} \right| \tag{1}$$

and

$$mes S_d(x) \leq C_2(S)\, mes \Gamma_d(x), \tag{2}$$

where $C_1(S)$and $C_2(S)$ are positive constants dependent only on S (if S a sphere, then $C_1(S) = \sqrt{2}$ and $C_2(S) = 2$). The following lemma is valid.

Lemma (see [3]) . *There exist the constants* $C_0' > 0$ and $C_1' > 0$ *independent of* h *for which for* $\forall l, j \in \{1, 2, ..., N(h)\}$, $j \neq l$ *and* $\forall y \in S_j^h$ *it is valid the following inequality*:

$$C_0' |y - x_l| \leq |x_j - x_l| \leq C_1' |y - x_l|. \tag{3}$$

For the function $g(x)$ continuous on S introduce the modulus of continuity of the form

$$\omega(g, \delta) = \delta \cdot \sup_{\tau \geq \delta} \frac{\overline{\omega}(g, \tau)}{\tau}, \ \delta > 0,$$

where

$$\omega(g, \tau) = \max_{\substack{|x-y| \leq \tau \\ x,y \in S}} |g(x) - g(y)|.$$

Let

$$P_l = \left\{ j \left| 1 \leq j \leq N(h), |x_l - x_j| \leq (R(h))^{\frac{1}{1+\alpha}} \right. \right\},$$

$$Q_l = \left\{ j \left| 1 \leq j \leq N(h), |x_l - x_j| > (R(h))^{\frac{1}{1+\alpha}} \right. \right\}.$$

It is valid the following

Theorem. *Let S be a Lyapunov surface with the exponent* $0 < \alpha \leq 1, \rho(x) -$ *be a continuously- differentiable function on* S, *and* $\int\limits_0^{diamS} \frac{\omega(\text{grad}\,\rho, t)}{t} dt < +\infty$. *Then the expression*

$$T^{N(h)}(x_l) = \sum_{\substack{j=1 \\ j \neq l}}^{N(h)} \frac{\partial}{\partial \overrightarrow{n}(x_l)} \left(\frac{\partial(\Phi_k(x_l, x_j) - \Phi_0(x_l, x_j))}{\partial \overrightarrow{n}(x_j)} \right) \rho(x_j)\, mes S_j^h -$$

$$-\frac{3}{4\pi} \sum_{\substack{j=1 \\ j \neq l}}^{N(h)} \frac{(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_j)) \cdot (\overrightarrow{x_l x_j}, \overrightarrow{n}(x_l))}{|x_l - x_j|^5} (\rho(x_j) - \rho(x_l))\, mes S_j^h +$$

$$+\frac{1}{4\pi} \sum_{j \in Q_l} \frac{(\overrightarrow{n}(x_l), \cdot (x_j))}{|x_l - x_j|^3} (\rho(x_j) - \rho(x_l))\, mes S_j^h$$

at the points $x_l, l = \overline{1, N(h)}$ *is a cubic formula for* $T(x)$, *and the following estimations are valid;*

$$\max_{l=\overline{1,N(h)}} \left| T(x_l) - T^{N(h)}(x_l) \right| \le M \cdot [\|\rho\|_\infty \cdot (R(h))^\alpha + \omega(\rho, R(h)) +$$

$$\left. + \|\mathrm{grad}\, \rho\|_\infty \cdot (R(h))^{\frac{\alpha}{1+\alpha}} + \int_0^{(R(h))^{\frac{1}{1+\alpha}}} \frac{\omega(\mathrm{grad}\, \rho, t)}{t} dt \right], \quad \text{if} \ \ 0 < \alpha < 1,$$

$$\max_{l=\overline{1,N(h)}} \left| T(x_l) - T^{N(h)}(x_l) \right| \le M \cdot [\|\rho\|_\infty \cdot R(h) \,|\cdot \ln(R(h))| + \omega(\rho, R(h)) +$$

$$\left. + \|\mathrm{grad}\, \rho\|_\infty \cdot (R(h))^{\frac{\alpha}{1+\alpha}} + \int_0^{(R(h))^{\frac{1}{1+\alpha}}} \frac{\omega(\mathrm{grad}\, \rho, t)}{t} dt \right], \quad \text{if} \ \ \alpha = 1.$$

Proof. It is known that (see [4])

$$T(x) = \int_S \frac{\partial}{\partial \overrightarrow{n}(x)} \left(\frac{\partial(\Phi_k(x,y) - \Phi_0(x,y))}{\partial \overrightarrow{n}(y)} \right) dS_y -$$

$$- \frac{3}{4\pi} \int_S \frac{(\overrightarrow{xy}, \overrightarrow{n}(y)) \cdot (\overrightarrow{xy}, \overrightarrow{n}(x))}{|x-y|^5} (\rho(y) - \rho(x)) \, dS_y +$$

$$+ \frac{1}{4\pi} \int_S \frac{(\overrightarrow{n}(y), (x_j))}{|x-y|^3} (\rho(y) - \rho(x)) \, dS_y. \tag{4}$$

Denote the additive integrals in equality (4) by $L(x), F(x)$ and $G(x)$, respectively.

The integral $L(x)$ is weakly – singular, therefore it is easy to prove that the expression

$$L^{N(h)}(x_l) = \sum_{\substack{j=1 \\ j \ne l}}^{N(h)} \frac{\partial}{\partial \overrightarrow{n}(x_l)} \left(\frac{\partial(\Phi_k(x_l, x_j) - \Phi_0(x_l, x_j))}{\partial \overrightarrow{n}(x_j)} \right) \rho(x_j) \, mesS_j^h \tag{5}$$

at the points $x_l, l = \overline{1, N(h)}$ is a cubic formula for the integral $L(x)$, and the following estimations are valid:

$$\max_{l=\overline{1,N(h)}} |r(L, x_l)| = \max_{l=\overline{1,N(h)}} \left| L(x_l) - L^{N(h)}(x_l) \right| \le$$

$$\le M \cdot \lfloor \|\rho\|_\infty (R(h))^\alpha + \omega(\rho, R(h)) \rfloor, \quad \text{if} \ \ 0 < \alpha < 1,$$

$$\max_{l=\overline{1,N(h)}} |r(L, x_l)| = \max_{l=\overline{1,N(h)}} \left| L(x_l) - L^{N(h)}(x_l) \right| \le$$

$$\le M \cdot [\|\rho\|_\infty R(h) \,|\ln(R(h))| + \omega(\rho, R(h))], if \ \ \alpha = 1.$$

Since the function $\rho(x)$ is continuously – differentiable, then there exists a point $y^* = x + \theta \cdot (y - x)$ (here $\theta = (\theta_1, \theta_2, \theta_3)$ and $\theta_i \in [0,1]\,, i = \overline{1,3}$) such that

$$\rho(y) - \rho(x) = \left(\mathrm{grad}\,\rho(y^*)\,, \overrightarrow{xy}\right), \; x, y \in S. \tag{6}$$

Then

$$\max_{l=\overline{1,N(h)}} |r(L, x_l)| \le M \cdot \left\lfloor \|\rho\|_\infty (R(h))^\alpha + \|\mathrm{grad}\,\rho\|_\infty \cdot R(h) \right\rfloor, \quad \text{if} \;\; 0 < \alpha < 1,$$

$$\max_{l=\overline{1,N(h)}} |r(L, x_l)| \le M \cdot \left[\|\rho\|_\infty R(h) \left|\ln(R(h))\right| + \|\mathrm{grad}\,\rho\|_\infty \cdot R(h) \right], \quad \text{if} \;\; \alpha = 1.$$

Let's construct the cubic formula for the integral $F(x)$. The expression

$$F^{N(h)}(x_l) = -\frac{3}{4\pi} \sum_{\substack{j=1 \\ j \ne l}}^{N(h)} \frac{\left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_j)\right) \cdot \left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_l)\right)}{|x_j - x_l|^5} \left(\rho(x_j) - \rho(x_l)\right) mes S_j^h \tag{7}$$

at the points $x_l, l = \overline{1, N(h)}$ is a cubic formula for the integral $F(x)$. Estimate the error by the cubic formula (7). Obviously,

$$r(F, x_l) = F(x_l) - F^{N(h)}(x_l) = -\frac{3}{4\pi} \int_{S_l^h} \frac{\left(\overrightarrow{x_l y}, \overrightarrow{n}(y)\right) \cdot \left(\overrightarrow{x_l y}, \overrightarrow{n}(x_l)\right)}{|x_l - y|^5} \left(\rho(y) - \rho(x_l)\right) d\sigma_y -$$

$$-\frac{3}{4\pi} \sum_{\substack{j=1 \\ j \ne l}}^{N(h)} \int_{S_j^h} \frac{\left(\overrightarrow{x_l y}, \overrightarrow{n}(y)\right) \cdot \left(\overrightarrow{x_l y}, \overrightarrow{n}(x_l)\right) - \left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_j)\right) \cdot \left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_l)\right)}{|x_j - y|^5} \left(\rho(y) - \rho(x_l)\right) d\sigma_y -$$

$$-\frac{3}{4\pi} \sum_{\substack{j=1 \\ j \ne l}}^{N(h)} \int_{S_j^h} \left(\frac{1}{|x_l - y|^5} - \frac{1}{|x_l - x_j|^5} \right) \left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_j)\right) \cdot \left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_l)\right) \times$$

$$\times \left(\rho(y) - \rho(x_l)\right) d\sigma_y - \frac{3}{4\pi} \sum_{\substack{j=1 \\ j \ne l}}^{N(h)} \int_{S_j^h} \frac{\left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_j)\right) \cdot \left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_l)\right)}{|x_l - x_j|^5} \times$$

$$\times \left(\rho(y) - \rho(x_j)\right) d\sigma_y = r_1(F, x_l) + r_2(F, x_l) + r_3(F, x_l) + r_4(F, x_l).$$

Taking into account (6), and applying the formula of reduction of the surface integral to double one, we have

$$|r_1(L, x_l)| \le M \cdot \|\mathrm{grad}\,\rho\|_\infty \int_S \frac{1}{|x_l - y|^{2-2\alpha}} d\sigma_y \le M \cdot \|\mathrm{grad}\,\rho\|_\infty \cdot (R(h))^{2\alpha}.$$

Let $y \in S_l^h$ and $j \ne l$, then taking into account (3) and (6) we get:

$$\left| \frac{\left(\overrightarrow{x_l y}, \overrightarrow{n}(y)\right) \cdot \left(\overrightarrow{x_l y}, \overrightarrow{n}(x_l)\right) - \left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_j)\right) \cdot \left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_l)\right)}{|x_l - y|^5} \cdot \left(\rho(y) - \rho(x_l)\right) \right| =$$

$$= \left| \frac{\left(\left(\overrightarrow{x_j y}, \overrightarrow{n}(y)\right) + \left(\overrightarrow{x_l x_j}, \overrightarrow{n}(y) - \overrightarrow{n}(x_j)\right)\right)\left(\overrightarrow{x_l y}, \overrightarrow{n}(x_l)\right)}{|x_l - y|^5} + \right.$$

$$\left. + \frac{\left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_j)\right)\left(\overrightarrow{(x_j y}, \overrightarrow{n}(x_l)\right) - \overrightarrow{n}(y)) + \left(\left(\overrightarrow{x_j y}, \overrightarrow{n}(y)\right)\right)}{|x_l - y|^5} \right| \times$$

$$\times \left|\left(\rho(y) - \rho(x_l)\right)\right| \le M \cdot \|\mathrm{grad}\, \rho\|_\infty \cdot \frac{(R(h))^\alpha}{|x_l - y|^{2-\alpha}}.$$

Hence we find

$$|r_2(F, x_l)| \le M \cdot (R(h))^\alpha \|\mathrm{grad}\, \rho\|_\infty \int\limits_{S \backslash S_l^h} \frac{1}{|x_l - y|^{2-\alpha}} d\sigma_y \le M \cdot \|\mathrm{grad}\, \rho\|_\infty \cdot (R(h))^\alpha.$$

We take into attention (3) and (6), get $\forall y \in S_j^h, j \ne l$

$$\left| \left(\frac{1}{|x_l - y|^5} - \frac{1}{|x_l - x_j|^5} \right) \cdot \left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_j)\right) \cdot \left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_l)\right) \cdot \left(\rho(y) - \rho(x_l)\right) \right| \le$$

$$\le M \cdot \|\mathrm{grad}\, \rho\|_\infty \cdot \frac{(R(h))}{|x_l - y|^{3-\alpha}}$$

and

$$\left| \frac{\left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_j)\right) \cdot \left(\overrightarrow{x_l x_j}, \overrightarrow{n}(x_l)\right)}{|x_j - x_j|^5} \cdot \left(\rho(y) - \rho(x_j)\right) \right| \le M \cdot \|\mathrm{grad}\, \rho\|_\infty \cdot \frac{(R(h))}{|x_l - y|^{3-\alpha}},$$

and so,

$$|r_m(F, x_l)| \le M \cdot (R(h)) \|\mathrm{grad}\, \rho\|_\infty \cdot \int\limits_{S \backslash S_l^h} \frac{1}{|x_l - y|^{3-\alpha}} d\sigma_y \le$$

$$\le M \cdot \|\mathrm{grad}\, \rho\|_\infty \cdot (R(h))^\alpha, if \quad 0 < \alpha < 1,$$

$$|r_m(F, x_l)| \le M \cdot \|\mathrm{grad}\, \rho\|_\infty (R(h)) \cdot |\ln(R(h))|, \ if \ \ \alpha = 1,$$

where $m = \overline{3,4}$.

Summing up the obtained estimations for the expression $r_i(F, x_l), i = \overline{1,4}$, we find

$$\max_{l=\overline{1,N(h)}} |r(F, x_l)| \le M \cdot \|\mathrm{grad}\, \rho\|_\infty (R(h))^\alpha, if \ \ 0 < \alpha < 1,$$

$$\max_{l=\overline{1,N(h)}} |r(F, x_l)| \le M \cdot \|\mathrm{grad}\, \rho\|_\infty R(h) |\ln R(h)|, \ if \ \ \alpha = 1.$$

Now let's construct a cubic formula for the integral $G(x)$. The expression

$$G^{N(h)}(x_l) = -\frac{1}{4\pi} \sum_{j \in Q_l} \frac{\left(\overrightarrow{n}(x_l), \overrightarrow{n}(x_j)\right)}{|x_j - x_l|^3} \cdot \left(\rho(x_j) - \rho(x_l)\right) \cdot mes S_j^h \qquad (8)$$

at the points $x_l, l=\overline{1,N(h)}$ is a cubic formula for the integral $G(x)$. Estimate the error by the cubic formula (8). Obviously,

$$r(G,x_l)=G(x_l)-G^{N(h)}(x_l)=\frac{1}{4\pi}\sum_{j\in Q_l}\int_{S_j^h}\left(\frac{\rho(y)-\rho(x_l)}{|x_l-y|^3}\cdot(\overrightarrow{n}(y),\overrightarrow{n}(x_l))-\right.$$

$$\left.-\frac{\rho(x_j)-\rho(x_l)}{|x_l-x_j|^3}\cdot(\overrightarrow{n}(x_j),\overrightarrow{n}(x_l))\right)dS_y+$$

$$+\frac{1}{4\pi}\cdot\int_{\bigcup\limits_{j\in P_l}S_j^h}\frac{\rho(y)-\rho(x_l)}{|x_l-y|^3}\cdot(\overrightarrow{n}(y),\overrightarrow{n}(x_l))\,dS_y=r_1(G,x_l)+r_2(G,x_l).$$

Let $y\in S_j^h, j\in Q_l$, then .

$$\left|\frac{\rho(y)-\rho(x_l)}{|x_l-y|^3}(\overrightarrow{n}(y),\overrightarrow{n}(x_l))-\frac{\rho(x_j)-\rho(x_l)}{|x_l-x_j|^3}\cdot(\overrightarrow{n}(x_j),\overrightarrow{n}(x_l))\right|\le$$

$$\left|\frac{(\rho(y)-\rho(x_l))\cdot(\overrightarrow{n}(y),\overrightarrow{n}(x_l))\cdot\left(|x_l-x_j|^3-|x_l-y|^3\right)}{|x_l-y|^3\cdot|x_l-x_j|^3}\right|+$$

$$+\frac{1}{|x_l-x_j|^3}\cdot|(\rho(y)-\rho(x_j))\cdot(\overrightarrow{n}(y),\overrightarrow{n}(x_l))+$$

$$+(\rho(x_j)-\rho(x_l))\cdot(\overrightarrow{n}(y)-\overrightarrow{n}(x_j),\overrightarrow{n}(x_l))|\le$$

$$\le M\cdot\|\mathrm{grad}\,\rho\|_\infty\cdot\left(\frac{R(h)}{|x_l-y|^3}+\frac{(R(h))^\alpha}{|x_l-y|^2}\right).$$

Hence we find

$$|r_1(G,x_l)|\le\frac{M}{4\pi}\cdot\|\mathrm{grad}\,\rho\|_\infty\cdot\left(R(h)\cdot\int_{(R(h))^{\frac{1}{1+\alpha}}}^{diamS}\frac{dt}{t^2}+(R(h))^\alpha\cdot\int_{(R(h))^{\frac{1}{1+\alpha}}}^{diamS}\frac{dt}{t}\right)\le$$

$$\le M\cdot\|\mathrm{grad}\,\rho\|_\infty(R(h))^{\frac{\alpha}{1+\alpha}}.$$

Since there exists a point $\widetilde{y}_l=x_l+\theta\cdot(y-x_l)$, such that

$$\rho(y)-\rho(x_l)=\left(\mathrm{grad}\,\rho(\widetilde{y}_l),\overrightarrow{x_l y}\right),\ \ y\in\bigcup_{j\in P_l}S_j^h. \tag{9}$$

we can represent the expression $r_2(G,x_l)$ in the form

$$r_2(G,x_l)\le\frac{1}{4\pi}\int_{\bigcup\limits_{j\in P_l}S_j^h}\frac{\left(\mathrm{grad}\,\rho(\widetilde{y}_l),\overrightarrow{x_l y}\right)}{|x_l-y|^3}\cdot(\overrightarrow{n}(y)-\overrightarrow{n}(x_l),\overrightarrow{n}(x_l))\,dS_y+$$

$$+\frac{1}{4\pi}\int\limits_{\underset{j\in P_l}{\cup}S_j^h}\frac{(\operatorname{grad}\rho(\widetilde{y}_l)-\operatorname{grad}\rho(x_l),x_ly)}{|x_l-y|^3}dS_y+\frac{1}{4\pi}\int\limits_{\underset{j\in P_l}{\cup}S_j^h}\frac{(\operatorname{grad}\rho(x_l),x_ly)}{|x_l-y|^3}dS_y=$$

$$=r_{2,1}(G,x_l)+r_{2,2}(G,x_l)+r_{2,3}(G,x_l)\,.$$

Let $y\in\partial\left(\underset{j\in P_l}{\cup}S_j^h\right)$. Then, obviously, there exist $k\in P_l$ and $m\in Q_l$ such that $y\in\partial S_k^h$ and $y\in\partial S_m^h$.Hence we have

$$|x_l-y|\le|x_l-x_k|+|x_k-y|\le(R(h))^{\frac{1}{1+\alpha}}+R(h)$$

and

$$|x_l-y|\ge|x_l-x_m|-|x_m-y|>(R(h))^{\frac{1}{1+\alpha}}-R(h)\,,$$

so,

$$(R(h))^{\frac{1}{1+\alpha}}-R(h)<|x_l-y|\le(R(h))^{\frac{1}{1+\alpha}}+R(h)\,,\ \forall y\in\partial\left(\underset{j\in P_l}{\cup}S_j^h\right). \tag{10}$$

Then

$$|r_{2,1}(G,x_l)|\le M\cdot\|\operatorname{grad}\rho\|_\infty\int\limits_0^{(R(h))^{\frac{1}{1+\alpha}}+R(h)}\frac{dt}{t^{l-\alpha}}\le M\cdot\|\operatorname{grad}\rho\|_\infty(R(h))^{\frac{\alpha}{1+\alpha}}$$

and

$$|r_{2,2}(G,x_l)|\le M\cdot\int\limits_0^{(R(h))^{\frac{1}{1+\alpha}}+R(h)}\frac{\omega(\operatorname{grad}\rho,t)}{t}dt.$$

It is known that (see [2]) $S_d(x_l)$ intersects the straight line, parallel to the normal $\overrightarrow{n}(x_l)$, at a unique point or don't intersect at all, i.e. the set $S_d(x_l)$ is uniquely projected on the set $\Omega_d(x_l)$ lying in the circle of radius d centered at the point x_l on the tangential plane $\Gamma(x_l)$ to S at the point x_l. On the piece $S_d(x_l)$ choose a local right system of coordinates (u,v,w) with origin at the point x_l where the axis w is directed along the normal $\overrightarrow{n}(x_l)$ (the axes u and ν will lie on the tangential plane $\Gamma(x_l)$). Then in these coordinates $S_d(x_l)$ we can give the equation

$$w=f(u,\nu)\,,\ (u,\nu)\in\Omega_d(x_l)\,,$$

moreover

$$f\in C^{1,\alpha}(\Omega_d(x_l))\ \text{ and } f(0,0)=0,\ \frac{\partial f(0,0)}{\partial u}=0,\ \frac{\partial f(0,0)}{\partial\nu}=0. \tag{11}$$

Denote by Ω_l^h the projection of the set $\underset{j\in P_l}{\cup}S_j^h$ on the tangential plane $\Gamma(x_l)$. Let $d_h=\min\limits_{\widetilde{y}\in\partial\Omega_l^h}|x_l-\widetilde{y}|$ and $O_{d_h}(x_l)=\{u^2+\nu^2<d_h\}\subset\Gamma(x_l)$ (obviously $O_{d_h}\subset\Omega_l^h$) . Since

$$\int\limits_{\underset{j\in P_l}{\cup}S_j^h}\frac{(\operatorname{grad}\rho(x_l),\overrightarrow{x_ly})}{|x_l-y|^3}dS_y=$$

$$= \int\limits_{\bigcup\limits_{j\in P_l} S_j^h} \frac{(y_1-x_{1,l})\cdot\frac{\partial\rho(x_l)}{\partial x_1}+(y_2-x_{2,l})\cdot\frac{\partial\rho(x_l)}{\partial x_2}+(y_3-x_{3,l})\cdot\frac{\partial\rho(x_l)}{\partial x_3}}{\left((y_1-x_{1,l})^2+(y_2-x_{2,l})^2+(y_3-x_{3,l})^3\right)^{\frac{3}{2}}}dS_y,$$

then by the formula of reduction of the surface integral to the repeated one, we can represent the expression $r_{2,2}(G,x_l)$ in the form

$$r_{2,2}(G,x_l) = \int\limits_{O_{d_h(x_l)}} \frac{\frac{\partial\rho(x_l)}{\partial x_1}\cdot u+\frac{\partial\rho(x_l)}{\partial x_2}\cdot\nu}{\left(\sqrt{u^2+\nu^2}\right)^3}dud\nu+$$

$$+\int\limits_{O_{d_h(x_l)}} \frac{\frac{\partial\rho(x_l)}{\partial x_1}\cdot f(u,v)}{\left(\sqrt{u^2+\nu^2+f^2(u,v)}\right)^3}\cdot\sqrt{1+\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial\nu}\right)^2}dud\nu+$$

$$+\int\limits_{O_{d_h(x_l)}} \frac{\frac{\partial\rho(x_l)}{\partial x_1}\cdot u+\frac{\partial\rho(x_l)}{\partial x_2}\cdot\nu}{\left(\sqrt{u^2+\nu^2+f^2(u,v)}\right)^3}\cdot\left(\sqrt{1+\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial\nu}\right)^2}-1\right)dud\nu+$$

$$+\int\limits_{O_{d_h(x_l)}} \left(\frac{\partial\rho(x_l)}{\partial x_1}\cdot u+\frac{\partial\rho(x_l)}{\partial x_2}\cdot\nu\right)\times$$

$$\times\left(\frac{1}{\left(\sqrt{u^2+\nu^2+f^2(u,v)}\right)^3}-\frac{1}{\left(\sqrt{u^2+\nu^2}\right)^3}\right)dud\nu+$$

$$+\int\limits_{\Omega_l^h\backslash O_{d_h(x_l)}} \frac{\frac{\partial\rho(x_l)}{\partial x_1}\cdot u+\frac{\partial\rho(x_l)}{\partial x_2}\cdot\nu+\frac{\partial\rho(x_l)}{\partial x_3}\cdot f(u,v)}{\left(\sqrt{u^2+\nu^2+f^2(u,v)}\right)^3}\sqrt{1+\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial\nu}\right)^2}dud\nu. \quad (12)$$

The first additive integral in equality (12) exists in the sense of the Cauchy principal value and equals zero. Indeed, passing to the polar system of coordinates, we find

$$\int\limits_{O_{d_h(x_l)}} \frac{\frac{\partial\rho(x_l)}{\partial x_1}\cdot u+\frac{\partial\rho(x_l)}{\partial x_2}\cdot\nu}{\left(\sqrt{u^2+\nu^2}\right)^3}dud\nu=\lim_{\varepsilon\to+0}\int\limits_{\varepsilon}^{d_h}\int\limits_0^{2\pi}\left(\frac{\frac{\partial\rho(x_l)}{\partial x_1}}{r}\cdot\cos\varphi+\frac{\frac{\partial\rho(x_l)}{\partial x_2}}{r}\cdot\sin\varphi\right)d\varphi dr=0.$$

Furthermore, taking into account the inequalities

$$|f(u,\nu)|\le M\cdot\left(\sqrt{u^2+\nu^2}\right)^{1+\alpha},\quad (u,\nu)\in O_{d_h(x_l)},$$

$$\left|\sqrt{1+f_u^2+f_\nu^2}-1\right|\le M\cdot\left(\sqrt{u^2+\nu^2}\right)^{2\alpha},\quad (u,\nu)\in O_{d_h(x_l)},$$

$$\left|\frac{1}{\left(\sqrt{u^2+\nu^2+f^2(u,v)}\right)^3}-\frac{1}{\left(\sqrt{u^2+\nu^2}\right)^3}\right|\le$$

$$\leq M\cdot\frac{1}{\left(\sqrt{u^2+\nu^2}\right)^{3-2\alpha}},\ (u,\nu)\in O_{d_h(x_l)},(u,\nu)\neq(0,0)\,,$$

and passing to the polar system of coordinates, we have:

$$\left|\int\limits_{O_{d_h(x_l)}}\frac{\frac{\partial\rho(x_l)}{\partial x_3}\cdot f(u,v)}{\left(\sqrt{u^2+\nu^2+f^2(u,v)}\right)^3}\cdot\sqrt{1+\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial \nu}\right)^2}dud\nu\right|\leq$$

$$\leq M\cdot\|\mathrm{grad}\,\rho\|_\infty\,(R(h))^{\frac{\alpha}{1+\alpha}}$$

$$\left|\int\limits_{O_{d_h(x_l)}}\frac{\frac{\partial\rho(x_l)}{\partial x_1}\cdot u+\frac{\partial\rho(x_l)}{\partial x_2}\cdot\nu}{\left(\sqrt{u^2+\nu^2+f^2(u,v)}\right)^3}\cdot\left(\sqrt{1+\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial \nu}\right)^2}-1\right)dud\nu\right|\leq$$

$$\leq M\cdot\|\mathrm{grad}\,\rho\|_\infty\,(R(h))^{\frac{2\alpha}{1+\alpha}}$$

$$\left|\int\limits_{O_{d_h(x_l)}}\left(\frac{\partial\rho(x_l)}{\partial x_1}\cdot u+\frac{\partial\rho(x_l)}{\partial x_2}\cdot\nu\right)\cdot\left(\frac{1}{\left(\sqrt{u^2+\nu^2+f^2(u,v)}\right)^3}-\right.\right.$$

$$\left.\left.-\frac{1}{\left(\sqrt{u^2+\nu^2}\right)^3}\right)dud\nu\right|\leq M\cdot\|\mathrm{grad}\,\rho\|_\infty\,(R(h))^{\frac{2\alpha}{1+\alpha}}\,.$$

Now estimate the last additive integral in equality (12). Since there exists a point $\widetilde{y}_h\in\Omega_l^h$ such that $d_h=|x_l-\widetilde{y}_h|$. Denote by $y_h\in\partial\left(\underset{j\in P_l}{\cup}S_j^h\right)$ preimage of the point $\widetilde{y}_h$. Obviously,

$$d_h=|x_l-y_h|\cos\left(x_ly_hx_l\widetilde{y}_h\right)=|x_l-y_h|\cdot\sqrt{1-\cos^2\alpha\left(x_ly_h,n\left(x_l\right)\right)}\geq$$

$$\geq|x_l-y_h|\cdot\sqrt{1-M^2\cdot|x_l-y_h|^{2\alpha}}\geq\left((R(h))^{\frac{1}{1+\alpha}}-R(h)\right)\times$$

$$\times\sqrt{1-M^2\cdot\left((R(h))^{\frac{1}{1+\alpha}}+R(h)\right)^{2\alpha}}\geq\left((R(h))^{\frac{1}{1+\alpha}}-R(h)\right)\times$$

$$\times\sqrt{1-M^2\cdot\left(2\,(R(h))^{\frac{1}{1+\alpha}}\right)^{2\alpha}}=\left((R(h))^{\frac{1}{1+\alpha}}-R(h)\right)\times$$

$$\times\sqrt{\left(1-2^\alpha\cdot M^2\cdot\left(2\,(R(h))^{\frac{\alpha}{1+\alpha}}\right)\cdot\left(1+2^\alpha\cdot M\cdot(R(h))^{\frac{\alpha}{1+\alpha}}\right)\right.}\geq$$

$$\geq\left((R(h))^{\frac{1}{1+\alpha}}-R(h)\right)\cdot\left(1-2^\alpha\cdot M\cdot(R(h))^{\frac{\alpha}{1+\alpha}}\right)\geq\left((R(h))^{\frac{1}{1+\alpha}}-R(h)\cdot(1+2^\alpha\cdot M\right).$$

Then,

$$\left|\int\limits_{\Omega_l^h\backslash O_{d_h(x_l)}}\frac{\frac{\partial\rho(x_l)}{\partial x_1}\cdot u+\frac{\partial\rho(x_l)}{\partial x_2}\cdot\nu+\frac{\partial\rho(x_l)}{\partial x_3}\cdot f(u,v)}{\left(\sqrt{u^2+\nu^2+f^2(u,v)}\right)^3}\times\sqrt{1+\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial \nu}\right)^2}dud\nu\right|\leq$$

$$\leq M\cdot\left\|\operatorname{grad}\rho\right\|_{\infty}\cdot\int\limits_{(R(h))^{\frac{1}{1+\alpha}}-R(h)(1+2^{\alpha}\cdot M)}^{(R(h))^{\frac{1}{1+\alpha}}+R(h)}\frac{dt}{t}\leq$$

$$\leq M\cdot\left\|\operatorname{grad}\rho\right\|_{\infty}\cdot\frac{R\left(h\right)\cdot\left(2+2^{\alpha}\cdot M\right)}{\left(\left(R\left(h\right)\right)^{\frac{1}{1+\alpha}}-R\left(h\right)\cdot\left(1+2^{\alpha}\cdot M\right)\right)}\leq M\cdot\left\|\operatorname{grad}\rho\right\|_{\infty}\cdot\left(\left(R\left(h\right)\right)^{\frac{\alpha}{1+\alpha}}.$$

As a result we get

$$\left|r_{2,3}\left(G,x_{l}\right)\right|\leq M\cdot\left\|\operatorname{grad}\rho\right\|_{\infty}\cdot\left(\left(R\left(h\right)\right)^{\frac{\alpha}{1+\alpha}}.$$

Summing up the obtained estimations for the expression $r_{2,i}\left(G,x_{l}\right),i=\overline{1,3}$, we find:

$$\left|r_{2}\left(G,x_{l}\right)\right|\leq M\cdot\left[\left\|\operatorname{grad}\rho\right\|_{\infty}\cdot\left(\left(R\left(h\right)\right)^{\frac{\alpha}{1+\alpha}}\right|+\int\limits_{0}^{(R(h))^{\frac{1}{1+\alpha}}+R(h)}\frac{\omega\left(\operatorname{grad}\rho,t\right)}{t}dt\right].$$

And summing up the obtained estimations for the expressions $r_{1}\left(G,x_{l}\right)$ and $r_{2}\left(G,x_{l}\right)$, we have :

$$\max_{l=\overline{1,N(h)}}\left|r(G,x_{l})\right|\leq M\cdot\left[\left\|\operatorname{grad}\rho\right\|_{\infty}\cdot\left(\left(R\left(h\right)\right)^{\frac{\alpha}{1+\alpha}}+\int\limits_{0}^{(R(h))^{\frac{1}{1+\alpha}}+R(h)}\frac{\omega\left(\operatorname{grad}\rho,t\right)}{t}dt\right].$$

Finally, summing up the obtained estimations for the expressions $r\left(L,x_{l}\right)$ $r\left(F,x_{l}\right)$ and $r\left(G,x_{l}\right)$, we get the proof of the theorem.

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