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ON SOME RESULTS ON STABILITY OF $b_{\widetilde{X}}$ -ATOMIC DECOMPOSITION

Abstract

In the paper, as the generalization of atomic decomposition in Banach space with respect to Banach space of scalars sequences, we introduce the notion of b -atomic decomposition in Banach space with respect to Banach space of vectors sequences and study its stability and obtain the results on stability of b -atomic decomposition. Some results obtained in the paper are the generalizations of appropriate known results on stability of atomic decomposition in Banach spaces.

The notion of frame in Hilbert spaces was introduced by R.Y. Duffin and A.C. Schaeffer when studying Fourier nonharmonic series [1] in the following way: the system $\{f_i\}_{i\in N}$ of Hilbert space H is a frame if there exist the constants $A>0$ and $B>0$ satisfying the condition

$$A\left\|f\right\|^2 \leq \sum_{i=1}^{\infty}\left|(f,f_i)\right|^2 \leq B\left\|f\right\|^2 \quad \text{for any} \quad f\in H.$$

The constants A and B are said to be frame bounds. A lot of papers have been devoted to frames and their applications. One can look through theory of frames in the monographs [2,3]. Frames find applications in signal processes, data compression, characterization of functional spaces and in many other fields. Concerning the problems of stability of frame in Hilbert space, the Paley –Wiener type theorems are known [4-7].

In [8], the notion of Banach frame and atomic decomposition in Banach spaces are introduced. Give the definition of atomic decomposition.

Definition 1. *Let X be a Banach space, X_d a Banach space of scalars sequences, $\{x_n\}_{n\in N}\subset X$ and $\{x_n^*\}_{n\in N}\subset X^*$ satisfy the conditions*

(i) $\{x_n^*(x)\}_{n\in N}\in X_d$ for all $x\in X$;

(ii) there exist $A>0$ and $B>0$ such that $A\left\|x\right\|_X \leq \left\|\{x_n^*(x)\}_{n\in N}\right\|_{X_d} \leq B\left\|x\right\|_X$ for all $x\in X$;

(iii) $x=\sum_{i=1}^{\infty}x_n^*(x)\,x_n$ for all $x\in X$.

Then the pair $\left(\{x_n^*\}_{n\in N};\{x_n\}_{n\in N}\right)$ is said to be atomic decomposition in X with respect to X_d with the bounds A and B.

The stability of Banach frames and atomic decompositions in Banach spaces was studied in the papers [9-12].

In the present paper we consider a bounded bilinear mapping b by means of which notion of b-atomic decomposition in Banach space with respect to Banach space of vectors sequences is introduced. The results on stability of b-atomic decomposition are obtained. These results are the generalizations of appropriate results on stability of atomic decomposition in Banach space.

Stability of b-atomic decomposition.

Let X, Y and Z be B-spaces with appropriate norms $\|\|_X$, $\|\|_Y$ and $\|\|_Z$, $b(x,y) : X \times Y \to Z$ be a binear mapping satisfying the condition:

$$\exists M > 0 : \|b(x,y)\|_Z \leq M \|x\|_X, \|y\|_Y, \quad \forall x \in X, \quad \forall y \in Y. \qquad (1)$$

Let $\widetilde{X}$ be a Banach space of vectors sequences X with coordinatewise linear operations. $\widetilde{X}$ is said to be KB -space if $\lim\limits_{n\to\infty} \left\|\{x_k - \chi_k(n) x_k\}_{k\in N}\right\|_{\widetilde{X}} = 0$, where $\chi_k(n) = \begin{cases} 1, & k \leq n \\ 0, & k > 1 \end{cases}$. $\widetilde{X}$ is called CB -space if

$$\widetilde{X}^* = \left\{ \{t_n\}_{n\in N} \subset X^* : (\{t_n\}, \{x_n\}) = \sum_{n=1}^{\infty} t_n(x_n), \{x_n\}_{n\in N} \in \widetilde{X} \right\}.$$

Let $\widetilde{Y}$ be KB-space over Y. Say that $\widetilde{X}$ is normally subjected to $\widetilde{Y}$ if from $\{x_n\}_{n\in N} \subset X$, $\{\varphi_n\}_{n\in N} \subset Y$, $\|x_n\|_X \leq \|\varphi_n\|_Y$ and $\{\varphi_n\}_{n\in N} \in \widetilde{Y}$ it follows that $\{x_n\}_{n\in N} \in \widetilde{X}$ and $\left\|\{x_n\}_{n\in N}\right\|_{\widetilde{X}} \leq \left\|\{\varphi_n\}_{n\in N}\right\|_{\widetilde{Y}}$.

Definition 2. *Let $\{\varphi_n\}_{n\in N} \subset Y$ and $\{\varphi_n^*\}_{n\in N} \subset L(Z,X)$ be such that the following conditions are fulfilled:*

1) $\{\varphi_n^*(z)\}_{n\in N} \in \widetilde{X}$ for any $z \in Z$;

2) there exist the numbers $A > 0$, $B > 0$ such that $A\|z\|_Z \leq \left\|\{\varphi_n^*(z)\}_{n\in N}\right\|_{\widetilde{X}} \leq B\|z\|_Z$ for any $z \in Z$;

3) $z = \sum\limits_{n=1}^{\infty} b(\varphi_n^*(z), \varphi_n)$ for any $z \in Z$.

Then the pair $\left(\{\varphi_n^*\}_{n\in N}, \{\varphi_n\}_{n\in N}\right)$ is said to be b-atomic decomposition in Z with respect to $\widetilde{X}$ with the bounds A and B.

The b-atomic decomposition in Z with respect to $\widetilde{X}$ is said to be $b_{\widetilde{X}}$-atomic decomposition in Z.

Now study perturbations of $b_{\widetilde{X}}$-atomic decomposition in Banach space.

Theorem 1. *Let $\widetilde{X}$ be KB-space, $\left(\{\varphi_n^*\}_{n\in N}, \{\varphi_n\}_{n\in N}\right)$ be $b_{\widetilde{X}}$-atomic decomposition in Z with the bounds A* and *B, the system $\{\psi_n\}_{n\in N} \subset Y$. Assume that there exist the numbers $\lambda, \beta, \mu \geq 0$ such that*

(a) $\max\{\beta; \lambda + \mu B\} < 1$;

(b) $\left\|\sum\limits_i b(x_i, \varphi_i - \psi_i)\right\|_Z \leq \lambda \left\|\sum\limits_i b(x_i, \varphi_i)\right\|_Z + \beta \left\|\sum\limits_i b(x_i, \psi_i)\right\|_Z + \mu \|\{x_i\}\|_{\widetilde{X}}$

for any finite $\{x_i\} \in \widetilde{X}$.

Then there exists $\{\psi_n^*\}_{n\in N} \subset L(Z,X)$ such that $\left(\{\psi_n^*\}_{n\in N}, \{\psi_n\}_{n\in N}\right)$ is $b_{\widetilde{X}}$-atomic decomposition in Z with the bounds

$$\frac{(1-\beta)A}{1+(\lambda+\mu B)} \quad \text{and} \quad \frac{(1+\beta)B}{1-(\lambda+\mu B)}.$$

Proof. At first prove the convergence of the series $\sum\limits_{n=1}^{\infty} b(\varphi_n^*(z), \psi_n)$ for any $z \in Z$. For any $m, p \in N$ we have

$$\left\|\sum_{n=m+1}^{m+p} b(\varphi_n^*(z), \psi_n)\right\|_Z = \left\|\sum_{n=m+1}^{m+p} b(\varphi_n^*(z), \varphi_n) + \sum_{n=m+1}^{m+p} b(\varphi_n^*(z), \varphi_n - \psi_n)\right\|_Z \leq$$

$$\leq \left\| \sum_{n=m+1}^{m+p} b\left(\varphi_n^*\left(z\right), \varphi_n\right) \right\|_Z + \left\| \sum_{n=m+1}^{m+p} b\left(\varphi_n^*\left(z\right), \varphi_n - \psi_n\right) \right\|_Z \leq$$

$$\leq (1+\lambda) \left\| \sum_{n=m+1}^{m+p} b\left(\varphi_n^*\left(z\right), \varphi_n\right) \right\|_Z + \beta \left\| \sum_{n=m+1}^{m+p} b\left(\varphi_n^*\left(z\right), \psi_n\right) \right\|_Z +$$

$$+\mu \left\| \left\{ \varphi_n^*\left(z\right) \left(\chi_n\left(m+p\right) - \chi_n\left(m\right)\right) \right\}_{n\in N} \right\|_{\widetilde{X}}, \quad z \in Z.$$

Hence

$$\left\| \sum_{n=m+1}^{m+p} b\left(\varphi_n^*\left(z\right), \psi_n\right) \right\|_Z \leq \frac{1+\lambda}{1-\beta} \left\| \sum_{n=m+1}^{m+p} b\left(\varphi_n^*\left(z\right), \varphi_n\right) \right\|_Z +$$

$$+\frac{\mu}{1-\beta} \left\| \left\{ \varphi_n^*\left(z\right) \left(\chi_n\left(m+p\right) - \chi_n\left(m\right)\right) \right\}_{n\in N} \right\|_{\widetilde{X}}, \quad z \in Z.$$

Consequently, the series $\sum_{n=1}^{\infty} b\left(\varphi_n^*\left(z\right), \psi_n\right)$ converges. Define the operator $T : Z \to Z$ by the formula $Tz = \sum_{n=1}^{\infty} b\left(\varphi_n^*\left(z\right), \psi_n\right)$. We get

$$\left\| z - Tz \right\|_Z = \left\| \sum_{n=1}^{\infty} b\left(\varphi_n^*\left(z\right), \varphi_n - \psi_n\right) \right\|_Z \leq \lambda \left\| \sum_{n=1}^{\infty} b\left(\varphi_n^*\left(z\right), \varphi_n\right) \right\|_Z +$$

$$+\beta \left\| \sum_{n=1}^{\infty} b\left(\varphi_n^*\left(z\right), \psi_n\right) \right\|_Z + \mu \left\| \left\{ \varphi_n^*\left(z\right) \right\}_{n\in N} \right\|_{\widetilde{X}} \leq \left(\lambda + \mu B\right) \left\| z \right\|_Z + \beta \left\| Tz \right\|_Z .$$

Then according to the condition, the operator T is a bounded inverse operator and

$$\frac{1-\beta}{1+\left(\lambda+\mu B\right)} \leq \left\| T^{-1} \right\| \leq \frac{1+\beta}{1-\left(\lambda+\mu B\right)}.$$

Define the operator $\psi_n^* : Z \to X$ from the formula $\psi_n^*\left(z\right) = \varphi_n^*\left(T^{-1}z\right)$. Show that $\left(\left\{\psi_n^*\right\}_{n\in N}, \left\{\psi_n\right\}_{n\in N}\right)$ is $b_{\widetilde{X}}$-atomic decomposition in Z. For $z \in Z$ we have $\left\{\psi_n^*\left(z\right)\right\}_{n\in N} = \left\{\varphi_n^*\left(T^{-1}z\right)\right\}_{n\in N} \in \widetilde{X}$ or $T^{-1}z \in Z$.

In what follows

$$\left\| \left\{\psi_n^*\left(z\right)\right\}_{n\in N} \right\|_{\widetilde{X}} = \left\| \left\{\varphi_n^*\left(T^{-1}z\right)\right\}_{n\in N} \right\|_{\widetilde{X}} \leq$$

$$\leq B \left\| T^{-1}z \right\|_Z \leq B \left\| T^{-1} \right\| \left\| z \right\|_Z \leq \frac{\left(1+\beta\right) B}{1-\left(\lambda+\mu B\right)} \left\| z \right\|_Z ,$$

$$\left\| \left\{\psi_n^*\left(z\right)\right\}_{n\in N} \right\|_{\widetilde{X}} = \left\| \left\{\varphi_n^*\left(T^{-1}z\right)\right\}_{n\in N} \right\|_{\widetilde{X}} \geq$$

$$\geq A \left\| T^{-1}z \right\|_Z \geq A \left\| T \right\|^{-1} \left\| z \right\|_Z \geq \frac{\left(1-\beta\right) A}{1+\left(\lambda+\mu B\right)} \left\| z \right\|_Z$$

$$z = T\left(T^{-1}z\right) = \sum_{n=1}^{\infty} b\left(\varphi_n^*\left(T^{-1}z\right), \psi_n\right) = \sum_{n=1}^{\infty} b\left(\psi_n^*\left(z\right), \psi_n\right) \text{ for any } z \in Z.$$

The theorem is proved.

From the theorem proved we get the generalization of the theorem on stability of atomic decomposition in Banach space (see [4]).

Corollary. *Let $\widetilde{X}$ be KB space, $\left(\{\varphi_n^*\}_{n\in N}, \{\varphi_n\}_{n\in N}\right)$ be $b_{\widetilde{X}}$-atomic decomposition in Z with the bounds A and B, the system $\{\psi_n\}_{n\in N} \subset Y$. Assume that there exist the numbers $\lambda, \mu \geq 0$ such that*

(a) $\lambda + \mu B < 1$;

(b) $\left\|\sum_i b(x_i, \varphi_i - \psi_i)\right\|_Z \leq \lambda \left\|\sum_i b(x_i, \varphi_i)\right\|_Z + \mu \|\{x_i\}\|_{\widetilde{X}}$ *for any finite* $\{x_i\} \in \widetilde{X}$.

Then there exists $\{\psi_n^*\}_{n\in N} \subset L(Z, X)$ such that $\left(\{\psi_n^*\}_{n\in N}, \{\psi_n\}_{n\in N}\right)$ is $b_{\widetilde{X}}$-atomic decomposition in Z with the bounds

$$\frac{A}{1 + (\lambda + \mu B)} \quad and \quad \frac{B}{1 - (\lambda + \mu B)}.$$

Definition 3. *The system $\{\varphi_n\}_{n\in N} \subset Y$ is said to be ω_b-linear independent in Z with respect to $\widetilde{X}$ if from $\sum_{n=1}^{\infty} b(x_n, \varphi_n) = 0$, $\{x_n\}_{n\in N} \in \widetilde{X}$ it follows that $x_n = 0$ for any $n \in N$.*

Let $f \in Z^*$ and $y \in Y$. Consider the mapping $< f, y >: X \to C$ by formula $< f, y > (x) = f(b(x, y))$. It is clear that $< f, y > \in X$, and

$$\|< f, y >\| \leq M \|f\|_{Z^*} \|y\|_Y .$$

Theorem 2. *Let $\widetilde{Y}$ be KB space of vectors sequences Y, $\widetilde{X}$ be CB space such that $\widetilde{X}^*$ is normally subjected to $\widetilde{Y}$, the systems $\{\varphi_n\} \subset Y$ and $\{\varphi_n^*\}_{n\in N} \subset K(Z, X)$ be such that $\left(\{\varphi_n^*\}_{n\in N}, \{\varphi_n\}_{n\in N}\right)$ is $b_{\widetilde{X}}$-atomic decomposition in Z with the bounds A and B, the system $\{\psi_n\}_{n\in N} \subset Y$ be ω_b linear-independent with respect to $\widetilde{X}$ and such that $\{\varphi_n - \psi_n\}_{n\in N} \in \widetilde{Y}$. Then there exists $\{\psi_n^*\}_{n\in N} \subset L(Z, X)$ such that $\left(\{\psi_n^*\}_{n\in N}, \{\psi_n\}_{n\in N}\right)$ is $b_{\widetilde{X}}$-atomic decomposition in Z.*

Proof. For any $m, p \in N$ and $z \in Z$ from the corollary of Han-Banach space there exists $f_{m,p} \in Z^* : \|f_{m,p}\| = 1$ such that

$$\left\|\sum_{n=m+1}^{m+p} b(\varphi_n^*, \varphi_n - \psi_n)\right\|_Z = f_{m,p}\left(\sum_{n=m+1}^{m+p} b(\varphi_n^*(z), \varphi_n - \psi_n)\right).$$

We have

$$\left\|\sum_{n=m+1}^{m+p} b(\varphi_n^*(z), \varphi_n - \psi_n)\right\|_Z \leq \left|\sum_{n=m+1}^{m+p} f_{m,p}(b(\varphi_n^*(z), \varphi_n - \psi_n))\right| \leq$$

$$\leq \left|\sum_{n=m+1}^{m+p} < f_{m,p}, \varphi_n - \psi_n > (\varphi_n^*(z))\right| =$$

$$= \left|\left(\{(\chi_n(m+p) - \chi_n(m)) < f_{m,p}, \varphi_n - \psi_n >\}_{n\in N}, \{\varphi_n^*(z)\}_{n\in N}\right)\right| \leq$$

$$\leq \left\| \{ (\chi_n (m+p) - \chi_n (m)) < f_{m,p}, \varphi_n - \psi_n > \}_{n\in N} \right\|_{\widetilde{X}^*} \left\| \{ \varphi_n^* (z) \}_{n\in N} \right\|_{\widetilde{X}} \leq$$
$$\leq M \| f_{m,p} \| \left\| \{ (\chi_n (m+p) - \chi_n (m)) (\varphi_n - \psi_n) \}_{n\in N} \right\|_{\widetilde{Y}} \left\| \{ \varphi_n^* (z) \}_{n\in N} \right\|_{\widetilde{X}} \leq$$
$$\leq MB \left\| \{ (\chi_n (m+p) - \chi_n (m)) (\varphi_n - \psi_n) \}_{n\in N} \right\|_{\widetilde{Y}} \| z \|_Z .$$

Consequently, the series $\sum_{n=1}^{\infty} b(\varphi_n^*(z), \psi_n)$ converges for any $z \in Z$. Consider the operator $T : Z \to Z$ determined from the formula $T(z) = \sum_{n=1}^{\infty} b(\varphi_n^*(z), \varphi_n - \psi_n)$, $z \in Z$. It is clear that T is a completely continuous operator. Therefore, the operator $F = I - T$ is a Fredholm operator, and $F(z) = \sum_{n=1}^{\infty} b(\varphi_n^*(z), \psi_n)$. Find $KerF$. If $z \in KerF$, then $\sum_{n=1}^{\infty} b(\varphi_n^*(z), \psi_n) = 0$. Then from ω_b-linear independence of $\{\psi_n\}_{n\in N}$ with respect to $\widetilde{X}$ we get $\varphi_n^*(z) = 0$ for any $n \in N$, and $z = 0$. Thus, $KerF = \{0\}$ and consequently F is boundedly invertible. Let $\psi_n^* : Z \to X$ be determined by the expression $\psi_n^*(z) = \varphi_n^*\left(F^{-1}(z)\right)$, $z \in Z$. Then $\psi_n^* \in L(Z, X)$ and $\{\psi_n^*(z)\}_{n\in N} = \left\{\varphi_n^*\left(F^{-1}(z)\right)\right\}_{n\in N} \in \widetilde{X}$, $z \in Z$ or $F^{-1}(z) \in Z$. Then for any $z \in Z$ we get

$$z = F\left(F^{-1}(z)\right) = \sum_{n=1}^{\infty} b\left(\varphi_n^*\left(F^{-1}(z)\right), \psi_n\right) = \sum_{n=1}^{\infty} b(\psi_n^*(z), \psi_n)$$

and

$$\left\| \{\psi_n^*(z)\}_{n\in N} \right\|_{\widetilde{X}} = \left\| \left\{ \varphi_n^*\left(F^{-1}(z)\right) \right\}_{n\in N} \right\|_Z \leq B \left\| F^{-1}(z) \right\|_Z \leq B \left\| F^{-1} \right\| \|z\|_Z ,$$

$$\left\| \{\psi_n^*(z)\}_{n\in N} \right\|_{\widetilde{X}} = \left\| \left\{ \varphi_n^*\left(F^{-1}(z)\right) \right\}_{n\in N} \right\|_Z \geq A \left\| F^{-1}(z) \right\|_Z \geq A \left\| F \right\|^{-1} \|z\|_Z .$$

The theorem is proved.

Theorem 3. *Let $\widetilde{Y}$ be KB-space of vectors sequences Y, $\widetilde{X}$ be CB - space such that $\widetilde{X}^*$ is normally subjected to $\widetilde{Y}$, the systems $\{\varphi_n\} \subset Y$ and $\{\varphi_n^*\}_{n\in N} \subset L(Z, X)$ be such that $\left(\{\varphi_n^*\}_{n\in N}, \{\varphi_n\}_{n\in N}\right)$ is $b_{\widetilde{X}}$ -atomic decomposition in Z with the bounds A and B, the system $\{\psi_n\}_{n\in N} \subset Y$ be such that $\left\| \{\varphi_n - \psi_n\}_{n\in N} \right\|_{\widetilde{Y}} < \frac{1}{MB}$. Then there exists $\{\psi_n^*\}_{n\in N} \subset L(Z, X)$ such that $\left(\{\psi_n^*\}_{n\in N}, \{\psi_n\}_{n\in N}\right)$ is $b_{\widetilde{X}}$ -atomic decomposition in Z.*

Proof. As in the proof of theorem 2, we get that the series $\sum_{n=1}^{\infty} b(\varphi_n^*(z), \psi_n)$ converges for any $z \in Z$, and

$$\left\| \sum_{n=1}^{\infty} b(\varphi_n^*(z), \varphi_n - \psi_n) \right\|_Z \leq M \quad B \left\| \{\varphi_n - \psi_n\}_{n\in N} \right\|_{\widetilde{Y}} \|z\|_Z .$$

Let the operator $T : Z \to Z$ be given by the formula $T(z) = \sum_{n=1}^{\infty} b(\varphi_n^*(z), \varphi_n - \psi_n)$, $z \in Z$. Then $\|T\| \leq M \quad B \left\| \{\varphi_n - \psi_n\}_{n\in N} \right\|_{\widetilde{Y}}$ and by the same token, $\|T\| < 1$.

Therefore, the operator $F = I - T$ is boundedly inverse. Further, the theorem is proved similar to the proof of theorem 2.

The theorem is proved

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